Renormalization in QCD

M.Praszalowicz

February 23, 2020

1 Fermion self-energy

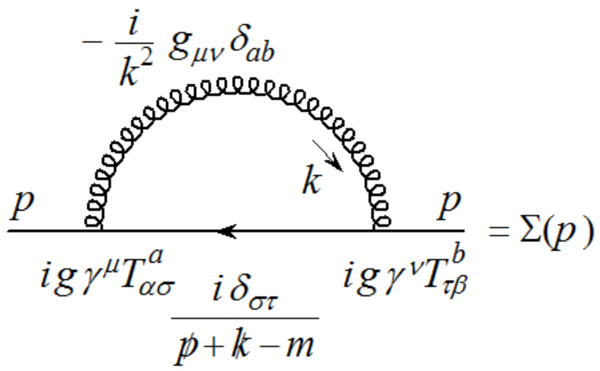

Figure 1: Feynman diagram corresponding to the quark self-energy.

We shall calculate diagram in Fig. 1 and using this example discuss renormalization in QCD. We have (in Feynman gauge)

$$\Sigma(p) = -g^2\mu^{4-d}C_F\delta_{\alpha\beta}\int \frac{d^dk}{(2\pi)^d}\frac{\gamma^\mu(\not p + \not k + m)\gamma_\mu}{[(p+k)^2 - m^2]\; k^2} \tag{1}$$

where we have anticipated that in order to regularize the integral in (1) we have to change the dimensionality of the integral

$$4 \to d = 4 - 2\varepsilon \tag{2}$$

where $\varepsilon \to 0_+$. The method of changing dimensionality of space-time, known as *dimensional regularization* proposed by Veltman and 't Hooft, has great advantage over some other regularization methods, namely it preserves gauge invariance. It becomes problematic in theories that involve γ_5 Dirac matrix in the interaction vertex, as in the case of weak interactions.

In the following we shall put $m = 0$, which creates a problem in the infrared that we will discuss later. Recall that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}. \tag{3}$$

Therefore

$$\begin{aligned}
\gamma^\mu(\not{p} + \not{k})\gamma_\mu &= g_{\mu\nu}\gamma^\mu\gamma^\tau\gamma^\nu(p+k)_\tau \\
&= g_{\mu\nu}\gamma^\mu\left(2g^{\tau\nu} - \gamma^\nu\gamma^\tau\right)(p+k)_\tau \\
&= 2(\not{p} + \not{k}) - d(\not{p} + \not{k}) \\
&= -2(1-\varepsilon)(\not{p} + \not{k}),
\end{aligned} \tag{4}$$

where we have used

$$g_{\mu\nu}g^{\mu\nu} = d. \tag{5}$$

Hence

$$\begin{aligned}
\Sigma(p) &= 2(1-\varepsilon)\, g^2\mu^{2\varepsilon}C_F\delta_{\alpha\beta}\int \frac{d^dk}{(2\pi)^d}\frac{\not{p} + \not{k}}{(p+k)^2\, k^2} \\
&= 2(1-\varepsilon)\, g^2\mu^{2\varepsilon}C_F\delta_{\alpha\beta}\left[\not{p}I + \gamma_\mu I^\mu\right].
\end{aligned} \tag{6}$$

We have to calculate two integrals

$$\{I, I^\mu\} = \int \frac{d^dk}{(2\pi)^d}\frac{1}{(p+k)^2\, k^2}\{1, k^\mu\}. \tag{7}$$

We shall introduce now Feynman parametrization for the propagators in (7):

$$\begin{aligned}
\frac{1}{(p+k)^2\, k^2} &= \int_0^1 dx \frac{1}{\left(k^2 + 2x\, p\cdot k + x\, p^2\right)^2} \\
&= \int_0^1 dx \frac{1}{\left((k^2 + 2x\, p\cdot k + x^2p^2) + x(1-x)\, p^2\right)^2}.
\end{aligned} \tag{8}$$

Changing variables

$$k^\mu \to k^\mu + xp^\mu \tag{9}$$

and introducing

$$M^2 = -x(1-x)\, p^2 \tag{10}$$

we arrive at:

$$\{I, I^\mu\} = \int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{1}{\left(k^2 - M^2\right)^2}\{1, k^\mu - xp^\mu\}. \tag{11}$$

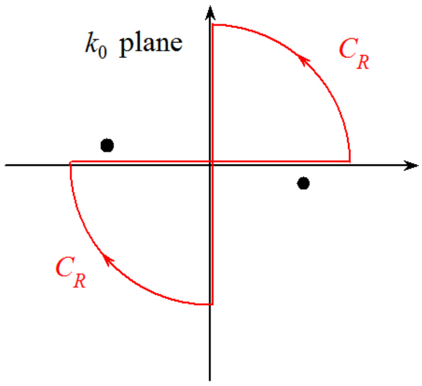

Figure 2: Integration contour over k_0. Black dots denote poles of Feynman propagators.

