The Lukacs-Olkin-Rubin theorem without invariance of the "quotient"

Bartosz Kołodziejek

Warsaw University of Technology

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Characterizations of matrix variate distributions

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Let $\Omega$ denote one of five irreducible symmetric cones:

- Positive definite symmetric real matrices $r \times r$, $r \geq 1$
- Positive definite Hermitian-complex matrices $r \times r$, $r \geq 2$
- Positive definite Hermitian-quaternionic matrices $r \times r$, $r \geq 2$
- Positive definite Hermitian-octonionic matrices $3 \times 3$
- Lorentz cone ($\mathbb{R} \times \mathbb{R}^n$, $n \geq 2$)

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Let $\mathcal{D} = \{x \in \Omega : I - x \in \Omega\}$
Matrix beta distribution on $\Omega$ with parameters $(p, q)$ ($p, q > \dim \Omega/r - 1)$:

$$
\beta(p, q)(d\mathbf{x}) = \frac{1}{\beta_{\Omega}(p, q)} (\det \mathbf{x})^{p-\dim \Omega/r} \det(l - \mathbf{x})^{q-\dim \Omega+/r} I_{\mathcal{D}}(\mathbf{x}) \, d\mathbf{x}.
$$
• Matrix beta distribution on $\Omega$ with parameters $(p, q)$ ($p, q > \dim \Omega/r - 1$):

$$\beta(p, q)(dx) = \frac{1}{\beta_{\Omega}(p, q)} (\det x)^{p-\dim \Omega/r} \det(I - x)^{q-\dim \Omega+/r} I_D(x) dx.$$ 

• (continuous) Wishart distribution on $\Omega$ with parameters $(p, a)$ ($p > \dim \Omega/r - 1, a \in \Omega$):

$$\gamma(p, a)(dx) = \frac{\det(a)^p}{\Gamma_{\Omega}(p)} (\det x)^{p-\dim \Omega/r} e^{-\langle a, x \rangle} I_{\Omega}(x) dx.$$
Probability distributions on $\Omega$

- **Matrix beta distribution on $\Omega$ with parameters $(p, q)$**
  $(p, q > \text{dim } \Omega/r - 1)$:
  \[
  \beta(p, q)(d\mathbf{x}) = \frac{1}{\beta_{\Omega}(p, q)} (\det \mathbf{x})^{p-\text{dim } \Omega/r} \det(I - \mathbf{x})^{q-\text{dim } \Omega+/r} l_D(\mathbf{x}) \, d\mathbf{x}.
  \]

- **(continuous) Wishart distribution on $\Omega$ with parameters $(p, \mathbf{a})$**
  $(p > \text{dim } \Omega/r - 1, \mathbf{a} \in \Omega)$:
  \[
  \gamma(p, \mathbf{a})(d\mathbf{x}) = \frac{\det(\mathbf{a})^p}{\Gamma_{\Omega}(p)} (\det \mathbf{x})^{p-\text{dim } \Omega/r} e^{-\langle \mathbf{a}, \mathbf{x} \rangle} l_{\Omega}(\mathbf{x}) \, d\mathbf{x}.
  \]

- **(continuous) matrix GIG distribution on $\Omega$ with parameters $(p, \mathbf{a}, \mathbf{b})$**
  $(p \in \mathbb{R}, \mathbf{a}, \mathbf{b} \in \Omega)$:
  \[
  \mu(p, \mathbf{a}, \mathbf{b})(d\mathbf{x}) = \frac{1}{K_p(\mathbf{a}, \mathbf{b})} (\det \mathbf{x})^{p-\text{dim } \Omega/r} e^{-\langle \mathbf{a}, \mathbf{x} \rangle - \langle \mathbf{b}, \mathbf{x}^{-1} \rangle} l_{\Omega}(\mathbf{x}) \, d\mathbf{x}.
  \]
GIG and Wishart characterization - Matsumoto-Yor property

**Theorem (Letac, Wesołowski (2000))**

Let $X$ and $Y$ be non-degenerate, independently distributed random variables. If $U = (X + Y)^{-1}$ and $V = X^{-1} - (X + Y)^{-1}$ are independent, then $X$ and $Y$ follow, respectively, GIG and Gamma distributions.

