Fuss-Catalan numbers in noncommutative probability

Wojtek Młotkowski (Wrocław)

Kraków, 05.07.2014
Abstract

The generalized Fuss-Catalan numbers are defined by \( \binom{np+r}{n} \frac{r}{np+r} \), where \( p, r \) are real parameters. They are moments of the Marchenko-Pastur law \( MP \) for \( p = 2, r = 1 \) (Catalan numbers) and, more generally, of the multiplicative free power \( MP \boxtimes p^{-1} \) for \( r = 1, p > 1 \). I will present properties of these numbers, of their generating functions and of corresponding probability distributions and relations with noncommutative probability.
The **Fuss numbers** are defined as

\[
\binom{np + 1}{n} \frac{1}{np + 1},
\]

where \( p \) is a real parameter and the generalized binomial coefficient is defined by

\[
\binom{a}{n} := \frac{a(a - 1) \ldots (a - n + 1)}{n!}.
\]

If \( p \) is natural then Fuss numbers have several combinatorial interpretations:
(1) The number of all such sequences \((a_1, a_2, \ldots, a_{np})\) that:
(a) \(a_i \in \{1, 1 - p\}\), (b) \(a_1 + a_2 + \ldots + a_s \geq 0\) for all \(s\) such that \(1 \leq s \leq np\) and (c) \(a_1 + a_2 + \ldots + a_{np} = 0\). Such sequences can be represented as special Dyck paths on the plane.

(2) The number of such noncrossing partitions \(\pi\) of the set \(\{1, 2, \ldots, pn\}\) that every block \(V \in \pi\) has \(p\) elements.

(3) The number of rooted trees with \(pn\) edges, such that every internal node has exactly \(p\) sons.

The case \(p = 2\), **Catalan numbers** \(\left(\begin{array}{c} 2n+1 \\ n \end{array}\right) \frac{1}{2n+1}\), is of particular interest. On the homepage of Richard P. Stanley there are 207 combinatorial interpretations.

For more general notion, see
The generating function, studied by Lambert (1750),

\[ B_p(z) := \sum_{n=0}^{\infty} \binom{np+1}{n} \frac{1}{np+1} z^n, \]  

(3)

is convergent in some neighborhood of 0 and satisfies

\[ B_p(z) = 1 + z B_p(z)^p. \]  

(4)

This means, that \( B_p(z) \) is the inverse function to the map \( w \mapsto \frac{w-1}{wp} \) in a neighborhood of 1, i.e.

\[ B_p\left( z(1 + z)^{-p} \right) = 1 + z \]  

(5)

around \( z = 0 \).
These functions $\mathcal{B}_p$ also satisfy a remarkable composition relation:

$$\mathcal{B}_p(z) = \mathcal{B}_{p-r}(z\mathcal{B}_p(z)^r).$$

(6)

Another formula:

$$\frac{\mathcal{B}_p(z)^{1+r}}{(1 - p)\mathcal{B}_p(z) + p} = \sum_{n=0}^{\infty} \binom{np + r}{n} z^n.$$  

(7)
Lambert (1770) found formula for Taylor expansion of the powers $B_p(z)^r$:

$$B_p(z)^r = \sum_{n=0}^{\infty} \binom{np+r}{n} \frac{r}{np+r} z^n. \quad (8)$$

We will call the coefficients $\binom{np+r}{n} \frac{r}{np+r}$ the two-parameter Fuss numbers, or Raney numbers.

If $p, r$ are natural then $\binom{np+r}{n} \frac{r}{np+r}$ is the number of all such sequences $(a_1, a_2, \ldots, a_{np+r})$ that:

(a) $a_i \in \{1, 1-p\}$,
(b) $a_1 + a_2 + \ldots + a_s > 0$ for all $s$ such that $1 \leq s \leq np + r$ and
(c) $a_1 + a_2 + \ldots + a_{np+r} = r$.