In order to calculate the integral over d^dk,which is the integral in Minkowski space, we observe that:

$$\left\{ \int\limits_{-\infty}^{\infty} + \int\limits_{C_R} + \int\limits_{+i\infty}^{-i\infty} \right\} dk^0 = 0. \tag{12}$$

Since the integral over C_R vanishes

$$\int\limits_{-\infty}^{\infty} dk^0 = - \int\limits_{+i\infty}^{-i\infty} dk^0 = i \int\limits_{-\infty}^{+\infty} dE \tag{13}$$

where $k^0 = iE$. Therefore the integral over d^dk in Minkowski space transforms into the Euclidean integral

$$\{I, I^\mu\} = i \int\limits_0^1 dx \int \frac{d^d\vec{k}}{(2\pi)^d} \frac{1}{\left(-\vec{k}^2 - M^2\right)^2} \{1, k^\mu - xp^\mu\} \tag{14}$$

where

$$\vec{k} = (E, k^1, k^2, \ldots, k^{d-1}). \tag{15}$$

1.1 Integrals in d dimensions

In order to calculate integrals (14) we shall introduce spherical coordinates in d dimensions. First we chose arbitrarily a d-th axis (equivalent of the z axis in three dimensions) and project on it $\vec{k}$ vector with $\cos\theta_{d-1}$. Therefore a projection on the $d-1$ dimensional subspace orthogonal do the d-th axis is $k\sin\theta_{d-1}$. Now we choose an axis in this $d-1$ dimensional subspace, the $d-1$ axis, and project on this axis this projection with $\cos\theta_{d-2}$. Next, a projection on the the $d-2$ dimensional subspace orthogonal do the d-th and $d-1$

axes involves $\sin\theta_{d-2}$. We continue this procedure until we "run out of dimensions" with the result:

$$\begin{aligned} k_d &= k\cos\theta_{d-1}, \\ k_{d-1} &= k\sin\theta_{d-1}\cos\theta_{d-2}, \\ &\dots \\ k_2 &= k\sin\theta_{d-1}\sin\theta_{d-2}\dots\cos\theta_1, \\ k_1 &= k\sin\theta_{d-1}\sin\theta_{d-2}\dots\sin\theta_1, \end{aligned} \tag{16}$$

where $\theta_1 \in (0, 2\pi), \quad \theta_{i>1} \in (0, \pi)$ and

$$\int d\Omega_d = \int \prod_{i=1}^{d-1} \left(\sin^{i-1}\theta_i d\theta_i\right) = 2\prod_{i=1}^{d-1}\left(\int_0^\pi \sin^{i-1}\theta_i d\theta_i\right). \tag{17}$$

If the integral depends only on k^2 we can perform angular integral with the help of the following formula

$$\int_0^\pi \sin^n\theta\, d\theta = B\left(\frac{1+n}{2}, \frac{1}{2}\right) \tag{18}$$

and the result reads

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}. \tag{19}$$

Let's check. In two dimensions $d = 2$

$$\int d\Omega_2 = \frac{2\pi}{\Gamma(1)} = 2\pi \tag{20}$$

is the length of a circle of radius $r = 1$. In $d = 3$

$$\int d\Omega_3 = \frac{2\pi^{3/2}}{\Gamma(3/2)}. \tag{21}$$

Now

$$\Gamma(1/2) = \sqrt{\pi},\ \Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{\sqrt{\pi}}{2} \tag{22}$$

and

$$\int d\Omega_3 = 4\pi, \tag{23}$$

etc. Since we are working in $d = 4 - 2\varepsilon$ dimensions

$$\int d\Omega_d = \frac{2\pi^{2-\varepsilon}}{\Gamma(2-\varepsilon)}. \tag{24}$$

1.2 Continuation

Angular integration nullifies k^μ and we arrive at[1]

$$
\begin{aligned}
\{I, I^\mu\} &= \frac{i}{\Gamma(2-\varepsilon)} \frac{2\pi^{2-\varepsilon}}{(2\pi)^{4-2\varepsilon}} \int\limits_0^1 dx \, \{1, -xp^\mu\} \int\limits_0^\infty dk \frac{k^{d-1}}{\left(k^2+M^2\right)^2} \\
&= \frac{i}{\Gamma(2-\varepsilon)} \frac{(4\pi)^\varepsilon}{2^3\pi^2} \int\limits_0^1 dx \, \{1, -xp^\mu\} \, M^{d-4} \int\limits_0^\infty d\left(\frac{k}{M}\right) \frac{(k/M)^{d-1}}{\left(1+(k/M)^2\right)^2} \\
&= \frac{i}{\Gamma(2-\varepsilon)} \frac{1}{2^3\pi^2} \left(\frac{4\pi}{-p^2}\right)^\varepsilon \int\limits_0^1 dx \, [x(1-x)]^{-\varepsilon} \{1, -xp^\mu\} \int\limits_0^\infty dr \frac{r^{d-1}}{(1+r^2)^2}, \qquad (25)
\end{aligned}
$$

