# QCD

problem set 1

1. QED. In QED the Maxwell equations can be derived from a variational principle, analogously to the equations of motion in classical mechanics. To this end we have to define degrees of freedom, in the QED case these are electromagnetic potentials expressed in terms of the potential four-vector $A^\mu(\boldsymbol{x}, t) = (\Phi, \boldsymbol{A})$, and the action. Before defining the action we have to introduce the field strength:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu. \tag{1}$$

Write explicitly the 4 dimensional matrix $F^{\mu\nu}$ in terms of the electromagnetic fields (we work in natural units):

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}, \qquad \boldsymbol{E} = -\boldsymbol{\nabla}\Phi - \frac{\partial \boldsymbol{A}}{\partial t}. \tag{2}$$

2. Show that the field tensor satisfies the Jacobi identity

$$\partial^\lambda F^{\mu\nu} + \partial^\nu F^{\lambda\mu} + \partial^\mu F^{\nu\lambda} \equiv 0. \tag{3}$$

Express this identity in terms of the electromagnetic fields (2), and show that it implies two out of four Maxwell equations.

3. Remaining two Maxwell equations can be derived from the variational principle. To this end we define first the *Lagrange density*

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\nu A_\nu \tag{4}$$

where $j^\nu$ denotes external electromagnetic current. Express $\mathcal{L}$ in terms of fields (2).

4. Now we define the action

$$S = \int d^4x \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - j^\nu A_\nu \right]. \tag{5}$$

Consider variation of the $A^\mu$ field

$$A^\mu \to A^\mu + (\delta A)^\mu \tag{6}$$

at fixed $j^\nu$. Calculate step by step $\delta S$ corresponding to transformation (6). At some point you will have to perform integration by parts, assuming that all fields vanish for large $x^\mu$. Show that the requirement $\delta S = 0$ is equivalent to the two "missing" Maxwell equations.

5. Show that the very specific transformation

$$(\delta A)^\mu = \partial^\mu \chi(\boldsymbol{x}, t), \tag{7}$$

called *gauge transformation*, leaves $F^{\mu\nu}$ invariant. What is $\delta S$ corresponding to the gauge transformation? How to interpret the obtained result?

6. In full theory the current $j^\mu$ is constructed from dynamical matter fields $\psi$, be it fermions or bosons. To describe matter fields the pertinent lagrangian density has to be added to (4). This density is at most quadratic in field derivatives $\partial^\mu \psi$ and is invariant under *global* transformation

$$\psi(x) \to \psi'(x) = e^{-i\theta}\psi(x), \tag{8}$$

it is however *not* invariant under local transformation where $\theta \to \theta(x)$ ($x$ being four-point), so called *gauge transformation*. In order to make the theory *gauge invariant* one replaces

$$\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu \tag{9}$$

where $q$ is a constant (in fact in the case of QED it corresponds to the electric charge of field $\psi$). Find transformation of the vector filed $A^\mu$ from the requirement

$$D'_\mu \psi'(x) = e^{-i\theta(x)} D_\mu \psi(x), \tag{10}$$

where $D'_\mu$ is simply $D_\mu(A'_\mu)$. Note that transformation (10) implies

$$D'_\mu = e^{-i\theta(x)} D_\mu e^{i\theta(x)}. \tag{11}$$

7. Yang-Mills fields. Suppose now that instead of a phase transformation (8), we impose a non-Abelian local SU($N$) transformation

$$\Psi(x) \to \Psi'(x) = U(x)\Psi(x) \tag{12}$$

where $\Psi$ is a $N$-component vector of fields and matrix $U$ can be parameterized as follows

$$U(x) = e^{-i\theta_m(x)T^m},$$

where $T^m$ are *generators* of the SU($N$) group (for SU(2) $T^m = \tau^m/2$). To make such theory gauge invariant we have to introduce $N$ vector fields $A^m_\mu$ and define the covariant derivative as follows

$$D_\mu = \partial_\mu + igT^m A^m_\mu(x) = \partial_\mu + ig\boldsymbol{A}_\mu(x). \tag{13}$$

Guided by Eq. (11) find transformation law for the $\boldsymbol{A}_\mu$ fields.