**Theorem (BK 2014)**

Let $X$ and $Y$ be independently distributed random variables valued in $\Omega$ with continuous densities, which are strictly positive on $\Omega$. If $U = (X + Y)^{-1}$ and $V = X^{-1} - (X + Y)^{-1}$ are independent, then $X$ and $Y$ follow, respectively, GIG and Wishart distributions.

Contributors: Matsumoto, Yor, Letac, Wesołowski, Bernadac, Seshadri, Koudou, Vallois,...
Theorem (Wesołowski, Seshadri (2003))

Let $X$ and $Y$ be non-degenerate, independently distributed random variables valued in $(0, 1)$. If $U = 1 - XY$ and $V = \frac{1-X}{U}$ are independent, then $X$ and $Y$ follow beta distribution.

Theorem (BK (2014))

Let $X$ and $Y$ be independently distributed random variables valued in $D$ with continuous densities, which are strictly positive on $D$. If $U = I - X^{1/2} \cdot Y \cdot X^{1/2}$ and $V = U^{-1/2} \cdot (I - X) \cdot U^{-1/2}$ are independent, then $X$ and $Y$ follow matrix beta distribution.
Theorem (Lukacs (1955))

Let $X$ and $Y$ be non-degenerate, independently distributed, positive random variables. If $U = X + Y$ and $V = \frac{X}{X+Y}$ are independent, then $X$ and $Y$ follow Gamma distributions with the same scale parameter.

Theorem (BK (2013))

Let $X$ and $Y$ be independently distributed random variables valued in $\Omega$ with continuous densities, which are strictly positive on $\Omega$. If $U = X + Y$ and $V = (X + Y)^{-1/2} \cdot X \cdot (X + Y)^{-1/2}$ are independent, then $X$ and $Y$ follow Wishart distributions with the same scale parameter ($a$).

Contributors: Lukacs, Olkin, Rubin, Casalis, Letac, Massam, Hassairi, Lajmi, Zine, Boutoria, Bobecka, Wesołowski,...
Each of the aforementioned characterizations has its analogue (sometimes only one implication is known) in free probability.

The "independence" is replaced with the "freeness" and characterized distributions by its "free" counterparts.

Proofs are very different!

Contributors: Bożejko, Bryc, Ejsmont, Franz, Szpojankowski, Wesołowski,...
The Lukacs-Olkin-Rubin theorem without invariance of the ”quotient”

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One may consider more general form of "quotient" $V$:

$$V_w = w(X + Y)^{-1} \cdot X \cdot w^T(X + Y)^{-1},$$

where $w(x) \cdot w(x)^T = x$ and $w(x)$ is invertible $r \times r$ matrix for any $x \in \Omega$.

$w(x)$ is generalized square root of $x$ (not necessarily symmetric).
Lukacs-Olkin-Rubin theorem

- One may consider more general form of "quotient" $V$:
  \[ V_w = w(X + Y)^{-1} \cdot X \cdot w^T (X + Y)^{-1}, \]
  where $w(x) \cdot w(x)^T = x$ and $w(x)$ is invertible $r \times r$ matrix for any $x \in \Omega$.
- $w(x)$ is generalized square root of $x$ (not necessarily symmetric).
- Mapping $w$ is called multiplication algorithm $(x \circ_w y = w(x) \cdot y \cdot w^T(x))$.
One may consider more general form of "quotient" $V$:

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**Theorem (Olkin and Rubin (1962), Casalis and Letac (1996))**

Let $X$ and $Y$ be non-degenerate, independently distributed random variables (+ technical assumptions). If $U = X + Y$ and $V_w$ are independent and distribution $V_w$ is invariant under rotations, then $X$ and $Y$ follow Wishart distributions.
One may consider more general form of "quotient" $V$:

$$V_w = w(X + Y)^{-1} \cdot X \cdot w^T (X + Y)^{-1},$$

where $w(x) \cdot w(x)^T = x$ and $w(x)$ is invertible $r \times r$ matrix for any $x \in \Omega$.

- $w(x)$ is generalized square root of $x$ (not necessarily symmetric).
- Mapping $w$ is called multiplication algorithm ($x \circ_w y = w(x) \cdot y \cdot w^T(x)$).