where $r = k/M$. Introducing

$$
t = r^2, \; dt = 2rdr \;\rightarrow\; dr = \frac{1}{2} dt \, t^{-1/2} \qquad (26)
$$

we arrive at

$$
\{I, I^\mu\} = \frac{i}{\Gamma(2-\varepsilon)} \frac{1}{2^4\pi^2} \left(\frac{4\pi}{-p^2}\right)^\varepsilon \int\limits_0^1 dx \, [x(1-x)]^{-\varepsilon} \{1, -xp^\mu\} \int\limits_0^\infty dt \, \frac{t^{1-\varepsilon}}{(1+t)^2}. \qquad (27)
$$

Two integrals over x and t are easy to calculate with the help of the following formulae:

$$
\begin{aligned}
\int\limits_0^\infty dt \, \frac{t^{x-1}}{(1+t)^{x+y}} &= B(x, y), \\
\int\limits_0^1 dx \, x^{\alpha-1}(1-x)^{\beta-1} &= B(\alpha, \beta). \qquad (28)
\end{aligned}
$$

In our case $x = 2-\varepsilon$, $y = \varepsilon$ and $\beta = 1-\varepsilon$ and $\alpha = 1-\varepsilon$ or $\alpha = 2-\varepsilon$ for I and I^μ respectively. Hence

$$
\begin{aligned}
\{I, I^\mu\} &= \frac{i}{\Gamma(2-\varepsilon)} \frac{1}{2^4\pi^2} \left(\frac{4\pi}{-p^2}\right)^\varepsilon \Big\{B(1-\varepsilon, 1-\varepsilon), -p^\mu B(2-\varepsilon, 1-\varepsilon)\Big\} B(2-\varepsilon, \varepsilon) \\
&= i \frac{1}{2^4\pi^2} \left(\frac{4\pi}{-p^2}\right)^\varepsilon \left\{ \frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)}, -p^\mu \frac{\Gamma(2-\varepsilon)}{\Gamma(3-2\varepsilon)} \right\} \Gamma(1-\varepsilon) \frac{\Gamma(\varepsilon)}{\Gamma(2)}. \qquad (29)
\end{aligned}
$$

[1] At the lecture we have split integration into an integral over dE and an integral over $d-1$ dimensional Euclidean space. Here we combine these two integrals into one integral over d dimensional Euclidean space.

1.3 Expanding Γ functions

Basic formulae:

$$\begin{aligned} z\Gamma(z) &= \Gamma(z+1), && (30)\\ \Gamma(1/2) &= \sqrt{\pi}. && (31)\end{aligned}$$

and

$$\Gamma(1-\varepsilon) = \exp\left(\gamma\varepsilon + \frac{\pi^2}{12}\varepsilon^2 + \ldots\right) \tag{32}$$

where γ is Euler constant. In the present calculation we work with accuracy $\mathcal{O}(\varepsilon)$.Therefore

$$\begin{aligned} \frac{\Gamma(1-\varepsilon)}{\Gamma(2-2\varepsilon)} &= \frac{\Gamma(1-\varepsilon)}{(1-2\varepsilon)\,\Gamma(1-2\varepsilon)} \simeq (1+2\varepsilon)e^{-\gamma\varepsilon}, && (33)\\ \frac{\Gamma(2-\varepsilon)}{\Gamma(3-2\varepsilon)} &= \frac{(1-\varepsilon)\Gamma(1-\varepsilon)}{2(1-\varepsilon)\,(1-2\varepsilon)\,\Gamma(1-2\varepsilon)} \simeq \frac{1}{2}(1+2\varepsilon)e^{-\gamma\varepsilon}, && (34)\\ \Gamma(1-\varepsilon)\frac{\Gamma(\varepsilon)}{\Gamma(2)} &= \Gamma(1-\varepsilon)\frac{\Gamma(1+\varepsilon)}{\varepsilon} \simeq \frac{1}{\varepsilon}. && (35)\end{aligned}$$