A sequence \( \{a_n\}_{n=0}^{\infty} \) is called **positive definite** if we have

\[
\sum_{i,j \geq 0} a_{i+j} c_i c_j \geq 0
\]  
(9)

for an arbitrary sequence \( \{c_i\}_{i \geq 0} \) of real numbers with finite support (i.e. \( c_i = 0 \) except for finitely many values of \( i \)). This is equivalent, that \( \{a_n\}_{n=0}^{\infty} \) is a **moment sequence** for some positive measure \( \mu \) on \( \mathbb{R} \), i.e.

\[
a_n = \int_{\mathbb{R}} x^n \, d\mu(x).
\]  
(10)
Positive definiteness of the Fuss numbers

For which parameters $p, r \in \mathbb{R}$ the sequence $\binom{n p + r}{n} \frac{r}{n p + r}$ is positive definite? If this is the case, the corresponding probability measure will be denoted $\mu(p, r)$.

The Catalan numbers are moments of the Marchenko-Pastur distribution $\text{MP} = \mu(2, 1)$:

$$
\binom{2n + 1}{n} \frac{1}{2n + 1} = \int_0^4 x^n \frac{1}{2\pi} \sqrt{\frac{4 - x}{x}} \, dx. \quad (11)
$$

Since $(-1)^n \binom{n p + r}{n} \frac{r}{n p + r} = \binom{n(1 - p) - r}{n} \frac{-(1 - p) - r}{n(1 - p) - r}$, the measure $\mu(1 - p, -r)$ is just reflection of $\mu(p, r)$. Also the case $r = 0$ is not interesting because $\mu(p, 0) = \delta_0$. Therefore from now on we assume that $p, r > 0$. 
Theorem

The sequence \( \binom{mp+r}{m} \frac{r}{mp+r} \) is positive definite if and only if \( p \geq 1, 0 < r \leq p \).

The “if” part was first proved by Młotkowski (2010) using free multiplicative convolution, then in Młotkowski, Penson, Życzkowski (2013) by using Mellin convolutions of beta measures. The “only if” part was proved by Młotkowski, Penson (2013). Alternative proofs are given by Liu-Pego (2014), Forrester-Liu (2014).

The corresponding probability measure \( \mu(p, r) \) has support \([0, p^p(p - 1)^{1-p}]\) and is absolutely continuous. For rational \( p > 0 \) the density function \( W_{p,r}(x) \) can be described in terms of the Meijer G-functions (Młotkowski, Penson, Życzkowski 2013).

• Młotkowski, Penson, Życzkowski, *Densities of the Raney distributions*, Documenta Mathematica 18 (2013), 1573-1596.


For \( p = \frac{k}{l} > 1, \ 0 < r \leq p \) we have 

\[
W_{p,r}(x) = \frac{r p^r}{x(p - 1)^{r + 1/2} \sqrt{2k\pi}} G_{k,k}^{k,0} \left( \frac{x^l}{c(p)^l} \middle| \begin{array}{c}
\alpha_1, \ldots, \alpha_k \\
\beta_1, \ldots, \beta_k
\end{array} \right),
\]  

(12)

where \( x \in (0, c(p)) \), where \( c(p) = p^p (p - 1)^{1-p} \) and the parameters \( \alpha_j, \beta_j \) are given by

\[
\alpha_j = \begin{cases}
\frac{j}{l} & \text{if } 1 \leq j \leq l, \\
\frac{r + j - l}{k - l} & \text{if } l + 1 \leq j \leq k,
\end{cases}
\]  

(13)

and

\[
\beta_j = \frac{r + j - 1}{k}, \quad 1 \leq j \leq k.
\]  

(14)
Theorem

\[ W_{p,r}(f_p(\phi)) = \frac{\sin \phi \sin r\phi (\sin(p-1)\phi)^{p-r-1}}{\pi (\sin p\phi)^{p-r}}, \]

where for \(0 < \phi < \pi/p\)

\[ f_p(\phi) = \frac{(\sin p\phi)^p}{\sin \phi (\sin(p-1)\phi)^{p-1}}. \]
The case $p=2$

