Elementary Particle Physics: theory and experiments

Course will concentrate on the modern view of particle physics with the emphasis on how theoretical concepts relate to recent experimental measurements.

Follow the course/slides from M. A. Thomson lectures at Cambridge University

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Course Synopsis

Books

M. Thomson, „Modern Particle Physics"

Other recommended books

- "Particle Physics", Martin and Shaw (Wiley): fairly basic but good.
- "Introductory High Energy Physics", Perkins (Cambridge): slightly below level of the course but well written.
- "Introduction to Elementary Physics", Griffiths (Wiley): about right level but doesn't cover the more recent material.
- "Quarks and Leptons", Halzen & Martin (Wiley): good graduate level textbook (slightly above level of this course).

Preliminaries: Natural Units

- S.I. UNITS: **kg m s** are a natural choice for "everyday" objects

- not very natural in particle physics
- instead use **Natural Units** based on the language of particle physics
  - **From Quantum Mechanics** - the unit of action : $\hbar$
  - **From relativity** - the speed of light: c
  - **From Particle Physics** - unit of energy: GeV (1 GeV ~ proton rest mass energy)

★ **Units become (i.e. with the correct dimensions):**

Energy	GeV	**Time**	$(\text{GeV}/\hbar)^{-1}$
Momentum	GeV/c	**Length**	$(\text{GeV}/\hbar c)^{-1}$
Mass	GeV/c^2	**Area**	$(\text{GeV}/\hbar c)^{-2}$

★ Simplify algebra by <u>setting</u>: $\boxed{\hbar = c = 1}$

- Now all quantities expressed in powers of GeV

Energy	GeV	**Time**	GeV^{-1}
Momentum	GeV	**Length**	GeV^{-1}
Mass	GeV	**Area**	GeV^{-2}

To convert back to S.I. units, need to restore missing factors of $\hbar$ and c

Preliminaries: Heaviside-Lorentz Units

- **Electron charge** <u>defined</u> by Force equation: $F = \dfrac{e^2}{4\pi\varepsilon_0 r^2}$

- In Heaviside-Lorentz units set $\boxed{\varepsilon_0 = 1}$

  and $F \rightarrow \dfrac{e^2}{4\pi r^2}$ **NOW: electric charge has dimensions** $[FL^2]^{\frac{1}{2}} = [EL]^{\frac{1}{2}} = [\hbar c]^{\frac{1}{2}}$

- **Since** $c = (\varepsilon_0 \mu_0)^{-\frac{1}{2}} = 1$ ➡ $\boxed{\mu_0 = 1}$

$$\boxed{\hbar = c = \varepsilon_0 = \mu_0 = 1}$$

★ **Unless otherwise stated, Natural Units are used**
$$E^2 = p^2 + m^2 , \quad \vec{p} = \vec{k}, \text{ etc.}$$

Elementary Particle Physics: theory and experiments

Short recap on the Standard Model

Feynman diagrams, units and kinematics

Decay Rates and Cross-sections

Review of the Standard Model

Particle Physics is the study of:

★ MATTER: the fundamental constituents of the universe
- the elementary particles

★ FORCE: the fundamental forces of nature, i.e. the interactions between the elementary particles

Try to categorise the **PARTICLES** and **FORCES** in as simple and fundamental manner possible

★Current understanding embodied in the **STANDARD MODEL**:
- **Forces** between particles due to exchange of **particles**
- Consistent with <u>all</u> current experimental data !
- But it is just a "**model**" with many unpredicted parameters, e.g. particle masses.
- As such it is not the ultimate theory (if such a thing exists), there are many mysteries.

★ In the Standard Model the fundamental "matter" is described by **point-like spin-1/2** fermions

	LEPTONS			QUARKS		
		q	m/GeV		q	m/GeV
First Generation	e^-	-1	0.0005	d	$-1/3$	0.3
	ν_1	0	≈ 0	u	$+2/3$	0.3
Second Generation	μ^-	-1	0.106	s	$-1/3$	0.5
	ν_2	0	≈ 0	c	$+2/3$	1.5
Third Generation	τ^-	-1	1.77	b	$-1/3$	4.5
	ν_3	0	≈ 0	t	$+2/3$	175

The masses quoted for the quarks are the "constituent masses", i.e. the effective masses for quarks confined in a bound state

- In the SM there are <u>three generations</u> – the particles in each generation are copies of each other differing **only** in mass. (not understood why three).
- The neutrinos are much lighter than all other particles (e.g. ν_1 has m<3 eV) – we now know that neutrinos have non-zero mass (don't understand why so small)

Forces in the Standard Model

★ **Forces mediated by the exchange of spin-1 Gauge Bosons**

Force	Boson(s)	J^P	m/GeV
EM (QED)	Photon γ	1^-	0
Weak	$W^\pm$ / Z	1^-	80 / 91
Strong (QCD)	8 Gluons g	1^-	0
Gravity (?)	Graviton?	2^+	0
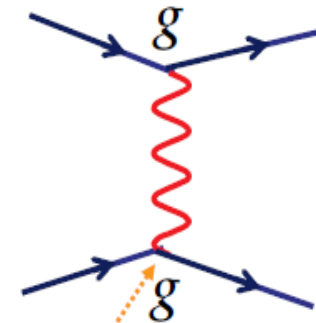
- **Fundamental interaction strength is given by charge g.**
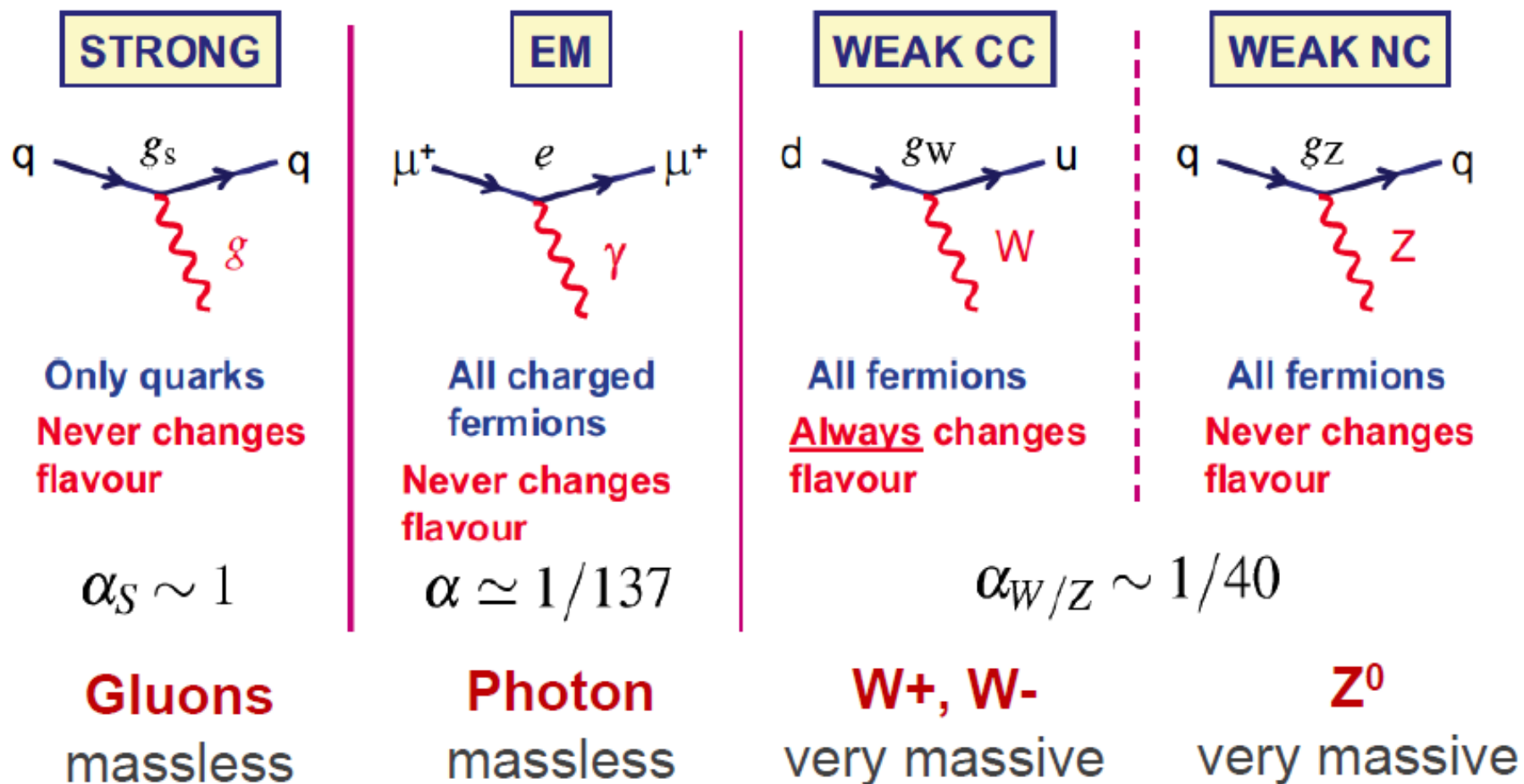- **Related to the <u>dimensionless</u> coupling "constant" α**