1.4 Continuation

We have

$$\{I, I^\mu\} = i\frac{1}{2^4\pi^2}\left(\frac{4\pi e^{-\gamma}}{-p^2}\right)^\varepsilon \left\{1, -\frac{1}{2}p^\mu\right\}(1+2\varepsilon)\frac{1}{\varepsilon} \tag{36}$$

and

$$\begin{aligned} \Sigma(p) &= (1-\varepsilon)\,g^2\mu^{2\varepsilon}C_F\delta_{\alpha\beta}\,i\frac{1}{2^4\pi^2}\left(\frac{4\pi e^{-\gamma}}{-p^2}\right)^\varepsilon \not{p}(1+2\varepsilon)\frac{1}{\varepsilon}\\ &= i\not{p}C_F\delta_{\alpha\beta}\frac{\alpha_s}{4\pi}\left(\frac{\mu^2 4\pi e^{-\gamma}}{-p^2}\right)^\varepsilon\left(\frac{1}{\varepsilon}+1\right) && (37)\end{aligned}$$

where

$$\alpha_s = \frac{g^2}{4\pi}. \tag{38}$$

Now the full propagator for a massless fermion reads (skipping color $\delta_{\alpha\beta}$)

$$S_F(p) = \frac{i}{\not{p}} + \frac{i}{\not{p}}\Sigma(p)\frac{i}{\not{p}} = \frac{i}{\not{p}}\left(1 - \frac{\alpha_s}{4\pi}C_F\left(\frac{\mu^2 4\pi e^{-\gamma}}{-p^2}\right)^\varepsilon\left(\frac{1}{\varepsilon}+1\right)\right). \tag{39}$$

At this point it is convenient to redefine the mass parameter entering (1) and (39):

$$\bar{\mu}^2 = \mu^2 4\pi e^{-\gamma}. \tag{40}$$

In practical terms renormalization proceeds by subtracting the infinity, in our case $1/\varepsilon$ pole, plus some finite parts. The minimal subtraction scheme (MS-bar) consits in subtracting the pole from (39): [2]

$$
\begin{aligned}
S_F^R(p) &= \frac{i}{\not p}\left(1-\frac{\alpha_s}{4\pi}C_F\left(\frac{\bar\mu^2}{-p^2}\right)^{\varepsilon}\left(\frac{1}{\varepsilon}+1\right)+\frac{\alpha_s}{4\pi}C_F\frac{1}{\varepsilon}\right) \\
&= \frac{i}{\not p}\left(1-\frac{\alpha_s}{4\pi}C_F\exp\left[\varepsilon\ln\left(\frac{\bar\mu^2}{-p^2}\right)\right]\left(\frac{1}{\varepsilon}+1\right)+\frac{\alpha_s}{4\pi}C_F\frac{1}{\varepsilon}\right) \\
&= \frac{i}{\not p}\left(1+\frac{\alpha_s}{4\pi}C_F\left(\ln\left(\frac{-p^2}{\bar\mu^2}\right)-1\right)\right). \qquad (41)
\end{aligned}
$$

We see that renormalized propagator is finite for $p^2 \neq 0$. For $p^2 = 0$ there is a logarithmic divergence, which is due to our approximation $m = 0$. Since this is an infrared divergence (more precisely a collinear one), we can regularize it by changing the dimension to $d = 4 + 2\kappa$, which effectively means that we replace $\varepsilon \to -\kappa$ and set $p^2 = 0$:

$$
\begin{aligned}
S_F^R(p) &= \frac{i}{\not p}\left(1-\frac{\alpha_s}{4\pi}C_F\left(\frac{-p^2}{\bar\mu^2}\right)^{\kappa}\left(-\frac{1}{\kappa}+1\right)-\frac{\alpha_s}{4\pi}C_F\frac{1}{\kappa}\right) \\
&\underset{p^2=0}{=} \frac{i}{\not p}\left(1-\frac{\alpha_s}{4\pi}C_F\frac{1}{\kappa}\right). \qquad (42)
\end{aligned}
$$

We see that the subtracted UV divergence is the only piece that survives in this limit. The pole $1/\kappa$ will be cancelled when we calculate full physical process including real emissions of gluons (photons) that are parallel to the quark line.

2 Renormalization

Let's observe that formal subtraction of the pole in (41) can be achieved by multiplication

$$
Z_2 S_F^R = S_F \qquad (43)
$$

where

$$
Z_2 = 1 - \frac{\alpha_s}{4\pi}C_F\frac{1}{\varepsilon} + \mathcal{O}(\alpha_s^2). \qquad (44)
$$

Here Z_2 and S_F are infinite for $\varepsilon \to 0$, while S_F^R is finite. Indeed, up to the first order in α_s we have

$$
\begin{aligned}
S_F^R &= \frac{i}{\not p}\left(1+\frac{\alpha_s}{4\pi}C_F\frac{1}{\varepsilon}\right)\left(1-\frac{\alpha_s}{4\pi}C_F\left(\frac{1}{\varepsilon}-\ln\left(\frac{-p^2}{\bar\mu^2}\right)+1\right)\right) \\
&= \frac{i}{\not p}\left(1+\frac{\alpha_s}{4\pi}C_F\left(\ln\left(\frac{-p^2}{\bar\mu^2}\right)-1\right)\right). \qquad (45)
\end{aligned}
$$