For $p = 2$, $r > 0$, the function $W_{2,r}$ is

$$W_{2,r}(x) = \frac{\sin \left( r \cdot \arccos \sqrt{\frac{x}{4}} \right)}{\pi x^{1-r/2}}, \quad (17)$$

$x \in (0, 4)$. In particular, for $r = 1/2, 1, 3/2, 2$ we have

$$W_{2,1/2}(x) = \frac{\sqrt{2 - \sqrt{x}}}{2\pi x^{3/4}}, \quad (18)$$

$$W_{2,1}(x) = \frac{1}{2\pi} \sqrt{\frac{4 - x}{x}}, \quad (19)$$

$$W_{2,3/2}(x) = \frac{(\sqrt{x} + 1) \sqrt{2 - \sqrt{x}}}{2\pi x^{1/4}}, \quad (20)$$

$$W_{2,2}(x) = \frac{1}{2\pi} \sqrt{x(4 - x)}. \quad (21)$$
The case $p=3$

Theorem

$$W_{3,1}(x) = \frac{3 \left(1 + \sqrt{1 - 4x/27}\right)^{2/3} - 2^{2/3}x^{1/3}}{2^{4/3}3^{1/2}\pi x^{2/3} \left(1 + \sqrt{1 - 4x/27}\right)^{1/3}}, \quad (22)$$

$$W_{3,2}(x) = \frac{9 \left(1 + \sqrt{1 - 4x/27}\right)^{4/3} - 2^{4/3}x^{2/3}}{2^{5/3}3^{3/2}\pi x^{1/3} \left(1 + \sqrt{1 - 4x/27}\right)^{2/3}}, \quad (23)$$

and, finally, $W_{3,3}(x) = x \cdot W_{3,1}(x)$, with $x \in (0, 27/4)$.

The case $p=3/2, r=1/2$

$$W_{3/2,1/2}(x) = \frac{\left(1 + \sqrt{1 - 4x^2/27}\right)^{2/3} - \left(1 - \sqrt{1 - 4x^2/27}\right)^{2/3}}{2^{5/3}3^{-1/2}\pi x^{2/3}}. \quad (24)$$

The dilation $D_{2\mu}(3/2, 1/2)$, with the density $W_{3/2,1/2}(x/2)/2$, is known as the **Bures distribution**.

The case $p=3/2$, $r=1$

$MP^{1/2}$ the free multiplicative square root of $MP$

$W_{3/2,1}(x) = 3^{1/2} \left( \frac{1 + \sqrt{1 - 4x^2/27}}{2^{4/3} \pi x^{1/3}} \right)^{1/3} - \left( \frac{1 - \sqrt{1 - 4x^2/27}}{2^{4/3} \pi x^{1/3}} \right)^{1/3}$

$+ 3^{1/2} x^{1/3} \left( \frac{1 + \sqrt{1 - 4x^2/27}}{2^{5/3} \pi} \right)^{2/3} - \left( \frac{1 - \sqrt{1 - 4x^2/27}}{2^{5/3} \pi} \right)^{2/3}$  (25)
5. Relations of the Fuss numbers with the free, Boolean and monotonic convolution

Młotkowski, Fuss-Catalan numbers in noncommutative probability, Documenta Mathematica 15 (2010).

Classical convolution of probability measures on $\mathbb{R}$:

$$\mu \ast \nu(E) := \int_{\mathbb{R}} \mu(E - y) \, d\nu(y).$$  \hfill (26)

Convolutions coming from noncommutative probability do not have such direct description.