e.g. QED
$$g_{em} = e = \sqrt{4\pi\alpha\varepsilon_0\hbar c}$$

★ **In Natural Units**
$$g = \sqrt{4\pi\alpha}$$

(both g and α are dimensionless, but g contains a "hidden" $\hbar c$)

★ **Convenient to express couplings in terms of α which, being genuinely dimensionless does not depend on the system of units (this is not true for the numerical value for e)**
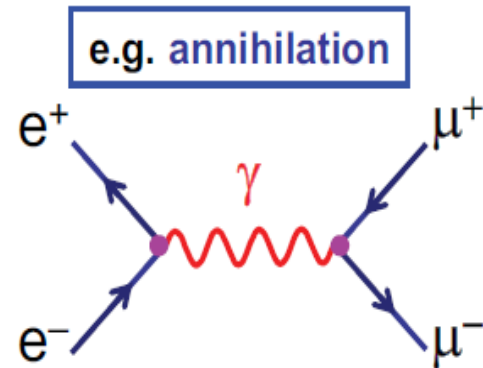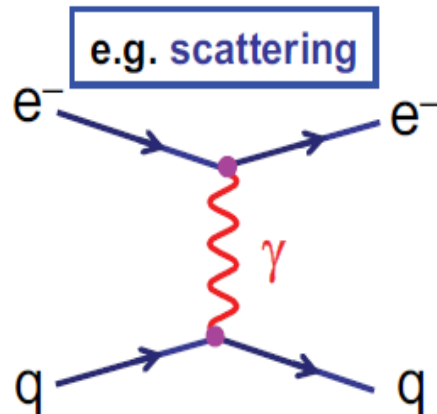
Standard Model vertices

The interaction of gauge bosons with fermions is described by the Standard Model

STRONG	EM	WEAK CC	WEAK NC
$q \xrightarrow{g_s} q$ g	$\mu^+ \xrightarrow{e} \mu^+$ γ	$d \xrightarrow{g_W} u$ W	$q \xrightarrow{g_Z} q$ Z
Only quarks **Never changes flavour**	**All charged fermions** **Never changes flavour**	**All fermions** **Always changes flavour**	**All fermions** **Never changes flavour**
$\alpha_S \sim 1$	$\alpha \simeq 1/137$	$\alpha_{W/Z} \sim 1/40$	
Gluons massless	**Photon** massless	**W+, W-** very massive	**Z^0** very massive

Feynman diagrams

★ **Particle interactions described in terms of Feynman diagrams**

e.g. scattering

e.g. annihilation

★ **IMPORTANT POINTS TO REMEMBER:**
- **"time" runs from left – right, only in sense that:**
    - ◆ **LHS of diagram is initial state**
    - ◆ **RHS of diagram is final state**
    - ◆ **Middle is "how it happened"**
- **anti-particle arrows in –ve "time" direction**
- **Energy, momentum, angular momentum, etc. conserved at all interaction vertices**
- **All intermediate particles are "virtual"**
    i.e. $E^2 \neq |\vec{p}|^2 + m^2$
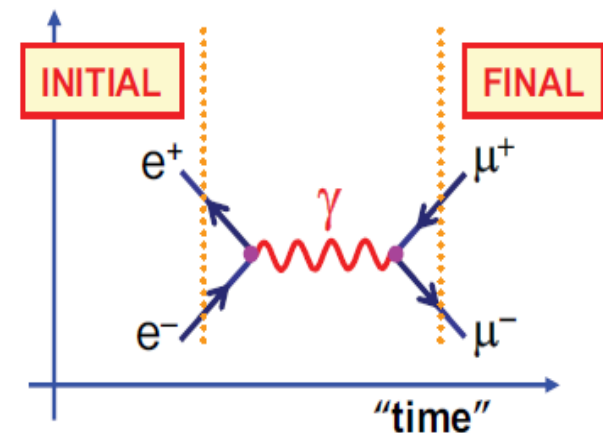
Special relativity and 4-vector notation

- Will use 4-vector notation with p^0 as the time-like component, e.g.

$$p^\mu = \{E, \vec{p}\} = \{E, p_x, p_y, p_z\} \qquad \text{(contravariant)}$$

$$p_\mu = g_{\mu\nu} p^\mu = \{E, -\vec{p}\} = \{E, -p_x, -p_y, -p_z\} \qquad \text{(covariant)}$$

with

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- In particle physics, usually deal with relativistic particles. **Require** all calculations to be Lorentz Invariant. L.I. quantities formed from 4-vector scalar products, e.g.

$$p^\mu p_\mu = E^2 - p^2 = m^2 \qquad \text{Invariant mass}$$

$$x^\mu p_\mu = Et - \vec{p}.\vec{r} \qquad \text{Phase}$$

- A few words on **NOTATION**

  Four vectors written as either: p^μ or p

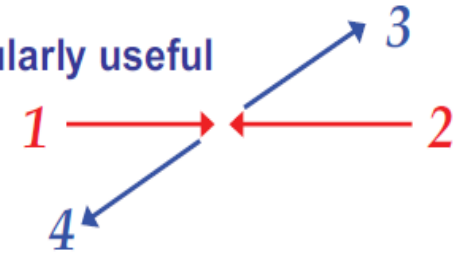 **Four vector scalar product:** $p^\mu q_\mu$ or $p.q$

  Three vectors written as: $\vec{p}$

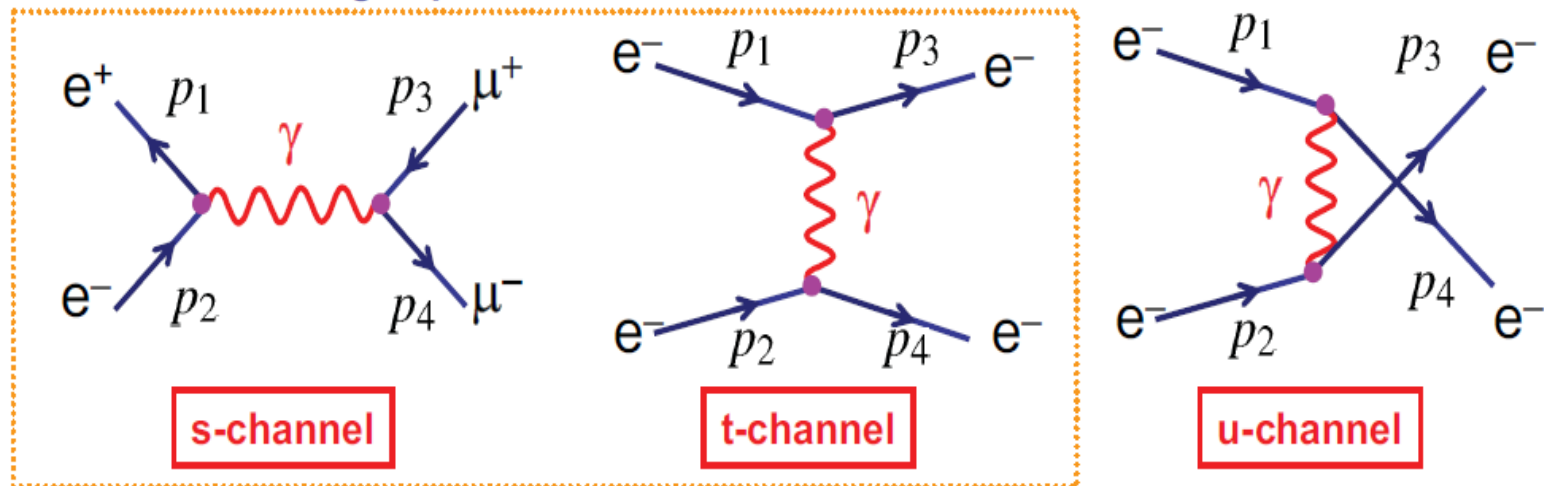 **Quantities evaluated in the centre of mass frame:** $\vec{p}^*, p^*$ etc.