[2] By redefinition of μ^2 we in fact subtract also $\ln(4\pi e^{-\gamma})$

Note that

$$\frac{1}{Z_2} \simeq 1 + \frac{\alpha_s}{4\pi} C_F \frac{1}{\varepsilon}. \tag{46}$$

On the other hand we know that fermion propagator is defined as (in configuration space)

$$S_F(x-y) = \left\langle 0 \left| T(\psi(x)\bar{\psi}(y)) \right| 0 \right\rangle \tag{47}$$

where T denotes time ordering. This suggests that in order to renormalize QCD (or QED) we have to redefine fermion fields by by factors $\sqrt{Z_2}$. This is indeed how renormalization is practically done (and proven).

Let us start from the QCD lagrangian in $d = 4 - 2\varepsilon$ dimensions where all parameters of the theory and fields are finite, but will eventually diverge when we take $\varepsilon \to 0$. Such fields and parameters are called *bare* and will be denoted with a sub/superscript (0). The QCD lagrangian reads

$$\mathcal{L} = \bar{\psi}_{(0)} \left(i\not{D} - m_{(0)} \right) \psi_{(0)} - \frac{1}{4} F^{a\,\mu\nu}_{(0)} F^{a}_{(0)\mu\nu} + \ldots \tag{48}$$

where dots denote gauge fixing terms, ghost field lagrangian, etc. Recall that

$$\begin{aligned} D_\mu \psi_{(0)} &= \left(\partial_\mu + i g_{(0)} T^a A^{a(0)}_\mu \right), \\ F^{a\,\mu\nu}_{(0)} &= \partial_\mu A^{a(0)}_\nu - \partial_\nu A^{a(0)}_\mu - g_{(0)} f^{abc} A^{b(0)}_\mu A^{c(0)}_\nu . \end{aligned} \tag{49}$$

Note that bare field lagrangian (48) leads to the canonical commutation rules.

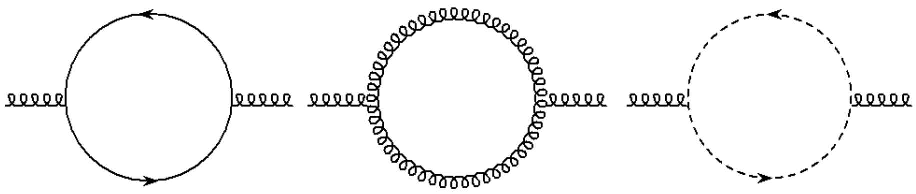

Figure 3: Feynman diagrams corresponding to the gluon self-energy: fermion loop, gluon loop and ghost loop.

Now we define renormalized fields ψ, A^a_μ that are finite in the limit $\varepsilon \to 0$:

$$\sqrt{Z_2}\psi = \psi_{(0)}, \ \sqrt{Z_3} A^a_\mu = A^{a(0)}_\mu, \text{ etc.} \tag{50}$$

Here Z_3 can be calculated from gluon self-energy depicted in Fig. 3:

$$Z_3 = 1 - \frac{\alpha_s}{4\pi} \left(\frac{2}{3} n_f - \frac{5}{3} C_A \right) \frac{1}{\varepsilon} + \ldots . \tag{51}$$

We see that Z_3 contains a term proportional to the number of active fermions (n_f) that comes from the first diagram and a term proportional to the Casimir of the adjoint representation of the gauge group (C_A) corresponding to the second diagram in Fig. 4 and to the ghost contribution of the third diagram. Note that in QED $C_A = 0$ and only fermion loop contributes. In pure Yang-Mills theory (no fermions) $n_f = 0$ and we have only the contribution proportional to C_A that changes the sign of the α_s term. This change of sign (still true for $n_f = 6$) with respect to QED has dramatic consequences for the UV behaviour of QCD. It is important to remember that renormalization constants Z_i are *gauge dependent*. For example in Landau gauge $Z_2 = 0$. In Eq.(51) Z_3 is in Feynman gauge.

