Estimation of Autocovariance matrices for Infinite Dimensional Vector Linear Process

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Applications...

- fMRI data: Hemodynamic Response of a large number of voxels for a few replications taken over time.

- Net Domestic Product, Net National Income etc. over a large number of countries: sometimes observations are available for a very short time period.

- Signal processing, pattern recognition, mathematical finance, weather forecasting, Earthquake prediction etc.
• Existing models:
  
  • Infinite dimensional vector moving average process,
  
  • Infinite dimensional vector autoregressive process (IVAR).

• A more general model is infinite dimensional vector linear process.

• Under **Causality** conditions, we can express above models as infinite dimensional vector linear process.
Consider the model

\[ X_{t,p}^{(n)} = \sum_{j=0}^{\infty} \psi_{j,p}^{(n)} \varepsilon_{t-j,p}^{(n)} t, \, n \geq 1 \text{ (almost surely)} \]

- \( X_{t,p}^{(n)} , \varepsilon_{t,p}^{(n)} \): \( p \)-dimensional vectors
- \( \varepsilon_{t,p}^{(n)} \): i.i.d. mean 0 and variance-covariance matrix \( \Sigma_p \)
- \( \psi_{j,p}^{(n)} \): coefficient matrices
- \( p(n) \rightarrow \infty \) as \( n \rightarrow \infty \).
• Triangular sequence:

\[
\begin{align*}
X^{(1)}_{1,p(1)} \\
X^{(2)}_{1,p(2)}, X^{(2)}_{2,p(2)} \\
X^{(3)}_{1,p(3)}, X^{(3)}_{2,p(3)}, X^{(3)}_{3,p(3)} \\
\vdots \\
X^{(n)}_{1,p(n)}, X^{(n)}_{2,p(n)}, X^{(n)}_{3,p(n)}, \ldots, X^{(n)}_{n,p(n)} \\
\vdots
\end{align*}
\]

• Sample at the \( n \)-th stage is the \( n \)-th row of this triangular sequence.
Let, \( \{X_{1,p}, X_{2,p}, \ldots X_{n,p}\} \) be \( p \)-dimensional stationary vectors with mean 0. The autocovariance matrix of order \( k \) is defined as

\[
\Gamma_{k,p} = E(X_t,pX'_{(t-k),p}) \quad \forall k = 1, 2, 3, \ldots
\]

The sample autocovariance matrix of order \( k \) is defined as

\[
\hat{\Gamma}_{k,p} = \frac{1}{n-k} \sum_{t=k+1}^{n} X_t,pX'_{(t-k),p} \quad \forall k = 1, 2, 3, \ldots (n-1).
\]
Problem...

• To estimate the autocovariance matrices $\Gamma_k$ consistently for infinite dimensional vector linear process.

• By a consistent estimator $\hat{A}$ of $A$, we mean

$$||\hat{A} - A||_2 = \sqrt{\lambda_{\text{max}}\{(\hat{A} - A)'(\hat{A} - A)\}} \xrightarrow{P} 0 \quad \text{as } n, p \to \infty.$$ 

• $\frac{\log p}{n} \to 0$
For any matrix $M$ of order $p$,

$$\|M\|_{(1,1)} = \max_j \sum_{i=1}^{p} |m_{ij}|, \quad \|M\|_2 = \sqrt{\lambda_{\max}(M'M)},$$

$$T(M, t) = \max_j \sum_{i:|i-j|>t} |m_{ij}|.$$

- For any matrix $M$ and $t \leq t'$ we have $T(M, t) \geq T(M, t')$. 
Classes for $\Sigma_p$ [Bickel and Levina, 2008]...

- $0 < \epsilon \leq \lambda_{min}(\Sigma_p) \leq \lambda_{max}(\Sigma_p) \leq \frac{1}{\epsilon} < \infty$

- $T(\Sigma, k) \leq Ck^{-\alpha}$ for all $k > 0$

$\Sigma_p = p$-th order principal minor of $\Sigma$

$\lambda_{max} = $ largest eigenvalue of $\Sigma_p$

$\lambda_{min} = $ smallest eigenvalue of $\Sigma_p$

and $\epsilon$ is independent of $p$. 
Classes for coefficient matrices...

Time Lag Criterion...

- Dependence decreases appropriately with the lag.

- Define \( \max(||\psi_j||_{(1,1)}, ||\psi_j'||_{(1,1)}) = r_j, j \geq 0 \).

- For some \( 0 < \beta < 1 \) and \( \lambda \geq 0 \),
  
  \[
  \sum_{j=0}^{\infty} r_j^{\beta} < \infty
  \]
  
  \[
  \sum_{j=0}^{\infty} r_j^{2(1-\beta)} j^\lambda < \infty
  \]

- Example: IVAR process under causality conditions satisfies \( r_j = \theta^j \ \forall j \) and for some \( 0 < \theta < 1 \).
Spatial lag criterion...

- Dependence among neighbours is stronger.

- For some $C, \alpha, \nu > 0$ and $0 < \eta < 1$,

\[ T \left( \psi_j, t \sum_{u=0}^{j} \eta^u \right) \leq C t^{-\alpha} r_j j^{\nu} \sum_{u=0}^{j} \eta^{-u\alpha} \]

\[ \sum_{j=k}^{\infty} r_j r_{j-k} j^{\nu} \frac{1}{\eta^{\alpha j}} < \infty \]

- Simplified form: $T(\psi_j, t) \leq C t^{-\alpha} j^{\nu}$
Banding of a matrix

For any matrix $M$ of order $p$ and $k > 0$, the banded version of $M$ is given by

$$B_k(M) = ((m_{ij} I(|i - j| \leq k))).$$
**Main result...**

**Theorem**

- \( \sup_{j \geq 1} E(e^{\lambda \varepsilon_{t,j}}) < \infty \) for all \( |\lambda| < \lambda_0 \)
- \( k_n = (n^{-1} \log p)^{-\frac{1}{2(\alpha+1)}} \)

\[ \| B_{k_n}(\hat{\Gamma}_k) - \Gamma_k \|_2 = O_p(k_n^{-\alpha} \| \Sigma_p \|_{(1,1)}) \]

**Remark**

If \( \| \Sigma_p \|_{(1,1)} \) is bounded and \( \frac{\log p}{n} \to 0 \), then we achieve the consistency.
References:


THANK YOU!