Mandelstam s, t and u

★ In particle scattering/annihilation there are three particularly useful **Lorentz Invariant** quantities: **s, t** and **u**

★ Consider the scattering process $1 + 2 \rightarrow 3 + 4$

★ (Simple) Feynman diagrams can be categorised according to the four-momentum of the exchanged particle

s-channel t-channel u-channel

• Can define **three** kinematic variables: **s, t** and **u** from the following four vector scalar products (squared four-momentum of exchanged particle)
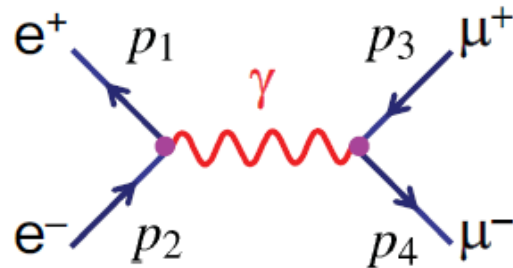
$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2$$

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Example: Mandelstam s, t and u

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2$$

Note: $$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

★ **e.g. Centre-of-mass energy, s:**

$$s = (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$$

- **This is a scalar product of two four-vectors** ➡ **Lorentz Invariant**

- **Since this is a L.I. quantity, can evaluate in any frame. Choose the most convenient, i.e. the centre-of-mass frame:**

$$p_1^* = (E_1^*, \vec{p}^*) \quad p_2^* = (E_2^*, -\vec{p}^*)$$

$$\Rightarrow \quad \boxed{s = (E_1^* + E_2^*)^2}$$

★ **Hence** $\sqrt{s}$ **is the total energy of collision in the centre-of-mass frame**

From Feynman diagrams to physics

Particle Physics = Precision Physics

★ **Particle physics is about building fundamental theories and testing their predictions against precise experimental data**

- **Dealing with fundamental particles and can make very precise theoretical predictions – not complicated by dealing with many-body systems**

- **Many beautiful experimental measurements**
  - ➡ **precise theoretical predictions challenged by precise measurements**

- **For all its flaws, the Standard Model describes all experimental data ! This is a (the?) remarkable achievement of late 20th century physics.**

Requires understanding of theory and experimental data

- **will use Feynman diagrams and associated Feynman rules to perform calculations for many processes**
- **hopefully gain a fairly deep understanding of the Standard Model and how it explains all current data**

Before we can start, need calculations for:

- **Interaction cross sections;**
- **Particle decay rates;**

Cross-sections and decay rates

- In particle physics we are mainly concerned with particle **interactions** and **decays**, i.e. transitions between states
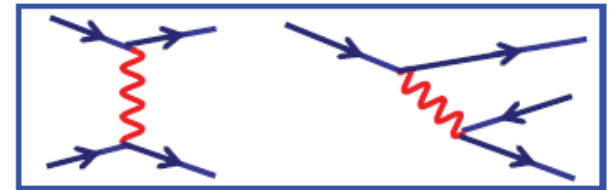
  - these are the experimental observables of particle physics

- Calculate **transition rates** from **Fermi's Golden Rule**

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

Γ_{fi} is number of transitions per unit time from initial state $|i\rangle$ to final state $\langle f|$ — **not Lorentz Invariant !**

T_{fi} is Transition **Matrix Element**

$$T_{fi} = \langle f|\hat{H}|i\rangle + \sum_{j\neq i} \frac{\langle f|\hat{H}|j\rangle\langle j|\hat{H}|i\rangle}{E_i - E_j} + \dots$$

$\hat{H}$ is the perturbing Hamiltonian

$\rho(E_f)$ is density of final states

★ Rates depend on **MATRIX ELEMENT** and **DENSITY OF STATES**

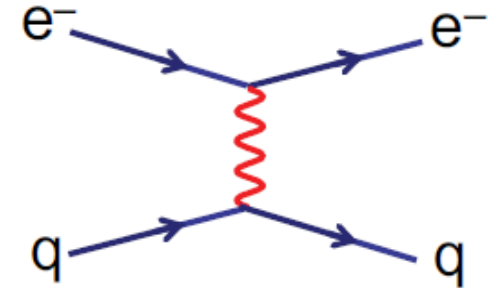the ME contains the fundamental particle physics   just kinematics
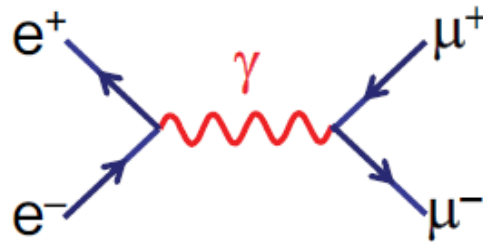
The first few lectures

★ **Aiming towards a proper calculation of decay and scattering processes**
  Will concentrate on:
  - $e^+e^- \rightarrow \mu^+\mu^-$
  - $e^-q \rightarrow e^-q$

  (e⁻q→e⁻q to probe
  proton structure)

△ **Need** <u>relativistic</u> **calculations of particle decay rates and cross sections:**

$$\sigma = \frac{|M_{fi}|^2}{\text{flux}} \times (\text{phase space})$$

△ **Need** <u>relativistic</u> **treatment of spin-half particles:**
  Dirac Equation
△ **Need** <u>relativistic</u> **calculation of interaction Matrix Element:**
  Interaction by particle exchange and Feynman rules
+ **and a few mathematical tricks along, e.g. the Dirac Delta Function**

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Particle decays

- **In general a given particle may decay to more than one decay mode. For example tau lepton decay into a number of final states:**

$$\tau^- \to e^- \bar{\nu}_e \nu_\tau, \quad \tau^- \to \mu^- \bar{\nu}_\mu \nu_\tau \quad \text{and} \quad \tau^- \to \nu_\tau + \text{hadrons}$$

- **The transition rate for each decay mode j can be calculated independently using Fermi's golden rule. The individual transition rates Γ_j are refered as partial decay rates or partial widths.**

- **The total decay rate is a simple sum of partial decay rates,** $\Gamma = \sum_j \Gamma_j.$

- **The lifetime of particle in its rest frame τ is refered to as its proper lifetime and is determined from the total decay rate** $\tau = \dfrac{1}{\Gamma}.$

- **The relative frequency of the particular decay mode is refered to as a branching ratio (or a branching fraction).**

$$BR(j) = \frac{\Gamma_j}{\Gamma}.$$

- **For example the branching ratio for the tau lepton decay:** $\tau^- \to e^- \bar{\nu}_e \nu_\tau$ **is** 0.17

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- **Consider the two-body decay**

$$i \rightarrow 1 + 2$$

The two body decay in the rest fame of particle i

- **Want to calculate the decay rate in first order perturbation theory using plane-wave descriptions of the particles (Born approximation):**

$$\psi_1 = N e^{i(\vec{p}.\vec{r} - Et)}$$
$$= N e^{-ip.x}$$

$$(\vec{k}.\vec{r} = \vec{p}.\vec{r} \quad \text{as} \quad \hbar = 1)$$

where N is the normalisation and $p.x = p^\mu x_\mu$

For decay rate calculation need to know:

