QCD

Problem set #5

Tuesday, November 18, 10:00, A-1-13

ANOMALY

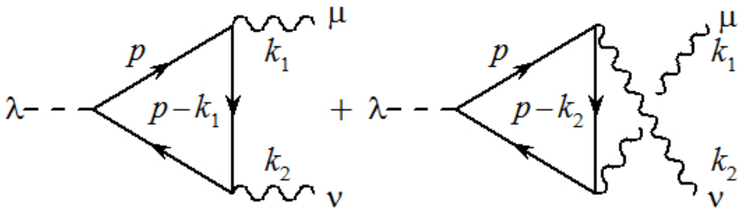

Figure 1: Loop diagrams contributing to the decay of axial-vector current (dashed line) to two photons.

Consider the following loop contribution to the decay of axial-vector current to two photons (Fig. 1):

$$\begin{aligned} T_{\mu\nu\lambda} &= -i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{(\not p - \not q) - m}\gamma_\nu\frac{i}{(\not p - \not k_1) - m}\gamma_\mu\right] \\ &\quad -i\int \frac{d^4p}{(2\pi)^4}\,\mathrm{Tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{(\not p - \not q) - m}\gamma_\mu\frac{i}{(\not p - \not k_2) - m}\gamma_\nu\right] \end{aligned} \tag{1}$$

where

$$q = k_1 + k_2. \tag{2}$$

Note that the second line in (1) and the first line are related by a replacement $\mu \longleftrightarrow \nu$ and $k_1 \longleftrightarrow k_2$. We expect that vector currents are consrved

$$k_1^\mu T_{\mu\nu\lambda} = k_2^\nu T_{\mu\nu\lambda} = 0 \tag{3}$$

and that the axial current is conserved in a massless limit

$$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu}. \tag{4}$$

In fact on general grounds we expect $T_{\mu\nu}$ to be obtained from $T_{\mu\nu\lambda}$ by replacing $\gamma_\lambda\gamma_5 \to \gamma_5$. In the following, we will show that it is impossible to satisfy both conservation laws (4) and (3) (Ward identities) at one loop level.

1. Calculate (4) using the following substitution

$$\begin{aligned} \not q\gamma_5 &= -\gamma_5\not q \\ &= \gamma_5\left[(\not p - \not q) - m\right] - \gamma_5\left[\not p - m\right] \\ &= \gamma_5\left[(\not p - \not q) - m\right] + \left[\not p - m\right]\gamma_5 + 2m\gamma_5. \end{aligned} \tag{5}$$

The result should read

$$q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} \tag{6}$$

where

$$\begin{aligned}
\Delta^{(1)}_{\mu\nu} &= \int \frac{d^4p}{(2\pi)^4} \operatorname{Tr}\left[\frac{i}{\not p - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not k_1) - m}\gamma_\mu - \frac{i}{(\not p - \not k_2) - m}\gamma_5\gamma_\nu \frac{i}{(\not p - \not q) - m}\gamma_\mu\right], \\
\Delta^{(2)}_{\mu\nu} &= \int \frac{d^4p}{(2\pi)^4} \operatorname{Tr}\left[\frac{i}{\not p - m}\gamma_5\gamma_\mu \frac{i}{(\not p - \not k_2) - m}\gamma_\nu - \frac{i}{(\not p - \not k_1) - m}\gamma_5\gamma_\mu \frac{i}{(\not p - \not q) - m}\gamma_\nu\right].
\end{aligned} \tag{7}$$

The question is: are $\Delta^{(1,2)}_{\mu\nu}$ equal zero?

2. Consider an integral that naively is equal to zero

$$\int_{-\infty}^{\infty} dx\,[f(x+a) - f(x)] \tag{8}$$

where f is a function that does not vanish at infinity:

$$f(\pm\infty) \neq 0. \tag{9}$$

Calculate (8) expanding in a up to a^2. What happens when $f'(\pm\infty) = 0$. Generalize this result to the n-dimensional Euclidean integral

$$\Delta(\vec a) = \int d^n\vec r\,[f(\vec r + \vec a) - f(\vec r)]\,.$$

Show that for the integral in Minkowski space the result reads

$$\Delta(a) = 2i\pi^2 a^\mu \lim_{R\to\infty} R^2 R_\mu\, f(R). \tag{10}$$

Note that the surface of a sphere in 4 dimensions reads $2\pi^2R^3$.

3. Calculate applying (10) the change of (1)

$$\Delta_{\mu\nu\lambda}(a) = T_{\mu\nu\lambda}(p \to p + a) - T_{\mu\nu\lambda} \tag{11}$$

if the integration momentum p is shifted by a four-vector

$$a = \alpha k_1 + (\alpha - \beta)k_2. \tag{12}$$

The result should read:

$$\Delta_{\mu\nu\lambda}(a) = -\frac{1}{(2\pi)^4} 2i\pi^2 a^\sigma \lim_{P\to\infty} P^2 P_\sigma \operatorname{Tr}\left[\not P\gamma_\lambda\gamma_5\not P\gamma_\nu\not P\gamma_\mu\right]\frac{1}{P^6} + (\mu \longleftrightarrow \nu, k_1 \leftrightarrow k_2)\,. \tag{13}$$

Taking symmetric limit

$$\lim_{P\to\infty}\frac{P^{\sigma}P^{\alpha}}{P^{2}}=\frac{1}{4}g^{\sigma\alpha} \tag{14}$$

and calculating the trace one should obtain

$$\Delta_{\mu\nu\lambda}(a)=\frac{\beta}{8\pi^{2}}\varepsilon_{\alpha\mu\nu\lambda}\left(k_{1}-k_{2}\right)^{\alpha}. \tag{15}$$

4. Calculate $\Delta_{\mu\nu}^{(1,2)}$using the same trick with shifting the integration varible. In this case, however, one has to observe that $p\to p-k_{2}$ for $\Delta_{\mu\nu}^{(1)}$. The result should read:

$$\Delta_{\mu\nu}^{(1)}=\Delta_{\mu\nu}^{(2)}=-\frac{1}{8\pi^{2}}\varepsilon_{\mu\nu\sigma\rho}\,k_{1}^{\sigma}k_{2}^{\rho}. \tag{16}$$

Summing all terms show that

$$\begin{aligned} q^{\lambda}T_{\mu\nu l}(\beta) &= q^{\lambda}\left(T_{\mu\nu l}(\beta)-T_{\mu\nu l}(0)\right)+q^{\lambda}T_{\mu\nu l}(0) \\ &= 2mT_{\mu\nu}-\frac{1-\beta}{4\pi^{2}}\varepsilon_{\mu\nu\sigma\rho}\,k_{1}^{\sigma}k_{2}^{\rho}. \end{aligned} \tag{17}$$

5. Apply the same procedure to calculate $k_{1}^{\mu}T_{\mu\nu l}(\beta)$ and show that

$$k_{1}^{\mu}T_{\mu\nu l}(\beta)=\frac{1+\beta}{8\pi^{2}}\varepsilon_{\nu\lambda\sigma\rho}k_{1}^{\sigma}k_{2}^{\rho}. \tag{18}$$

We see that it is impossible to maintain both Ward identities (17) and (18) by a suitable choice of β. Because we know that vector current (charge) is conserved, we are forced to choose $\beta=-1$.Then

$$q^{\lambda}T_{\mu\nu l}=2mT_{\mu\nu}-\frac{1}{2\pi^{2}}\varepsilon_{\mu\nu\sigma\rho}\,k_{1}^{\sigma}k_{2}^{\rho}, \tag{19}$$

which means tht axial current is *anomalous.*