Unfortunately bare langrangian (48) expressed in terms of of the renormalized fields has wrong normalization:

$$\begin{aligned}\mathcal{L} &= Z_2\bar{\psi}\left(i\partial\!\!\!/ - m_{(0)}\right)\psi - Z_2\sqrt{Z_3}g_{(0)}\bar{\psi}T^a A\!\!\!/^a\psi \\ &-\frac{Z_3}{4}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)^2 + \frac{Z_3^{3/2}g_{(0)}}{2}\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right)f^{abc}A^b_\mu A^c_\nu \\ &-\frac{Z_3^2 g_{(0)}^2}{4}\left(f^{abc}A^b_\mu A^c_\nu\right)^2 + \ldots \end{aligned} \tag{52}$$

Now we add to (52) so called counter terms, which are arranged to cancel UV divergences term by term in perturbation theory. If we need only a finite number of such terms, such theory is *renormalizable*. For clarity, let's concentrate only on terms with massless fermions:

$$\mathcal{L}_R = \bar{\psi}i\partial\!\!\!/\psi + (Z_2-1)\bar{\psi}i\partial\!\!\!/\psi - g\mu^\varepsilon\bar{\psi}T^a A\!\!\!/^a\psi - \left(Z_2\sqrt{Z_3}g_{(0)} - g\mu^\varepsilon\right)\bar{\psi}T^a A\!\!\!/^a\psi + \ldots. \tag{53}$$

Note that, looking at (44), counter terms like $Z_2 - 1$ contain essentially only poles in ε and are of the order of g^2. Secondly, we denote by g a finite renormalized coupling in $d = 4$ dimensions, so we have multiplied it by a scale parameter μ^ε like in (1).

Now, the perturbative calculation from Sect. 1 should be understood as an expansion in *renormalized* coupling with new Feynman rules corresponding to the counter terms. $Z_2 - 1$ counter term is of the order of g^2, and we require that another counter term corresponding to the last term in (53) should be zero <u>in the lowest order</u>, as we want only gauge interaction term (third term in (53)) between fermions and A fields to be present in the lagrangian in the leading order. Since all renormalization constants Z_i start with unity, we have the following relation between bare and renormalized couplings up to first order in g^2:

$$g_{(0)} = g\mu^\varepsilon\left(1 + g^2\frac{\tilde{\beta}}{\varepsilon} + \ldots\right) \tag{54}$$

where $\tilde{\beta}$ is to be calculated. Therefore in the lowest order in g (or $g_{(0)}$) there is basically no difference between the two. However already at the level of g^2 ($\tilde{\beta}$ in Eq.(54)) we need to include not only the contributions from the external lines (encoded in the counter terms present in (53)), but also genuine vertex corrections depicted in Figs. 4 and 5. We see that we can calculate quantum corrections to the gauge coupling either from the quark-gluon vertex or from the 3g vertex. Needless to say that the result *must be the same*.

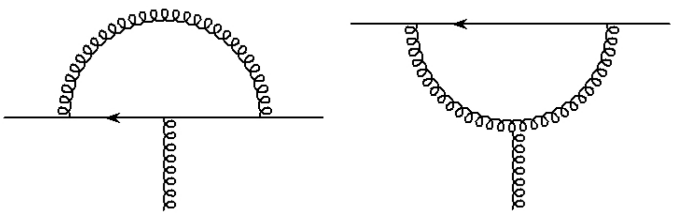

Figure 4: Feynman diagrams corresponding to the corrections to the quark-gluon vertex.

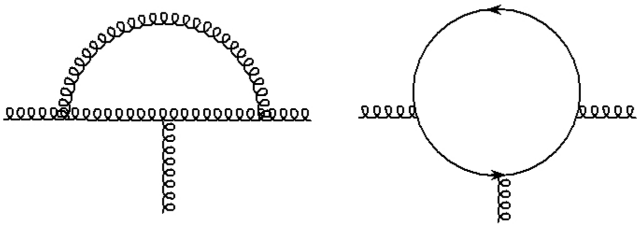

Figure 5: Some of the Feynman diagrams corresponding to the corrections to the 3-gluon vertex.

Loop corrections from diagrams of Fig. 4 or 5 yield poles in $1/\varepsilon$ and therefore we need to add a new counter term to (53). There are no more counter terms required to renormalize the coupling (there are some more for the mass, which we have set to zero in this note) and the final result from all contributions gives the following relation between the bare and renormalized couplings:

$$g_{(0)} = g\mu^{\varepsilon}\left(1 - \frac{\alpha_s}{4\pi}\left(\frac{11}{6}C_A - \frac{1}{3}n_f\right)\frac{1}{\varepsilon} + \dots\right). \tag{55}$$

Note that this $\mathcal{O}(\alpha_s)$ result is gauge invariant (although higher order terms denoted here by dots are not).