- **Wave-function normalisation**
- **Transition matrix element from perturbation theory**
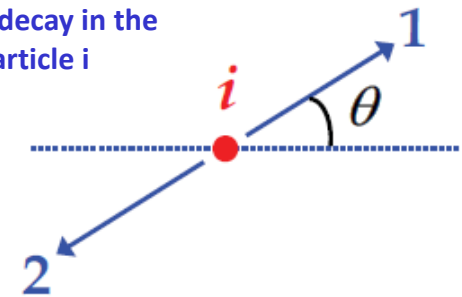- **Expression for the density of states**

All in a Lorentz Invariant form

★ **First consider wave-function normalisation**

- **Non-relativistic: normalised to one particle in a cube of side** a
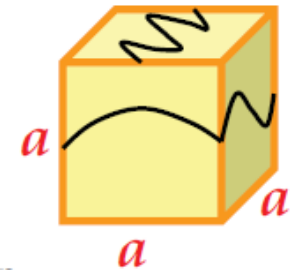
$$\int \psi \psi^* \mathrm{d}V = N^2 a^3 = 1 \quad \Rightarrow \quad N^2 = 1/a^3$$

Non-relativistic phase-space (revision)

- Apply boundary conditions $(\vec{p} = \hbar \dot{k})$:
- Wave-function vanishing at box boundaries

  ➡ quantised particle momenta:

$$p_x = \frac{2\pi n_x}{a}; \ p_y = \frac{2\pi n_y}{a}; \ p_z = \frac{2\pi n_z}{a}$$
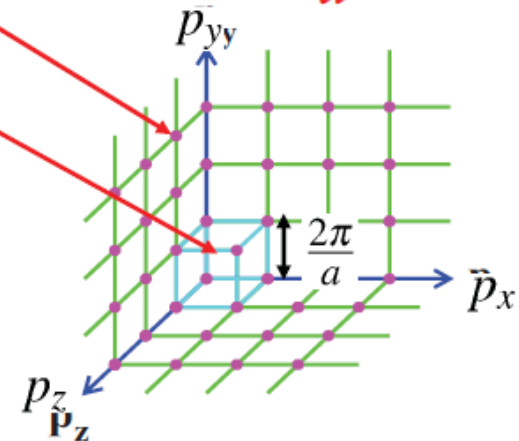
- Volume of single state in momentum space:

$$\left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$

- Normalising to **one particle/unit volume gives number of states** in element: $\mathrm{d}^3\vec{p} = \mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z$

$$\mathrm{d}n = \frac{\mathrm{d}^3\vec{p}}{\frac{(2\pi)^3}{V}} \times \frac{1}{V} = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}$$

- Therefore density of states in Golden rule:

$$\rho(E_f) = \left|\frac{\mathrm{d}n}{dE}\right|_{E_f} = \left|\frac{\mathrm{d}n}{\mathrm{d}|\vec{p}|}\frac{\mathrm{d}|\vec{p}|}{dE}\right|_{E_f}$$
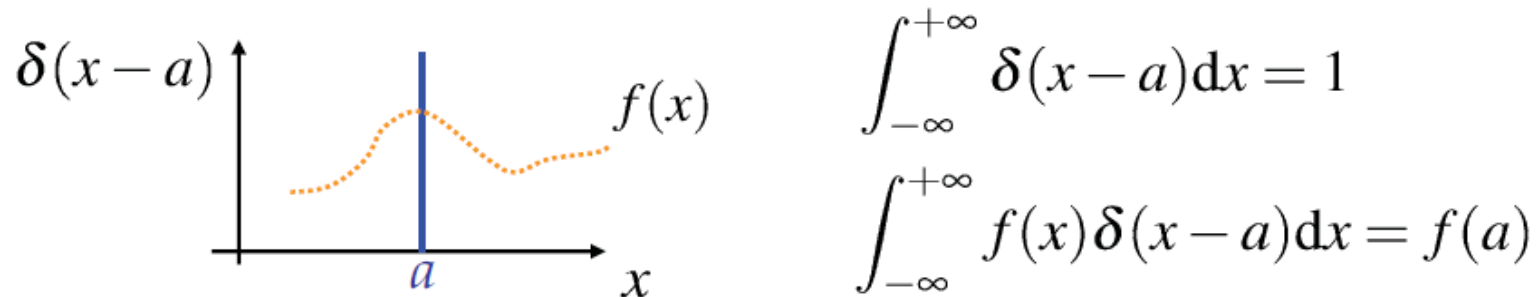
with

$$p = \beta E$$

- Integrating over an elemental shell in momentum-space gives

$$(\mathrm{d}^3\vec{p} = 4\pi p^2 \mathrm{d}p)$$

$$\rho(E_f) = \frac{4\pi p^2}{(2\pi)^3} \times \beta$$

Dirac δ function

- In the relativistic formulation of decay rates and cross sections we will make use of the Dirac δ function: "infinitely narrow spike of unit area"

$$\int_{-\infty}^{+\infty} \delta(x-a)\mathrm{d}x = 1$$

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)\mathrm{d}x = f(a)$$

- Any function with the above properties can represent $\delta(x)$

e.g. $\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x^2}{2\sigma^2}\right)}$ (an infinitesimally narrow Gaussian)

- In relativistic quantum mechanics delta functions prove extremely useful for integrals over phase space, e.g. in the decay $a \to 1+2$

$$\int \dots \delta(E_a - E_1 - E_2)\mathrm{d}E \qquad \text{and} \qquad \int \dots \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2)\mathrm{d}^3\vec{p}$$

express energy and momentum conservation

★ **We will soon need an expression for the delta function of a function** $\delta(f(x))$
   • **Start from the definition of a delta function**

$$\int_{y_1}^{y_2} \delta(y)\mathrm{d}y = \begin{cases} 1 & \text{if } y_1 < 0 < y_2 \\ 0 & \text{otherwise} \end{cases}$$

   • **Now express in terms of** $y = f(x)$ **where** $f(x_0) = 0$ **and then change variables**

$$\int_{x_1}^{x_2} \delta(f(x)) \frac{\mathrm{d}f}{\mathrm{d}x} \mathrm{d}x = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

   • **From properties of the delta function (i.e. here only non-zero at** x_0 **)**

$$\left.\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|\right|_{x_0} \int_{x_1}^{x_2} \delta(f(x))\mathrm{d}x = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$
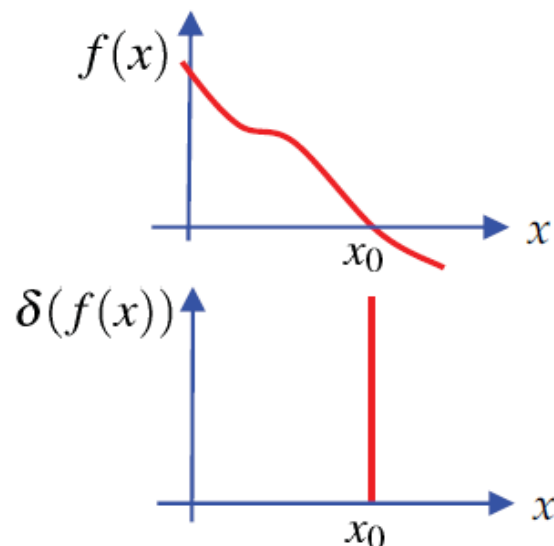
   • **Rearranging and expressing the RHS as a delta function**

$$\int_{x_1}^{x_2} \delta(f(x))\mathrm{d}x = \frac{1}{|\mathrm{d}f/\mathrm{d}x|_{x_0}} \int_{x_1}^{x_2} \delta(x-x_0)\mathrm{d}x$$

$$\boxed{\delta(f(x)) = \left.\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|^{-1}\right|_{x_0} \delta(x-x_0)}$$

$f(x)$

x_0

x

$\delta(f(x))$

x_0

x

(1)

The Golden Rule revisited

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

- Rewrite the expression for density of states using a delta-function

$$\rho(E_f) = \left|\frac{dn}{dE}\right|_{E_f} = \int \frac{dn}{dE} \delta(E - E_i) dE \qquad \text{since } E_f = E_i$$
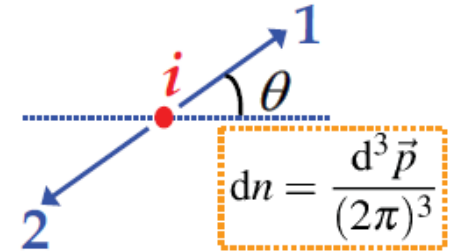
Note : integrating over all final state energies but energy conservation now taken into account explicitly by delta function

