QCD

Problem set #10

Monday, January 13, 10:00

1. Finish the calculation of the color factors, i.e. calculate so called Casimir operator

$$\sum_{a,j} T^a_{ij} T^a_{jk} = C\delta_{ik}$$

for the adjoint representations of the SU(N) group.

2. Real scalar field lagrangian density reads as follows:

$$\mathcal{L}(x) = \frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) - \frac{1}{2}m^2\phi^2(x).$$

Calculate Hamiltonian. Canonical equal-time quantization rules for real scalar field operators read:

$$\left[\hat{\phi}(t,\vec{x}), \hat{\pi}(t,\vec{x}\,')\right] = i\delta^{(3)}(\vec{x}-\vec{x}\,')$$

and all other possible commutators are zero. Using decomposition

$$\hat{\phi}(t,\vec{x}) = \int \frac{d^3\vec{k}}{(2\pi)^2\sqrt{2\omega_k}} \left[e^{-i\,kx}\hat{a}(\vec{k}) + e^{+i\,kx}\hat{a}^\dagger(\vec{k})\right]$$

show that the canonical quatization rules are satisfied if

$$\left[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}\,')\right] = (2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}\,')$$

and the remaining two commutators vanish.

3. For a system of SU(3) scalar fields $\hat{\phi}_i(x)$ with $i = 1, 2, 3$ that satisfy the following commutation rules

$$\left[\hat{\phi}_i(t,\vec{x}), \hat{\pi}_j(t,\vec{x}\,')\right] = i\delta^{(3)}(\vec{x}-\vec{x}\,')\delta_{ij}$$

one defines charge operators

$$\hat{Q}^a(t) = -i\int d^3\vec{x}\,\hat{\pi}_i(t,\vec{x})T^a_{ij}\hat{\phi}_j(t,\vec{x})$$

where matrices T^a satisfy SU(3) commutation relations:

$$\left[T^a, T^b\right] = if^{abc}T^c.$$

Prove that

$$\left[\hat{Q}^a(t), \hat{Q}^b(t)\right] = if^{abc}\hat{Q}^c(t).$$

4. In the case of fermion fields, commutation relations from problem 2 are replaced by anticommutation relations:

$$\{q_{\alpha,k}(t,\vec{x}), q_{\beta,l}(t,\vec{x}\,')\} = \delta^{(3)}(\vec{x}-\vec{x}\,')\delta_{\alpha\beta}\delta_{kl}$$

where α, β stand for Dirac indices and k, l denote SU(3) matrices. Relevant charges are defined as

$$\begin{aligned} \hat{Q}^a_{L,R}(t) &= \int d^3\vec{x}\, q^\dagger_{L,R}(t,\vec{x}) T^a q_{L,R}(t,\vec{x}), \\ \hat{Q}_V(t) &= \int d^3\vec{x}\, \left[q^\dagger_L(t,\vec{x}) q_L(t,\vec{x}) + q^\dagger_R(t,\vec{x}) q_R(t,\vec{x})\right] \end{aligned}$$

where $T^a = \lambda^a/a$ are SU(3) generators (Gell-Mann matrices). Making use of the identity (prove it!)

$$\{ab, cd\} = a\{b,c\}d - ac\{b,d\} + \{a,c\}bd - c\{a,d\}b$$

show that

$$\begin{aligned} \left[\hat{Q}^a_L, \hat{Q}^b_L\right] &= if^{abc}\hat{Q}^c_L, \\ \left[\hat{Q}^a_R, \hat{Q}^b_R\right] &= if^{abc}\hat{Q}^c_R, \\ \left[\hat{Q}^a_L, \hat{Q}^b_R\right] &= 0, \\ \left[\hat{Q}^a_{L,R}, \hat{Q}^b_V\right] &= 0. \end{aligned}$$