For a compactly supported measure $\mu$ on $\mathbb{R}$ we define its moment generating function:

$$M_\mu(z) := \int_{\mathbb{R}} \frac{1}{1 - xz} \, d\mu(x) = \sum_{n=0}^{\infty} z^n \int_{\mathbb{R}} x^n \, d\mu(x).$$  \hfill (27)
Additive free convolution is defined in the following way. For a probability measure $\mu$ we define its free $R$-transform $R_\mu(z)$ by the formula:

$$M_\mu(z) = R_\mu(zM_\mu(z)) + 1.$$  \hspace{1cm} (28)

If $R_\mu(z) = \sum_{k=1}^{\infty} r_k(\mu)z^k$ then $r_k(\mu)$ are called free cumulants of $\mu$. Then the free convolution $\mu \boxplus \nu$ can be defined as the unique probability measure which satisfies

$$R_{\mu \boxplus \nu}(z) = R_\mu(z) + R_\nu(z).$$  \hspace{1cm} (29)

We also define free power $\mu \boxplus t$ by $R_{\mu \boxplus t}(z) := tR_\mu(z)$. This is well defined at least for $t \geq 1$. If $\mu \boxplus t$ is defined for all $t > 0$ then we say that $\mu$ is infinitely divisible with respect to the additive free convolution.
As a consequence of the formula \( B_p(z) = B_{p-r}(zB_p(z)^r) \) we have

**Theorem**

*For the free additive transform of \( \mu(p, r) \) we have*

\[
R_{\mu(p,r)}(z) = B_{p-r}(z)^r - 1, \quad (30)
\]

*hence the free cumulants of \( \mu(p, r) \) are equal to \( \left( \frac{n(p-r)+r}{n} \right)^r \frac{r}{n(p-r)+r} \).*

Note that if \( 0 \leq 2r \leq p, \ r + 1 \leq p \) then the cumulants of \( \mu(p, r) \) are moments of the measure \( \mu(p - r, r) \), which implies:

**Corollary**

*If \( 0 \leq 2r \leq p, \ r + 1 \leq p \) then \( \mu(p, r) \) is \( \boxplus \)-infinitely divisible.*
The **free S-transform** (or the **free multiplicative transform**) of a probability measure $\mu$ on $\mathbb{R}_+ := [0, +\infty)$ is defined by the relation

$$R_{\mu}(zS_{\mu}(z)) = z \quad \text{or} \quad M_{\mu}(z(1 + z)^{-1}S_{\mu}(z)) = 1 + z. \quad (31)$$

Then the **multiplicative free convolution** $\mu_1 \boxtimes \mu_2$ and the **multiplicative free power** $\mu \boxtimes t$ are defined by

$$S_{\mu_1 \boxtimes \mu_2}(z) := S_{\mu_1}(z)S_{\mu_2}(z) \quad \text{and} \quad S_{\mu \boxtimes t}(z) := S_{\mu}(z)^t, \quad (32)$$

the latter is well defined at least for $t \geq 1$.

From the relation $\mathcal{B}_p(z(1 + z)^{-p}) = 1 + z$ we get:
For $r \neq 0$ the $S$-transform of the measure $\mu(p, r)$ is equal to

$$S_{\mu(p,r)}(z) = \frac{(1 + z)^{\frac{1}{r}} - 1}{z}(1 + z)^{\frac{r-p}{r}}. \quad (33)$$

Consequently

$$\mu(1 + p_1, 1) \otimes \mu(1 + p_2, 1) = \mu(1 + p_1 + p_2, 1), \quad (34)$$

and more generally

$$\mu(p_1, r) \otimes \mu(1 + p_2, 1) = \mu(p_1 + rp_2, r). \quad (35)$$

We have also

$$\mu(1 + p, 1) \otimes t = \mu(1 + tp, 1). \quad (36)$$

The Fuss numbers $\left(\frac{np+1}{n}\right)^{\frac{1}{np+1}}$ are moments of $MP \otimes p-1$, $p \geq 1$. 
The **Boolean convolution** $\mu_1 \uplus \mu_2$ and the Boolean power $\mu^{\uplus t}$ can be defined by putting

$$
\frac{1}{M_{\mu_1 \uplus \mu_2}(z)} = \frac{1}{M_{\mu_1}(z)} + \frac{1}{M_{\mu_2}(z)} - 1,
$$

$$
M_{\mu^{\uplus t}}(z) = \frac{M_{\mu}(z)}{(1 - t)M_{\mu}(z) + t},
$$

the latter is well defined for all $t > 0$.