- Hence the golden rule becomes: $\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E) dn$

the integral is over **all** "allowed" final states of **any energy**

- For **dn** in a two-body decay, only need to consider one particle : mom. conservation fixes the other

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E_1 - E_2) \frac{d^3\vec{p}_1}{(2\pi)^3}$$

$$dn = \frac{d^3\vec{p}}{(2\pi)^3}$$

- However, can include momentum conservation explicitly by integrating over the momenta of **both** particles and using another δ-**fn**

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \underbrace{\delta(E_i - E_1 - E_2)}_{\text{Energy cons.}} \underbrace{\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)}_{\text{Mom. cons.}} \underbrace{\frac{d^3\vec{p}_1}{(2\pi)^3} \frac{d^3\vec{p}_2}{(2\pi)^3}}_{\text{Density of states}}$$

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Lorentz invariant phase-space

- In non-relativistic QM normalise to one particle/unit volume: $\int \psi^* \psi \, dV = 1$
- When considering relativistic effects, volume <u>contracts</u> by $\gamma = E/m$
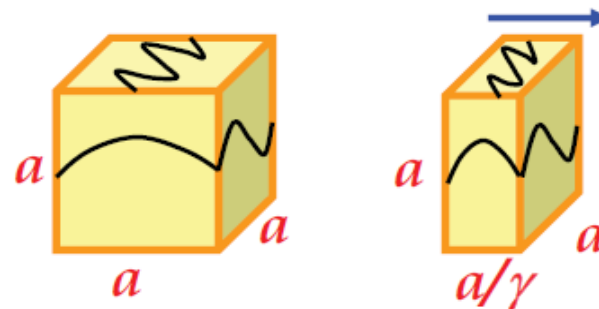
- Particle density therefore increases by $\gamma = E/m$
  - ★ Conclude that a relativistic invariant wave-function normalisation needs to be **proportional to E particles per unit volume**
- Usual convention: **Normalise to 2E particles/unit volume** $\quad \int \psi'^* \psi' \, dV = 2E$
- Previously used ψ normalised to 1 particle per unit volume $\quad \int \psi^* \psi \, dV = 1$
- Hence $\psi' = (2E)^{1/2} \psi$ is normalised to $2E$ per unit volume
- <u>**Define**</u> **Lorentz Invariant Matrix Element**, M_{fi} , **in terms of the wave-functions** normalised to $2E$ particles per unit volume

$$M_{fi} = \langle \psi'_1 . \psi'_2 ... | \hat{H} | ... \psi'_{n-1} \psi'_n \rangle = (2E_1 . 2E_2 . 2E_3 .... 2E_n)^{1/2} T_{fi}$$

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Decay rate calculations

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2)\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}$$
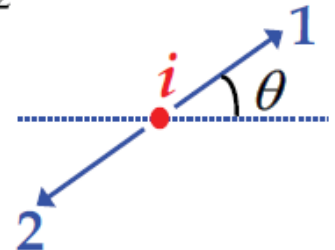
★ **Because the integral is Lorentz invariant (i.e. frame independent) it can be evaluated in any frame we choose. The C.o.M. frame is most convenient**

• **In the C.o.M. frame** $E_i = m_i$ **and** $\vec{p}_i = 0$ ⟹

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\delta^3(\vec{p}_1 + \vec{p}_2)\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}$$

• **Integrating over** $\vec{p}_2$ **using the** δ**-function:**

$$\Rightarrow \Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\frac{d^3\vec{p}_1}{4E_1 E_2}$$

now $E_2^2 = (m_2^2 + |\vec{p}_1|^2)$ **since the** δ**-function imposes** $\vec{p}_2 = -\vec{p}_1$

• **Writing** $d^3\vec{p}_1 = p_1^2 dp_1 \sin\theta d\theta d\phi = p_1^2 dp_1 d\Omega$
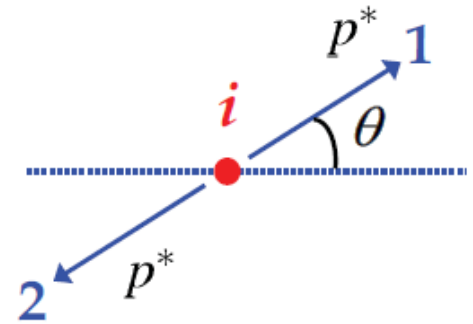
For convenience, here $|\vec{p}_1|$ is written as p_1

$$\Rightarrow \Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \delta\left(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right)\frac{p_1^2 dp_1 d\Omega}{E_1 E_2}$$

- **Which can be written in the form**

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 g(p_1)\delta(f(p_1))\mathrm{d}p_1\mathrm{d}\Omega \qquad (2)$$

where $g(p_1) = p_1^2/(E_1 E_2) = p_1^2(m_1^2 + p_1^2)^{-1/2}(m_2^2 + p_1^2)^{-1/2}$

and $f(p_1) = m_i - (m_1^2 + p_1^2)^{1/2} - (m_2^2 + p_1^2)^{1/2}$

Note:
- $\delta(f(p_1))$ imposes energy conservation.
- $f(p_1) = 0$ determines the C.o.M momenta of the two decay products
  i.e. $f(p_1) = 0$ for $p_1 = p^*$

★ **Eq. (2) can be integrated using the property of** δ **– function derived earlier (eq. (1))**

$$\int g(p_1)\delta(f(p_1))\mathrm{d}p_1 = \frac{1}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}} \int g(p_1)\delta(p_1 - p^*)\mathrm{d}p_1 = \frac{g(p^*)}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}}$$

where p^* is the value for which $f(p^*) = 0$

- **All that remains is to evaluate** $\mathrm{d}f/\mathrm{d}p_1$

$$\frac{\mathrm{d}f}{\mathrm{d}p_1} = -\frac{p_1}{(m_1^2 + p_1^2)^{1/2}} - \frac{p_1}{(m_2^2 + p_1^2)^{1/2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1\frac{E_1 + E_2}{E_1 E_2}$$

giving:

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \frac{E_1 E_2}{p_1(E_1 + E_2)} \frac{p_1^2}{E_1 E_2} \right|_{p_1 = p^*} d\Omega$$

$$= \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \frac{p_1}{E_1 + E_2} \right|_{p_1 = p^*} d\Omega$$

- But from $f(p_1) = 0$, i.e. energy conservation: $E_1 + E_2 = m_i$

$$\Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 E_i m_i} \int |M_{fi}|^2 d\Omega$$

In the particle's <u>rest frame</u> $E_i = m_i$

$$\boxed{\frac{1}{\tau} = \Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 d\Omega} \qquad (3)$$

VALID FOR <u>ALL</u> TWO-BODY DECAYS !

- p^* can be obtained from $f(p_1) = 0$

$$(m_1^2 + p^{*2})^{1/2} + (m_2^2 + p^{*2})^{1/2} = m_i$$

$$p^* = \frac{1}{2m_i} \sqrt{\left[(m_i^2 - (m_1 + m_2)^2\right] \left[m_i^2 - (m_1 - m_2)^2\right]}$$
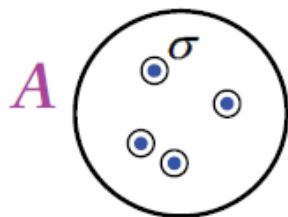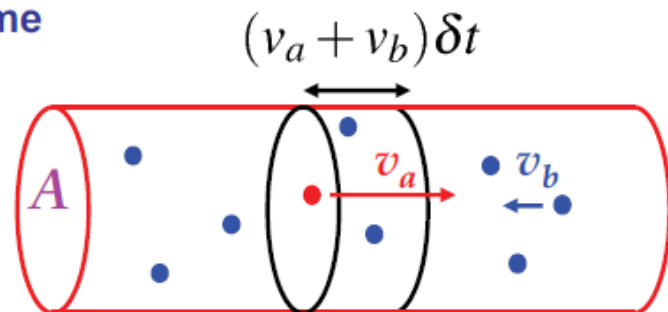
Interaction cross section

