Deformed q-Laguerre weight and difference equations

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Random Matrices:Theory and Applications
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- Introduction

- Summary of Results

- $q-$Ladder operators, compatibility condition-s, sum rule.

- A $t-$ deformation of the $q^{-1}-$ Laguerre weight.

- Coupled difference equations, akin to $\alpha q-PIV$ and $\alpha q - PV$.

Brief History-Statistical Mechanics of Large matrices

- Wigner 1950's, Statistical theory of nuclear energy levels; Dyson, Mehta and others, 1960's–1970's.
- Wishart and Tsu, 1930's, on Multi-variate statistics.
- Early string theory, Gross and Witten (1980), Gross and Migdal, Brezin and Kazakov, Douglas and Shanker, (1990s). Later string theory, Dijkgraff, Vafa,...(2000—)
- Tracy-Widom distributions, 1990's.
- Transport in disordered systems.
- Practical side; wireless communications, high dimensional statistical inference, principle component analysis with application to genomics.

- “It is not clear what we mean when we say Painlevé equations are integrable”

- “substitute Painlevé by discrete Painlevé.”

- Hankel (or moment) matrices, are basic objects in orthogonal polynomials.

• Notation, $0 < q < 1$, $(a;q)_\infty = \prod_{j=0}^{\infty}(1 - aq^j)$
• For $x \in [0, \infty)$
weight $w(x,t;q) = \frac{x^\alpha}{(-(1-q)x;q)_\infty(-(1-q)t/x;q)_\infty}$, $t \geq 0$, $\alpha > -1$,
special cases

$$w(x,t;1^-) = x^\alpha \, \mathrm{e}^{-x} \, \mathrm{e}^{-t/x},$$

$t \to 0$, give rise to $q^{-1}-$ Laguerre weight introduced by D. Moak.
• The moments, μ_j, are

$$\int_0^\infty x^j \, \mathrm{e}^{-c(\ln x)^2/2} dx = \mathrm{O}(\mathrm{e}^{j^2/(2c)})$$

The partition function:

$$D_n[w] = \frac{1}{n!} \int_{(0,\infty)^n} \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 \prod_{l=1}^{n} w(x_l) dx_l$$

The partition functions as Hankel determinants;

$$\begin{aligned} D_n[w] &= \det(\mu_{i+j})_{0 \leq i,j \leq n-1} \\ &:= \det\left(\int_0^\infty x^{i+j} w(x) dx\right)_{0 \leq i,j \leq n-1}. \end{aligned}$$

• The problem: computation of Hankel determinant for large n

• Introduce monic orthogonal polynomials,

$$P_n(x) = x^n + p(n)x^{n-1} + ...$$

i.e.

$$\int_0^\infty P_j(x)P_k(x)w(x)dx = h_j\delta_{jk}$$

$$D_n[w] = \prod_{j=0}^{n-1} h_j$$

•

$$w(x,t;q) = \mathrm{e}^{-c(\ln\, x)^2}, \text{decreases slowly near } \infty$$

$$= \mathrm{e}^{-c'(\ln\, x)^2}, \text{decreases slowly near } 0.$$

• **Notations and results.**

$$D_q f(x) = \frac{f(x) - f(qx)}{q(1-x)},$$

$$D_q f(x) \to f'(x), \quad q \to 1^-$$

Handy, since, for the weight considered here,

$$u(x) := -\frac{D_{q^{-1}} w(x)}{w(x)}$$

is rational in x.

Chain rule and integration by parts

$$D_q(f(x)g(x)) = f(qx) D_q g(x) + g(x) D_q f(x)$$

$$\int_0^\infty f(x) D_q g(x) dx = -\frac{1}{q} \int_0^\infty g(x) D_{q^{-1}} f(x) dx.$$

Theorem. The orthogonal polynomials satisfy

$q-$difference relations.

