Appendix B

The Relation between Minkowski and Euclidean Actions

The Minkowski action leads to canonical quantization and it is used to calculate matrix elements of the evolution operator e^{-iHt}. The Euclidean action is used to calculate the partition function $tre^{-\beta H}$. Lattice calculations are formulated in terms of the Euclidean action. In Minkowski space $g_{\mu\nu} = (1,-1,-1,-1)$ and $\det g = -1$, whereas in Euclidean space $g_{\mu\nu} = (1,1,1,1) = \delta_{\mu\nu}$ and $\det g = +1$. As a rule of the thumb, a Euclidean action can be transformed into a Minkowski action by the following substitutions:

Euclidean	$\to$	Minkowski
$g_{\mu\nu} = diag\,(1,1,1,1) = \delta_{\mu\nu}$	$\to$	$g_{\mu\nu} = diag\,(1,-1,-1,-1)$
$(t, \vec{r})$	$\to$	$(it, \vec{r})$
$(\partial_t, \partial_i) = (\partial_t, \nabla_i)$	$\to$	$(-i\partial_t, \partial_i) = (-i\partial_t, -\nabla_i)$
(A_0, A_i)	$\to$	(iA^0, A^i)
$\left(j_0, \vec{j}\right)$	$\to$	$\left(ij_0, \vec{j}\right)$
$A_\mu A_\mu$	$\to$	$-A^\mu A_\mu$
$\partial^2 = \partial_0^2 + \partial_i^2 = \partial_t^2 + \partial_i^2$	$\to$	$-\partial^2 = -\partial_\mu \partial^\mu$
$(\partial \wedge A)_{0i} = \partial_0 A_i - \partial_i A_0$	$\to$	$-i\,(\partial \wedge A)^{0i} = -i\,(\partial^0 A^i - \partial^i A^0)$
$(\partial \wedge A)_{ij} = \partial_i A_j - \partial_j A_i$	$\to$	$-\,(\partial \wedge A)^{ij} = -\partial^i A^j + \partial^j A^i$
$\frac{1}{4}\,(\partial \wedge A)_{\mu\nu}^2$	$\to$	$\frac{1}{4}\,(\partial \wedge A)_{\mu\nu}\,(\partial \wedge A)^{\mu\nu}$
$\varepsilon_{\mu\nu\alpha\beta} = \varepsilon^{\mu\nu\alpha\beta}$	$\to$	$\varepsilon^{\mu\nu\alpha\beta} = -\varepsilon_{\mu\nu\alpha\beta}$
$\varepsilon_{0123} = \varepsilon^{0123} = 1$	$\to$	$\varepsilon^{0123} = -\varepsilon_{0123} = 1$
$\overline{\partial \wedge A}_{0i} = \frac{1}{2}\varepsilon_{0ijk}\,(\partial \wedge A)_{jk}$	$\to$	$-\frac{1}{2}\varepsilon^{0ijk}\,(\partial \wedge A)_{jk} = -\overline{\partial \wedge A}^{0i}$

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Euclidean	$\rightarrow$	Minkowski
$\overline{\partial\wedge A}_{ij}=\varepsilon_{ij0k}\left(\partial\wedge A\right)_{0k}$	$\rightarrow$	$i\overline{\partial\wedge A}^{ij}=i\varepsilon^{ij0k}\left(\partial\wedge A\right)_{0k}$
$\frac{1}{4}\overline{\partial\wedge A}_{\mu\nu}\overline{\partial\wedge A}_{\mu\nu}$	$\rightarrow$	$-\frac{1}{4}\overline{\partial\wedge A}_{\mu\nu}\overline{\partial\wedge A}^{\mu\nu}$
$\int d^4x=\int dtdxdydz$	$\rightarrow$	$\int d^4x=\int dtdxdydz$
$(action)$	$\rightarrow$	$-(action)$
$\vec{E}$	$\rightarrow$	$-i\vec{E}$
$\vec{H}$	$\rightarrow$	$\vec{H}$

(B.1)

For example, the Minkowski Landau-Ginzburg action (3.8) of a dual superconductor is:

$$I_j\left(B,S,\varphi\right)=$$

$$\int d^4x\left(-\frac{1}{2}\left(\partial\wedge B+\bar{G}\right)^2+\frac{g^2S^2}{2}\left(B+\partial\varphi\right)^2+\frac{1}{2}\left(\partial S\right)^2-\frac{1}{2}b\left(S^2-v^2\right)^2\right) \tag{B.2}$$

whereas the Euclidean action is:

$$I_j\left(B,S,\varphi\right)=$$

$$\int d^4x\left(\frac{1}{2}\left(\partial\wedge B+\bar{G}\right)^2+\frac{g^2S^2}{2}\left(B-\partial\varphi\right)^2+\frac{1}{2}\left(\partial S\right)^2+\frac{1}{2}b\left(S^2-v^2\right)^2\right) \tag{B.3}$$

The table (B.1) can be used to recover the Minkowski action (B.2) from the Euclidean action (B.3). The change in sign of the action is chosen such that the partition function can be written in terms of a functional integral of the Euclidean action, in the form:

$$Z=e^{-\beta H}=\int D\left(B,S,\varphi\right)e^{-I_j(B,S,\varphi)} \tag{B.4}$$

In general however, the functional integrals need to be adapted to the acting constraints..

We can choose to represent the Euclidean field tensor $F^{\mu\nu}=F_{\mu\nu}$ in terms of Euclidean electric and magnetic fields $\vec{E}$ and $\vec{H}$ thus:

$$F^{\mu\nu}=\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -H_z & H_y \\ E_y & H_z & 0 & -H_x \\ E_z & -H_y & H_x & 0 \end{pmatrix}$$

$$\overline{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F_{\alpha\beta} = \begin{pmatrix} 0 & -H_x & -H_y & -H_z \\ H_x & 0 & E_z & -E_y \\ H_y & -E_z & 0 & E_x \\ H_z & E_y & -E_x & 0 \end{pmatrix} \tag{B.5}$$

If we want to express the Euclidean field tensor as $F = \partial \wedge A$ then the relation between the Euclidean and Minkowski electric and magnetic fields is the one given at the end of table B.1.The Euclidean electric and magnetic fields $\vec{E}$ and $\vec{H}$ are expressed in terms of the Euclidean gauge potential $A_\mu = \left(\phi, \vec{A}\right)$ as follows:

$$\vec{E} = -\partial_t \vec{A} + \vec{\nabla}\phi \qquad \vec{H} = -\vec{\nabla} \times \vec{A} \tag{B.6}$$

In the Euclidean formulation, $\varepsilon^2 = G$ and the duality transformation of antisymmetric tensors is reversible without a change in sign:

$$\bar{S}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}S_{\alpha\beta} \qquad S_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\bar{S}_{\alpha\beta} \tag{B.7}$$

The projectors K and E are defined by (A.42) with $g_{\mu\nu} = \delta_{\mu\nu}$ and we have:

$$K^2 = K = \varepsilon K \varepsilon \qquad E^2 = E \qquad KE = 0 \qquad K + E = G \tag{B.8}$$

with:

$$G_{\mu\nu,\alpha\beta} = \left(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\mu\beta}\delta_{\nu\alpha}\right) \tag{B.9}$$