- **Calculation of the interaction cross-section is slightly more complicated, it is necessary to account for the flux of the initial-state particles.**

- **In the simplest case one can imagine a beam of particle of type a, with flux ϕ_a, crossing a region of space in which there are n_b particles per unit volume. The interaction rate will be**

$$r_b = n_b\, \sigma\, \phi_a$$

- **The fundamental physics is contained in σ, which has dimensions of an area and is termed cross-section.**

- **Consider a single particle of type a with velocity, v_a, traversing a region of area A containing n_b particles of type b per unit volume**

$(v_a + v_b)\delta t$

In time δt a particle of type a traverses region containing $n_b(v_a + v_b)A\delta t$ particles of type b

A v_a v_b

A σ

★ **Interaction probability obtained from effective cross-sectional area occupied by the**

$n_b(v_a + v_b)A\delta t$ **particles of type b**

- **Interaction Probability =** $\dfrac{n_b(v_a + v_b)A\delta t\,\sigma}{A} = n_b v \delta t \sigma$ $[v = v_a + v_b]$

➡ **Rate per particle of type a = $n_b\,v\,\sigma$**

- **Consider volume V, total reaction rate =** $(n_b v \sigma).(n_a V) = (n_b V)(n_a v)\sigma$

$= N_b \phi_a \sigma$

- **As anticipated:**   **Rate = Flux x Number of targets x cross section**

Cross-section definition

$$\sigma = \frac{\text{no of interactions per unit time per target}}{\text{incident flux}}$$

Flux = number of incident particles/ unit area/unit time

- The "cross section", σ, can be thought of as the **effective** cross-sectional area of the target particles for the interaction to occur.
- In general this has nothing to do with the physical size of the target although there are exceptions, e.g. neutron absorption
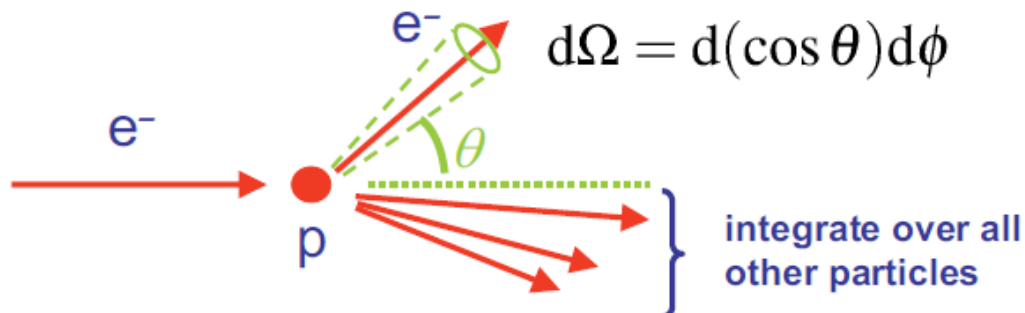
here σ is the projective area of nucleus

Differential Cross section

$$\frac{d\sigma}{d\Omega} = \frac{\text{no of particles per sec/per target into } d\Omega}{\text{incident flux}}$$

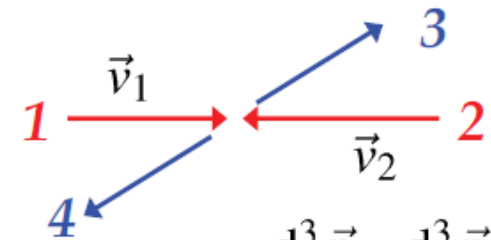or generally

$$\frac{d\sigma}{d\dots}$$

$$d\Omega = d(\cos\theta)d\phi$$

e^-

e^-

θ

p

integrate over all other particles

with $$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$

Cross-section calculations

- **Consider scattering process**

$$1 + 2 \rightarrow 3 + 4$$

- **Start from Fermi's Golden Rule:**

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4) \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \frac{d^3\vec{p}_3}{(2\pi)^3} \frac{d^3\vec{p}_4}{(2\pi)^3}$$

where T_{fi} is the transition matrix for a **normalisation of 1/unit volume**

- **Now** $\text{Rate/Volume} = (\text{flux of } 1) \times (\text{number density of } 2) \times \sigma$

$$= n_1(v_1 + v_2) \times n_2 \times \sigma$$

- **For 1 target particle per unit volume** $\text{Rate} = (v_1 + v_2)\sigma$

$$\sigma = \frac{\Gamma_{fi}}{(v_1 + v_2)}$$

$$\sigma = \frac{(2\pi)^4}{v_1 + v_2} \int |T_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4) \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \frac{d^3\vec{p}_3}{(2\pi)^3} \frac{d^3\vec{p}_4}{(2\pi)^3}$$

the parts are not Lorentz Invariant

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- To obtain a Lorentz Invariant form use wave-functions normalised to $2E$ particles per unit volume

$$\psi' = (2E)^{1/2}\psi$$

- Again define L.I. Matrix element $M_{fi} = (2E_1\,2E_2\,2E_3\,2E_4)^{1/2}T_{fi}$

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2 (v_1+v_2)}\int |M_{fi}|^2\delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{2E_3}\frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

- The integral is now written in a Lorentz invariant form
- The quantity $F = 2E_1 2E_2(v_1+v_1)$ can be written in terms of a four-vector scalar product and is therefore also Lorentz Invariant (the Lorentz Inv. Flux)

$$F = 4\left[(p_1^\mu p_{2\mu})^2 - m_1^2 m_2^2\right]^{1/2} \qquad \text{(see appendix I)}$$

- Consequently cross section is a Lorentz Invariant quantity

Two special cases of Lorentz Invariant Flux:

- Centre-of-Mass Frame

$$
\begin{aligned}
F &= 4E_1 E_2 (v_1 + v_2) \\
&= 4E_1 E_2 (|\vec{p}^*|/E_1 + |\vec{p}^*|/E_2) \\
&= 4|\vec{p}^*|(E_1 + E_2) \\
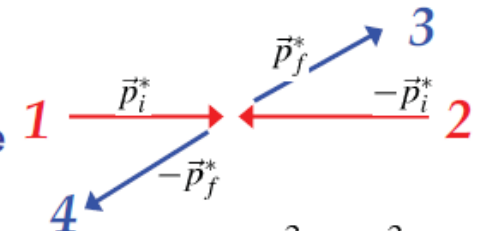&= 4|\vec{p}^*|\sqrt{s}
\end{aligned}
$$

- Target (particle 2) at rest

$$
\begin{aligned}
F &= 4E_1 E_2 (v_1 + v_2) \\
&= 4E_1 m_2 v_1 \\
&= 4E_1 m_2 (|\vec{p}_1|/E_1) \\
&= 4m_2 |\vec{p}_1|
\end{aligned}
$$

- We will now apply above Lorentz Invariant formula for the interaction cross section to the most common cases used in the course. First consider $2 \to 2$ scattering in C.o.M. frame

- Start from

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1+v_2)} \int |M_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{d^3\vec{p}_3}{2E_3}\frac{d^3\vec{p}_4}{2E_4}$$

- Here $\vec{p}_1 + \vec{p}_2 = 0$ and $E_1 + E_2 = \sqrt{s}$

$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \int |M_{fi}|^2 \delta(\sqrt{s}-E_3-E_4)\delta^3(\vec{p}_3+\vec{p}_4)\frac{d^3\vec{p}_3}{2E_3}\frac{d^3\vec{p}_4}{2E_4}$$
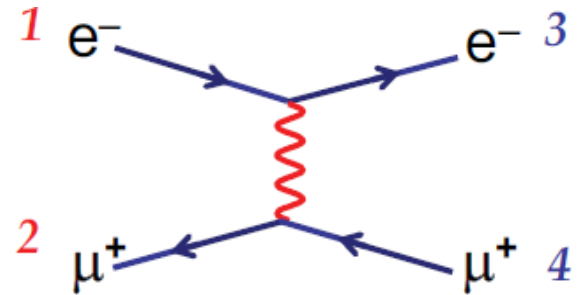
★ The integral is exactly the same integral that appeared in the particle decay calculation but with m_a replaced by $\sqrt{s}$