3 Renormalization group

When changing the number of dimensions to regularize UV divergent integrals we have introduced an arbitrary parameter μ to take care of the dimensionality of the renormalized coupling g. Unfortunately this parameter survives in the expressions for the renormalized quantities (see e.g. Eq.(45)) and also in the final expressions for the observables. It seems therefore that the renormalized theory has lost its predictive power, since by changing μ we can get an arbitrary numerical value for a given observable. The only way to get out of this trap is to require that the numerical values of renormalized parameters (coupling g or mass m) change with μ to compensate explicit dependence of given observable on μ. This makes the theory invariant under the change of μ and this invariance is called renormalization group (RG) invariance.

3.1 β function

This means that $g = g(\mu)$. On the other han the bare coupling $g_{(0)}$ (recall that we are still in $d = 4-2\varepsilon$ dimensions), which at first sight is a function of μ: $g_{(0)} = g_{(0)}(\mu, g(\mu), \varepsilon)$ should not depend on μ. This means that the full derivative of $g_{(0)}$ over μ (in practice we will take derivative over $\ln\mu^2$) should be zero:

$$0 = \frac{d}{d\ln\mu^2} g_{(0)}(\mu, g(\mu), \varepsilon) = \frac{\partial g_{(0)}}{\partial \ln\mu^2} + \frac{\partial g_{(0)}}{\partial g}\frac{dg(\mu)}{d\ln\mu^2}. \tag{56}$$

In order to calculate the first derivative in (56) let's observe that

$$\mu^\varepsilon = \exp\left(\frac{1}{2}\varepsilon\ln\mu^2\right) \tag{57}$$

hence

$$\frac{d}{d\ln\mu^2}\mu^\varepsilon = \frac{1}{2}\varepsilon\mu^\varepsilon \tag{58}$$

and finally

$$\frac{dg(\mu)}{d\ln\mu^2} = -\frac{1}{2}\varepsilon\frac{g_{(0)}}{\partial g_{(0)}/\partial g}. \tag{59}$$

This is the renormalization group equation for the renormalized coupling constant $g(\mu)$. On the right hand side of (59) we need a pole part of $\frac{g_{(0)}}{\partial g_{(0)}/\partial g}$ only to get finite result in the limit $\varepsilon \to 0$.

Customarily we work with g^2 rather than with g. Let's define

$$a_s(\mu) = \frac{g^2(\mu)}{16\pi^2} = \frac{\alpha_s(\mu)}{4\pi}. \tag{60}$$

We have

$$\frac{da_s(\mu)}{d\ln\mu^2} = \frac{2g(\mu)}{16\pi^2}\frac{dg(\mu)}{d\ln\mu^2} = -\frac{g(\mu)}{16\pi^2}\varepsilon\frac{g_{(0)}}{\partial g_{(0)}/\partial g} \underset{\varepsilon=0}{=} \beta(a_s). \tag{61}$$

Let's calculate β from (55) remembering (38):

$$\begin{aligned}
\beta(a_s) &= -\frac{g}{16\pi^2}\varepsilon\frac{g\mu^\varepsilon\left(1-\frac{\alpha_s}{4\pi}\left(\frac{11}{6}C_A-\frac{1}{3}n_f\right)\frac{1}{\varepsilon}\right)}{\mu^\varepsilon\left(1-\frac{3\alpha_s}{4\pi}\left(\frac{11}{6}C_A-\frac{1}{3}n_f\right)\frac{1}{\varepsilon}\right)} \\
&= -a_s\varepsilon\frac{\left(1-\frac{3\alpha_s}{4\pi}\left(\frac{11}{6}C_A-\frac{1}{3}n_f\right)\frac{1}{\varepsilon}\right)+\frac{2\alpha_s}{4\pi}\left(\frac{11}{6}C_A-\frac{1}{3}n_f\right)\frac{1}{\varepsilon}}{1-\frac{3\alpha_s}{4\pi}\left(\frac{11}{6}C_A-\frac{1}{3}n_f\right)\frac{1}{\varepsilon}} \\
&= -a_s\varepsilon - a_s^2\left(\frac{11}{3}C_A-\frac{2}{3}n_f\right)+\ldots \\
&\underset{\varepsilon=0}{=} -a_s^2\left(\frac{11}{3}C_A-\frac{2}{3}n_f\right)+\ldots.
\end{aligned} \tag{62}$$

In order to arrive at the last line of (62) we have neglected a_s^3 and higher terms. For higher orders we need to compute two loop diagrams, so (62) is a one loop expression for the so called *beta function* of QCD.

3.2 Solving RG equation for the gauge coupling

In the last subsection we have derived the renormalization group equation for the gauge coupling

$$\frac{da_s}{d\ln\mu^2} = \beta(a_s) \tag{63}$$

where β function has power series expansion

$$\beta(a_s) = -\beta_0 a_s^2 - \beta_1 a_s^3 + \ldots \tag{64}$$

For β_0 we have obtained an explicit form

$$\beta_0 = \frac{11}{3}C_A - \frac{2}{3}n_f \tag{65}$$

which is positive for $n_f < 16$. This is the case of QCD where $n_f = 6$. The values of the coefficients up to β_3 can be found in the book of John Collins for example.