From the formula

$$
\frac{B_p(z)^{1+r}}{(1 - p)B_p(z) + p} = \sum_{n=0}^{\infty} \binom{np + r}{n} z^n.
$$

we see that for $p \geq 1$ the numbers $\binom{np}{n}$ are moments of $\mu(p, 1)^{\uplus p}$. 

Wojtek Młotkowski

Fuss-Catalan numbers

05.07.2014
A by-product: Consider the dilation $D_{1/p} \mu(p, 1)^{\oplus p}$. Its moments are $\binom{np}{n} \frac{1}{p^n}$. It is easy to check that

$$
\left( \binom{np}{n} \frac{1}{p^n} \right) \frac{n^n}{n!} \rightarrow \frac{n^n}{n!}
$$

as $p \rightarrow \infty$. This proves that the sequence $\frac{n^n}{n!}$ is positive definite. The corresponding measure was described by Sakuma and Yoshida.

The **Monotonic convolution** is an associative, noncommuting operation \( \mu_1 \triangleright \mu_2 = \mu \) on probability measures on \( \mathbb{R} \) which is defined by

\[
M_\mu(z) = M_{\mu_1}(zM_{\mu_2}(z)) \cdot M_{\mu_2}(z). \tag{41}
\]

Then the formula \( B_p(z) = B_{p-r}(zB_p(z)^r) \) leads to:

\[
\mu(a, b) \triangleright \mu(a + r, r) = \mu(a + r, b + r), \tag{42}
\]

\( a \geq 1, 0 \leq b \leq a, r > 0. \)

**Corollary**

*If* \( 0 \leq r \leq p - 1 \) *then the sequence* \( \{ \binom{mp + r}{m} \}_{m=0}^{\infty} \) *is positive definite as the moment sequence of*

\[
\mu(p - r, 1)^{\triangleright p} \triangleright \mu(p, r).
\]
In fact we have:

**Theorem**

The sequence \( \{ \binom{mp+r}{m} \}_{m=0}^{\infty} \) is positive definite if and only if \( p \geq 1 \) and \( -1 \leq r \leq p - 1 \).


Denote the corresponding measure by \( \nu(p, r) \).
We know already that if \( 0 \leq r \leq p - 1 \) then

\[
\nu(p, r) = \mu(p - r, 1)^{\oplus p} \triangleright \mu(p, r).
\]
For $c > 0$ define probability measure $\eta(c)$ by

$$\eta(c) := c \cdot x^{c-1} \, dx, \quad x \in [0, 1].$$

with moments $\left\{ \frac{c^n}{n+c} \right\}_{n=0}^{\infty}$.

From the formula

$$\binom{np + r - 1}{n} \cdot \frac{r}{n(p-1) + r} = \binom{np + r}{n} \frac{r}{np + r}$$

we get Mellin convolution relation:

**Theorem**

For $p > 1$, $0 < r \leq p$ we have

$$\nu(p, r - 1) \circ \eta \left( \frac{r}{p - 1} \right) = \mu(p, r).$$
Theorem

For $p > 1$ we have

$$\nu(p, -1) = D_{c(p)} \left( \frac{1}{p} \delta_0 + \frac{p - 1}{p} \delta_1 \right)^{\boxtimes p},$$

(43)

where $c(p) = p^p (p - 1)^{1-p}$, and

$$\nu(p, 0) = D_p \left( \left( \frac{1}{p} \delta_0 + \frac{p - 1}{p} \delta_1 \right)^{\boxtimes p/(p-1)} \right)^{\boxtimes p-1}.$$

(44)

$D$ denotes dilation: $D_c(\mu)(E) := \mu \left( \frac{1}{c} E \right)$. 