$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}}\frac{|\vec{p}_f^*|}{4\sqrt{s}} \int |M_{fi}|^2 d\Omega^*$$

$$\boxed{\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 d\Omega^*}$$

(4)

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- **In the case of elastic scattering** $|\vec{p}_i^*| = |\vec{p}_f^*|$

$$\sigma_{\text{elastic}} = \frac{1}{64\pi^2 s} \int |M_{fi}|^2 d\Omega^*$$

- **For calculating the total cross-section (which is Lorentz Invariant) the result on the previous page (eq. (4)) is sufficient. However, it is not so useful for calculating the differential cross section in a rest frame other than the C.o.M:**

$$d\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} |M_{fi}|^2 d\Omega^*$$

because the angles in $d\Omega^* = d(\cos\theta^*)d\phi^*$ **refer to the C.o.M frame**

- **For the last calculation in this section, we need to find a L.I. expression for** $d\sigma$

★ **Start by expressing** $d\Omega^*$ **in terms of Mandelstam** *t*
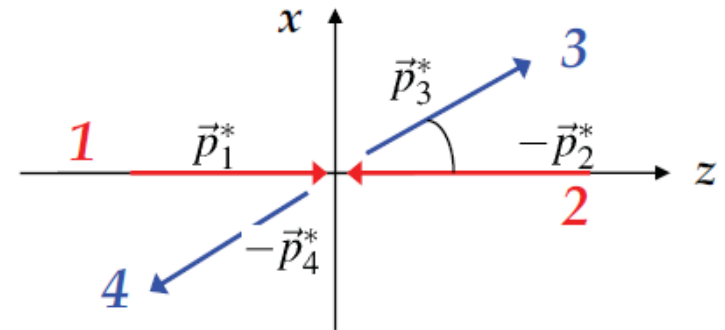**i.e. the square of the four-momentum transfer**

$$t = q^2 = (p_1 - p_3)^2$$

$$q^\mu = p_1^\mu - p_3^\mu$$
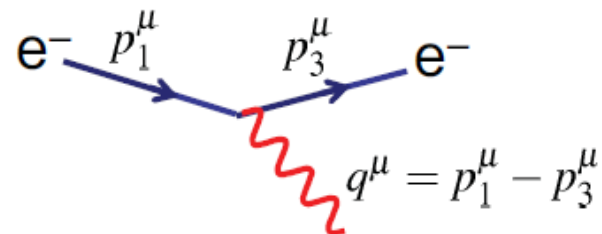
Product of four-vectors therefore L.I.

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- **Want to express** $\mathrm{d}\Omega^*$ **in terms of Lorentz Invariant** $\mathrm{d}t$

  where $\quad t \equiv (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1 \cdot p_3 = m_1^2 + m_3^2 - 2p_1 \cdot p_3$

- **In C.o.M. frame:**

$$
\begin{aligned}
p_1^{*\mu} &= (E_1^*, 0, 0, |\vec{p}_1^*|) \\
p_3^{*\mu} &= (E_3^*, |\vec{p}_3^*|\sin\theta^*, 0, |\vec{p}_3^*|\cos\theta^*) \\
p_1^{\mu} p_{3\mu} &= E_1^* E_3^* - |\vec{p}_1^*||\vec{p}_3^*|\cos\theta^* \\
t &= m_1^2 + m_3^3 - E_1^* E_3^* + 2|\vec{p}_1^*||\vec{p}_3^*|\cos\theta^*
\end{aligned}
$$

$\quad$ **giving** $\qquad \mathrm{d}t = 2|\vec{p}_1^*||\vec{p}_3^*|\mathrm{d}(\cos\theta^*)$

$\quad$ **therefore** $\quad \mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^* = \dfrac{\mathrm{d}t\,\mathrm{d}\phi^*}{2|\vec{p}_1^*||\vec{p}_3^*|}$

hence $\quad \mathrm{d}\sigma \quad = \quad \dfrac{1}{64\pi^2 s}\dfrac{|\vec{p}_3^*|}{|\vec{p}_1^*|}|M_{fi}|^2 \mathrm{d}\Omega^* = \dfrac{1}{2\cdot 64\pi^2 s|\vec{p}_1^*|^2}|M_{fi}|^2 \mathrm{d}\phi^* \mathrm{d}t$

- **Finally, integrating over** $\mathrm{d}\phi^*$ **(assuming no** ϕ^* **dependence of** $|M_{fi}|^2$ **) gives:**

$$
\boxed{\dfrac{\mathrm{d}\sigma}{\mathrm{d}t} = \dfrac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2}
$$

Lorentz invariant differential cross-section

- All quantities in the expression for $d\sigma/dt$ are Lorentz Invariant and therefore, it applies to **any rest frame**. It should be noted that $|\vec{p}_i^*|^2$ is a constant, fixed by energy/momentum conservation

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]$$

- As an example of how to use the invariant expression $d\sigma/dt$ we will consider elastic scattering in the laboratory frame in the limit where we can neglect the mass of the incoming particle $E_1 \gg m_1$

E_1 → m_2 ●   **e.g. electron or neutrino scattering**

In this limit $|\vec{p}_i^*|^2 = \dfrac{(s - m_2)^2}{4s}$

$$\boxed{\frac{d\sigma}{dt} = \frac{1}{16\pi(s - m_2^2)^2}|M_{fi}|^2}$$ $(m_1 = 0)$

2->2 body scattering in the Lab Frame

- **The other commonly occurring case is scattering from a fixed target in the Laboratory Frame (e.g. electron-proton scattering)**

- **First take the case of elastic scattering at high energy where the mass of the incoming particles can be neglected:** $m_1 = m_3 = 0, \quad m_2 = m_4 = M$
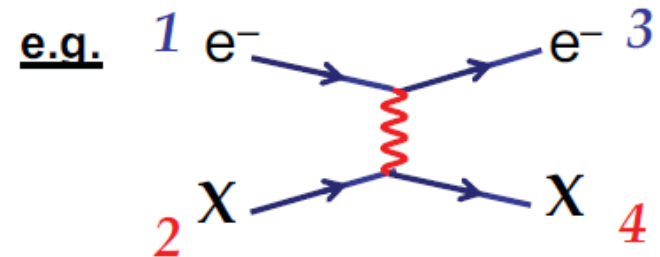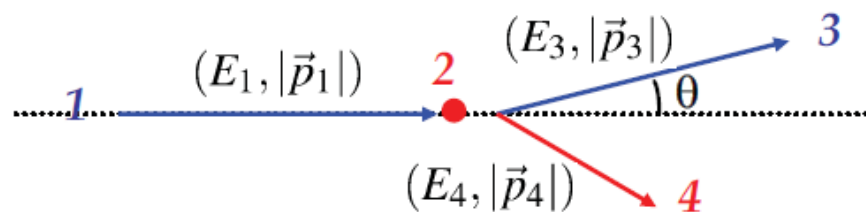
- **Wish to express the cross section in terms of scattering angle of the e⁻**

$$d\Omega = 2\pi d(\cos\theta)$$

therefore $\quad \dfrac{d\sigma}{d\Omega} = \dfrac{d\sigma}{dt}\dfrac{dt}{d\Omega} = \dfrac{1}{2\pi}\dfrac{dt}{d(\cos\theta)}\dfrac{d\sigma}{dt}$

Integrating over $d\phi$

- **The rest is some rather tedious algebra…. start from four-momenta**

$$p_1 = (E_1, 0, 0, E_1), \; p_2 = (M, 0, 0, 0), \; p_3 = (E_3, E_3\sin\theta, 0, E_3\cos\theta), \; p_4 = (E_4, \vec{p}_4)$$

so here $\quad t = (p_1 - p_3)^2 = -2p_1\cdot p_3 = -2E_1E_3(1-\cos\theta)$

But from (E,p) conservation $\quad p_1 + p_2 = p_3 + p_4$

and, therefore, can also express t **in terms of particles 2 and 4**

$$t = (p_2 - p_4)^2 = 2M^2 - 2p_2 \cdot p_4 = 2M^2 - 2ME_4$$
$$= 2M^2 - 2M(E_1 + M - E_3) = -2M(E_1 - E_3)$$

Note E_1 is a constant (the energy of the incoming particle) so

$$\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} = 2M\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}$$