$D_q P_n(x) = \beta_n \, A_n(x) P_{n-1}(x) - B_n(x) P_n(x)$

$B_{n+1}(x) + B_n(x) = (x - \alpha_n) A_n(x)$
$+ (q-1)x \sum_{j=0}^n A_j(x) - u(qx), \qquad (qS_1)$

$\beta_{n+1} A_{n+1}(x) - \beta_n A_{n-1}(x) = 1 + (x - \alpha_n) B_{n+1}(x)$
$- (qx - \alpha_n) B_n(x), \quad (qS_2)$

Sum rule
$\beta_n A_n(x)A_{n-1}(x) = B_n(x)^2 + u(qx)B_n(x)$
$+ [1 + (1-q)xB_n(x)]\sum_{j=0}^{n-1} A_j(x), \quad (qS_2')$
Reduces to that found by Magnus.
Remark The sum rule appeared to be new,

- $A_n(x) = \dfrac{1}{h_n}\displaystyle\int_0^\infty \frac{u(qx)-u(y)}{qx-y} P_n(y)P_n(y/q)w(y)dy$
- $B_n(x) = \dfrac{1}{h_{n-1}}\displaystyle\int_0^\infty \frac{u(qx)-u(y)}{qx-y} P_n(y)P_{n-1}(y/q)w(y)dy$

Recent examples, $x \in [0,\infty)$

$$w(x) = \frac{x^\alpha(-p/x^2;q^2)}{(-x^2;q^2)_\infty(-q^2/x^2;q^2)_\infty},\ p \in [0, q^{-\alpha}),\ \alpha \geq 0.$$

recurrence coefficients are related to the $q-$ discrete Painleve V, (Boelen and Van Assche, 2013.)
If $p = 0$, the recurrence coefficients are related to a particular $q - PIII$, (Askey, 1989.)

$$w(x) = \frac{x^\alpha(-p_1/x^2;q^2)_\infty(-p_2/x^2;q^2)_\infty}{(-x^2;q^2)(-q^2/x^2;q^2)_\infty},$$

$$p_1p_2 < q^{2-\alpha},\ p_1 > 0,\ p_2 > 0, \alpha \geq 0,$$

(Filipuk and Smet, 2013), second order, second degree difference equation for the recurrence coefficients.

• Expressions obtained employing orthogonal polynomials and ladder operators.

Lemma. Let $\{P_n\}$ be the monic polynomials orthogonal with respect to

$$w(x,t;q) = \frac{x^\alpha}{(-(1-q)x;q)_\infty(-(1-q)t/x;q)_\infty}.$$

Furthermore, let,

$$R_n := \frac{1}{h_n}\int_0^\infty P_n(y)P_n(y/q)w(y,t;q)\frac{dy}{y},$$

$$r_n := \frac{1}{h_{n-1}}\int_0^\infty P_n(y)P_{n-1}(y/q)w(y,t;q)\frac{dy}{y}$$

Then the recurrence coefficients α_n and β_n are

$$q^{2n+\alpha}\,\alpha_n = \frac{1-q^n}{1-q} + \frac{1-q^{n+\alpha+1}}{q(1-q)} + q^{n-1}t(R_n + (1-q)S_{n-1}),$$

$$q^{2n-1}\beta_n = q^{-2n-2\alpha}\frac{1-q^n}{1-q}\frac{1-q^{n+\alpha}}{1-q} + \frac{(1-q^n)t}{q^{1+\alpha}} + q^{n-\alpha-1}tr_n$$
$$+q^{-2\alpha-n-1}tS_{n-1},$$

where $S_{n-1} = \sum_{j=0}^{n-1} R_j$.

Remark The sum, S_n, maybe expressed in terms of r_n and R_n.

easy solution for $t = q/(1-q)^2$

Theorem Let

$$x_n = \frac{q^{n+\alpha}(1-q)}{R_n}, \;\; y_n = q^n(1-r_n), \;\; T = \frac{(1-q)^2}{q}t,$$

Then x_n and y_n satisfy the following

$$(x_n y_n - 1)(x_{n-1} y_n - 1) = q^{2n+\alpha} T \frac{(y_n - 1)(y_n - 1/T)}{q^n - y_n}$$

$$(x_n y_n - 1)(x_n y_{n+1} - 1) = -q^{2n+\alpha+1} \frac{(x_n - 1)(x_n - T)}{x_n}$$

The right sides of $\alpha q - PIV$ and $\alpha q - PV$ are larger rational functions in x_n and y_n.

• $p(n)$ satisfies a second order non-linear difference equation.

Comment

Behavior of x_n and y_n and therefore the recurrence coefficients at large n?

Behavior of $p(n)$ at large n?

Ultimately D_n for large n,

$$\log D_n \sim \mathrm{O}(n^3) + \text{lower order} + \text{constant} + .. \quad ?$$