Equation (63) can be solved by integrating both sides over the interval (μ_0, μ):

$$\ln\frac{\mu^2}{\mu_0^2} = \int\limits_{a_s(\mu_0)}^{a_s(\mu)} \frac{d\alpha_s}{\beta(\alpha_s)}. \tag{66}$$

In one loop approximation we have

$$\int\limits_{a_s(\mu_0)}^{a_s(\mu)} \frac{d\alpha_s}{\beta(\alpha_s)} = \frac{1}{\beta_0}\left(\frac{1}{a_s(\mu)} - \frac{1}{a_s(\mu_0)}\right). \tag{67}$$

We can therefore rewrite (66) as

$$\frac{1}{a_s(\mu)} - \beta_0 \ln\mu^2 = \frac{1}{a_s(\mu_0)} - \beta_0\ln\mu_0^2 = -\beta_0 \ln\Lambda_{\rm QCD}^2 \tag{68}$$

where in the last step the unknown value of the gauge coupling $g(\mu_0)$ (i.e. $a_s(\mu_0)$) has been replaced by a logarithm of a dimensional constant denoted by $\Lambda_{\rm QCD}$. This constant (or the QCD charge g) cannot be calculated and has to be measured. In QCD it is equal to 150 – 200 MeV (this depends what order of perturbative expansion we are working at). The asymptotic value (we work in perturbative expansion assuming that g is small) reads therefore:

$$a_s(\mu) = \frac{1}{\beta_0 \ln\frac{\mu^2}{\Lambda_{\rm QCD}^2}}. \tag{69}$$

For $\mu \to \infty$ we have that $a_s(\mu) \to 0$. So in this limit the interaction vanishes and the theory is *asymptotically free.*

Alternatively, we can rewrite solution (68) in a way that relates the coupling at one scale to the coupling at another scale:

$$a_s(\mu) = \frac{a_s(\mu_0)}{1 + \beta_0 a_s(\mu_0) \ln(\mu^2/\mu_0^2)}. \tag{70}$$

For positive β_0, which is the case of QCD, $a_s(\mu \to \infty) \to 0$, however for negative β_0, which is the case of QED, the coupling develops a pole at some μ, so called Landau pole. The coupling is growing with μ until it reaches a pole. This effect is, however, very small at typical momentum transfers characteristic for atomic or experimental particle physics.

4 Final remarks

Even though we have started from the theory with no mass scale (for massless fermions), after renormalization we ended up with a theory with a mass scale: $\Lambda_{\rm QCD}$. This phenomenon is called *dimensional transmutation* and is independent of the way we regularize the theory (one might suspect that it is due to the way we corrected dimensionality of the gauge coupling). The dependence of g and also other parameters of the theory like masses on the scale parameter μ^2 leads to the concept of *running coupling constant* (*running mass*). In a typical QCD calculation we can choose μ^2 at will, and a typical choice is that μ^2 corresponds to the large momentum transfer present in a given process. See for example Eq. (45)

$$S_F^R = \frac{i}{\not{p}} \left(1 + \frac{\alpha(\mu^2)}{4\pi} C_F \left(\ln \left(\frac{-p^2}{\bar{\mu}^2} \right) - 1 \right) \right) \tag{71}$$

where the choice of $\bar{\mu}^2 \sim -p^2$ (provided $-p^2 \gg \Lambda^2_{\rm QCD}$) nullifies potentially large logarithm. Obviously S_F^R is not an observable, however the above reasoning applies to the measurable quantities like cross-sections or decay widths.

One might be worried that the change of scale in (71) changes the numerical value of the quark propagator in plain contradiction with the RG invariance. One should, however, keep in mind that RG invariance concerns *full theory* and (71) is only a one loop approximation. Today when two, three or even four loop calculations for different observables are available, one can convince onself that increasing the accuracy of perturbative calculations reduces sensitivity to the choice of scale.

While asymptotic freedom is a welcome feature that justifies the use of perturbative expansion in QCD, the growth of the coupling constant for small momenta $\mu^2 \lesssim \Lambda^2_{\rm QCD}$ invalidates the use of perturbative expansion in this kinematical range, and is a clear signal of another important feature of QCD, namely *confinement*. Roughly speaking confinement corresponds to the fact that no free quarks have been ever observed. The interaction strength between two color sources increases with distance, so that finally either the sources have to bounce back or the quark antiquark pair is created from the vacuum. While the growth of the coupling constant for small momentum transfers (large distances by the uncertainty principle) is an important hint of the confinement, until now there exits no non-perturbative proof of confinement in QCD.