• **Equating the two expressions for t gives**
$$\boxed{E_3 = \frac{E_1 M}{M + E_1 - E_1\cos\theta}}$$

so
$$\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = \frac{E_1 M}{(M + E_1 - E_1\cos\theta)^2} = E_1^2 M\left(\frac{E_3}{E_1 M}\right)^2 = \frac{E_3^2}{M}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{2\pi}2M\frac{E_3^2}{M}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi}\frac{1}{16\pi(s - M^2)^2}|M_{fi}|^2$$

using $\quad s = (p_1 + p_2)^2 = M^2 + 2p_1 \cdot p_2 = M^2 + 2ME_1$

Particle 1 massless
$$\Rightarrow (p_1^2 = 0)$$

gives $\quad (s - M^2) = 2ME_1$

$$\Rightarrow \boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2|M_{fi}|^2}$$

In limit $m_1 \rightarrow 0$

In this equation, E_3 is a function of θ:

$$E_3 = \frac{E_1 M}{M + E_1 - E_1 \cos\theta}$$

giving
$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1 \cos\theta}\right)^2 |M_{fi}|^2 \qquad (m_1 = 0)$$

General form for 2→2 Body Scattering in Lab. Frame

★ The calculation of the differential cross section for the case where m_1 can not be neglected is longer and contains no more "physics" (see appendix II). It gives:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{p_1 m_1} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \cdot |M_{fi}|^2$$

Again there is **only one independent variable**, θ, which can be seen from conservation of energy

$$E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \underbrace{\sqrt{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta + m_4^2}}_{\vec{p}_4 = \vec{p}_1 - \vec{p}_3}$$

i.e. $|\vec{p}_3|$ is a function of θ

Summary

★ **Used a Lorentz invariant formulation of Fermi's Golden Rule to derive decay rates and cross-sections in terms of the Lorentz Invariant Matrix Element** (wave-functions normalised to 2E/Volume)

Main Results:

★ **Particle decay:**

$$\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 d\Omega$$

Where p^* is a function of particle masses

$$p^* = \frac{1}{2m_i} \sqrt{[(m_i^2 - (m_1+m_2)^2][m_i^2 - (m_1-m_2)^2]}$$

★ **Scattering cross section in C.o.M. frame:**

$$\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 d\Omega^*$$

★ **Invariant differential cross section (valid in all frames):**

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s |\vec{p}_i^*|^2} |M_{fi}|^2$$

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s - (m_1+m_2)^2][s - (m_1-m_2)^2]$$

Summary cont.

★ **Differential cross section in the lab. frame ($m_1=0$)**

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2 \quad \Longleftrightarrow \quad \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M+E_1-E_1\cos\theta}\right)^2 |M_{fi}|^2$$

★ **Differential cross section in the lab. frame ($m_1\neq 0$)**

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\cdot\frac{1}{|\vec{p}_1|m_1}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1+m_2)-E_3|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2$$

with $\quad E_1+m_2 = \sqrt{|\vec{p}_3|^2+m_3^2}+\sqrt{|\vec{p}_1|^2+|\vec{p}_3|^2-2|\vec{p}_1||\vec{p}_3|\cos\theta+m_4^2}$

Summary of the summary:

★ **Have now dealt with kinematics of particle decays and cross sections**
★ **The fundamental particle physics is in the matrix element**
★ **The above equations are the basis for all calculations that follow**

Appendix I: Lorentz Invariant Flux

Collinear collision:

a ———————→   ←——————— b

$$v_a, \vec{p}_a \qquad\qquad v_b, \vec{p}_b$$

$$
\begin{aligned}
F = 2E_a 2E_b (v_a + v_b) \ &= \ 4E_a E_b \left(\frac{|\vec{p}_a|}{E_a} + \frac{|\vec{p}_b|}{E_b} \right) \\
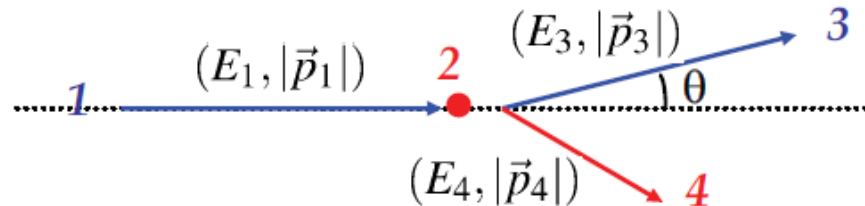&= \ 4(|\vec{p}_a| E_b + |\vec{p}_b| E_a)
\end{aligned}
$$

To show this is Lorentz invariant, first consider

$$p_a \cdot p_b = p_a^\mu p_{b\mu} = E_a E_b - \vec{p}_a \cdot \vec{p}_b = E_a E_b + |\vec{p}_a||\vec{p}_b|$$

Giving

$$
\begin{aligned}
F^2/16 - (p_a^\mu p_{b\mu})^2 \ &= \ (|\vec{p}_a| E_b + |\vec{p}_b| E_a)^2 - (E_a E_b + |\vec{p}_a||\vec{p}_b|)^2 \\
&= \ |\vec{p}_a|^2 (E_b^2 - |\vec{p}_b|^2) + E_a^2 (|\vec{p}_b|^2 - E_b^2) \\
&= \ |\vec{p}_a|^2 m_b^2 - E_a^2 m_b^2 \\
&= \ -m_a^2 m_b^2 \\
F \ &= \ 4 \left[(p_a^\mu p_{b\mu})^2 - m_a^2 m_b^2 \right]^{1/2}
\end{aligned}
$$

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$$p_1 = (E_1, 0, 0, |\vec{p}_1|), \ p_2 = (M, 0, 0, 0), \ p_3 = (E_3, E_3 \sin\theta, 0, E_3 \cos\theta), \ p_4 = (E_4, \vec{p}_4)$$

again

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\frac{dt}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt}$$

But now the invariant quantity t:

$$
\begin{aligned}
t &= (p_2 - p_4)^2 = m_2^2 + m_4^2 - 2p_2 \cdot p_4 = m_2^2 + m_4^2 - 2m_2 E_4 \\
&= m_2^2 + m_4^2 - 2m_2(E_1 + m_2 - E_3)
\end{aligned}
$$

$$\Longrightarrow \quad \frac{dt}{d(\cos\theta)} = 2m_2 \frac{dE_3}{d(\cos\theta)}$$

Which gives
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m_2}{\pi} \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} \frac{\mathrm{d}\sigma}{\mathrm{d}t}$$

To determine $\mathrm{d}E_3/\mathrm{d}(\cos\theta)$, first differentiate $E_3^2 - |\vec{p}_3|^2 = m_3^2$

$$2E_3 \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = 2|\vec{p}_3| \frac{\mathrm{d}|\vec{p}_3|}{\mathrm{d}(\cos\theta)} \qquad \text{(AII.1)}$$

Then equate $\quad t = (p_1 - p_3)^2 = (p_4 - p_2)^2 \quad$ **to give**

$$m_1^2 + m_3^2 - 2(E_1 E_3 - |\vec{p}_1||\vec{p}_3|\cos\theta) = m_4^2 + m_2^2 - 2m_2(E_1 + m_2 - E_3)$$

Differentiate wrt. $\cos\theta$

$$(E_1 + m_2)\frac{\mathrm{d}E_3}{\mathrm{d}\cos\theta} - |\vec{p}_1|\cos\theta \frac{\mathrm{d}|\vec{p}_3|}{\mathrm{d}\cos\theta} = |\vec{p}_1||\vec{p}_3|$$

Using (1) ➡ $\quad \dfrac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = \dfrac{|\vec{p}_1||\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \qquad$ (AII.2)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m_2}{\pi} \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{m_2}{\pi} \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} \frac{1}{64\pi s|\vec{p}_i^*|^2} |M_{fi}|^2$$

It is easy to show $|\vec{p}_i^*|\sqrt{s} = m_2|\vec{p}_1|$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} \frac{m_2}{64\pi^2 m_2^2 |\vec{p}_1|^2} |M_{fi}|^2$$

and using (AII.2) obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{p_1 m_1} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \cdot |M_{fi}|^2$$