Theorem (Olkin and Rubin (1962), Casalis and Letac (1996))

Let $X$ and $Y$ be non-degenerate, independently distributed random variables (+ technical assumptions). If $U = X + Y$ and $V_w$ are independent and distribution $V_w$ is invariant under rotations, then $X$ and $Y$ follow Wishart distributions.
Density of \( w \)-Wishart distribution on \( \Omega \) with parameters:

\[
\gamma_w(dx) \propto f(x) e^{-\langle a, x \rangle} l_{\Omega}(x) \, dx,
\]

where \( f \) is continuous \( w \)-multiplicative Cauchy function, that is:

\[
f(x)f(y) = f(w(x) \cdot y \cdot w^T(x)) \quad (x, y) \in \Omega^2.
\]
Density of $w$-Wishart distribution on $\Omega$ with parameters:

$$\gamma_w(dx) \propto f(x) e^{-\langle a, x \rangle} I_\Omega(x) dx,$$

where $f$ is continuous $w$-multiplicative Cauchy function, that is:

$$f(x)f(y) = f(w(x) \cdot y \cdot w^T(x)) \quad (x, y) \in \Omega^2.$$

If $w(x) = x^{1/2}$, then there exists $p \in \mathbb{R}$ such that

$$f(x) = (\det x)^{p - \dim \Omega / r}.$$
Density of $w$-Wishart distribution on $\Omega$ with parameters:

$$\gamma_w(dx) \propto f(x)e^{-\langle a, x \rangle} l_\Omega(x) \, dx,$$

where $f$ is continuous $w$-multiplicative Cauchy function, that is:

$$f(x)f(y) = f(w(x) \cdot y \cdot w^T(x)) \quad (x, y) \in \Omega^2.$$

- If $w(x) = x^{1/2}$, then there exists $p \in \mathbb{R}$ such that $f(x) = (\det x)^{p - \dim \Omega/r}$.
- If $w(x) = t_x$ (lower triangular matrix from Cholesky decomposition of $x = t_x \cdot t_x^T$), then there exists vector $s \in \mathbb{R}^r$ such that $f(x) = \prod_{k=1}^r (\det^{(k)} x)^{s_k}$, where $\det^{(k)} x$ is the $k$th principal minor of $x$. 

Bartosz Kołodziejek

Characterizations of matrix variate distributions
It was observed (Hassairi, Lajmi, Zine (2008)) that the invariance property is not of technical nature only.

**Theorem (BK (2014))**

Let $X$ and $Y$ be independently distributed random variables valued in $\Omega$ with continuous densities, which are strictly positive on $\Omega$. If $U = X + Y$ and $V_w$ are independent, then $X$ and $Y$ follow $w$-Wishart distribution.

In particular:

- If $w(x) = x^{1/2}$ then $X$ and $Y$ follow Wishart distribution;
- If $w(x) = t_x$ then $X$ and $Y$ follow Riesz distribution (generalization of Wishart);
Proof:

Functional equation on $\Omega$ with four unknown continuous functions (densities): $(x, y \in \Omega)$

$$f_X(x)f_Y(y) = \det(x + y)^{\text{dim} \Omega / r} f_U(x + y)f_V(w(x + y)^{-1} \cdot x \cdot w^T(x + y)^{-1})$$
**Proof:**

Functional equation on $\Omega$ with four unknown continuous functions (densities): $(x, y \in \Omega)$

\[
f_x(x)f_y(y) = \det(x + y)^{\dim \Omega/r} f_u(x + y)f_v(w(x + y)^{-1} \cdot x \cdot w^T(x + y)^{-1})
\]

**Solution:** There exist continuous $w$-multiplicative functions $f$ and $g$ such that for any $x \in \Omega$ and $u \in D$:

\[
\begin{align*}
f_x(x) &= f(x)e^{-\langle a, x \rangle}, \\
f_y(x) &= g(x)e^{-\langle a, x \rangle}, \\
f_u(x) &= f(x)g(x)(\det x)^{-\dim \Omega/r} e^{-\langle a, x \rangle}, \\
f_v(u) &= f(u)g(I - u)
\end{align*}
\]
Thank you for your attention!