

INTRODUCTION TO DATA SCIENCE

This lecture is
based on course by E. Fox and C. Guestrin, Univ of Washington

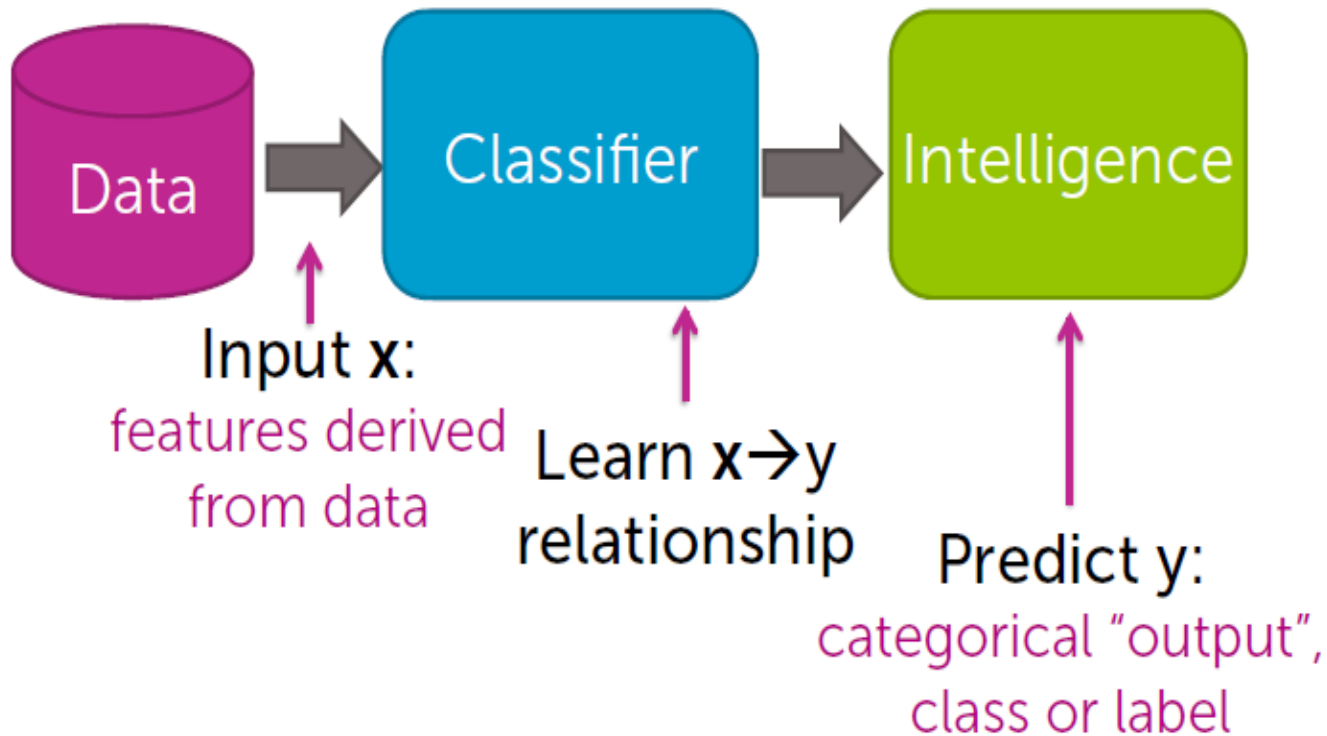
9/11, 16/11, 23/11
2022

WFAiS UJ, Informatyka Stosowana
I stopień studiów

What is a classification?

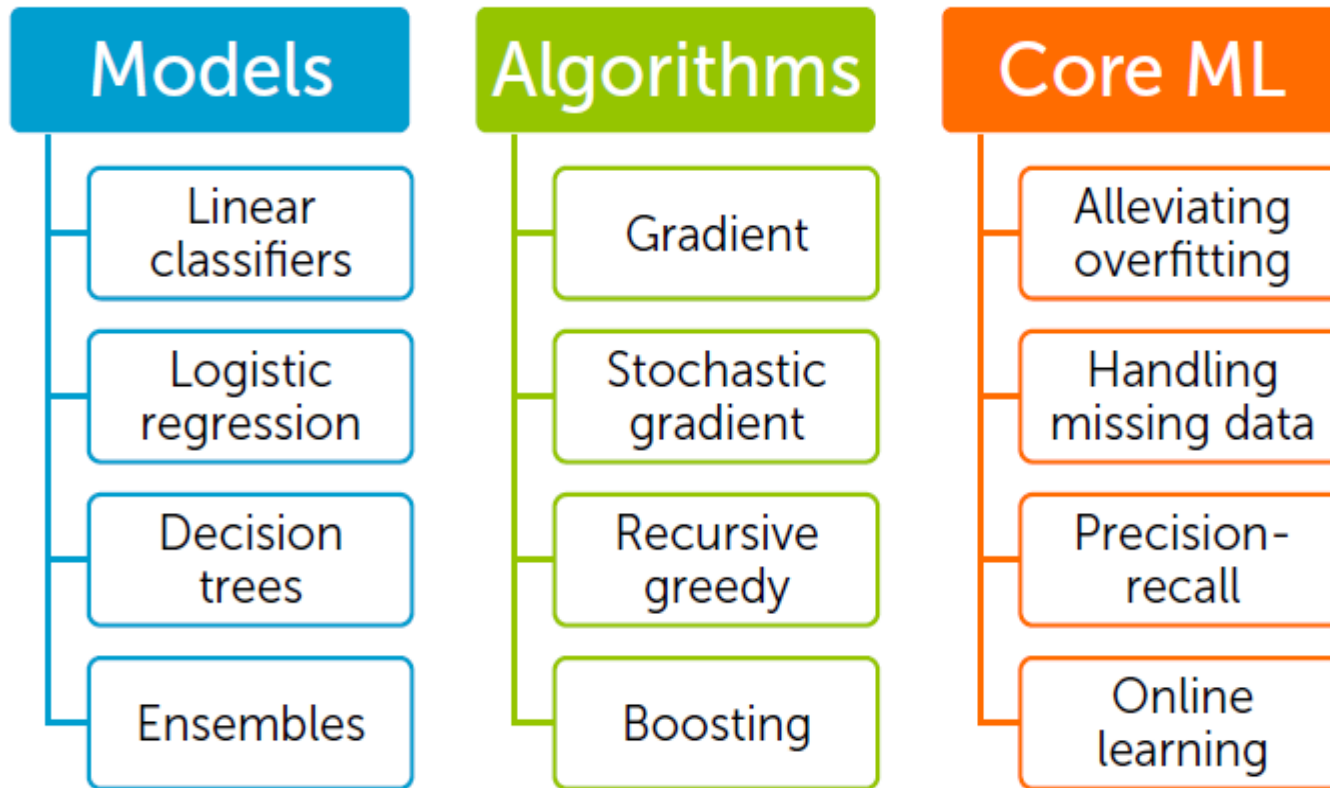
2

From features to predictions



Overview of the content

3



Linear classifier


An intelligent restaurant review system

5

It's a big day & I want to book a table at a nice Japanese restaurant



Seattle has many
★★★★
sushi restaurants



What are people
saying about
the food?
the ambiance?...



Reviews

6

★★★★☆ 428 reviews
\$\$ · Japanese, Sushi Bars



Positive reviews not positive about everything

Sample review:

Watching the chefs create incredible edible art made the experience very unique.

My wife tried their ramen and it was pretty forgettable.

All the sushi was delicious!
Easily best sushi in Seattle.



Classifying sentiment of review

7

Easily best sushi in Seattle.

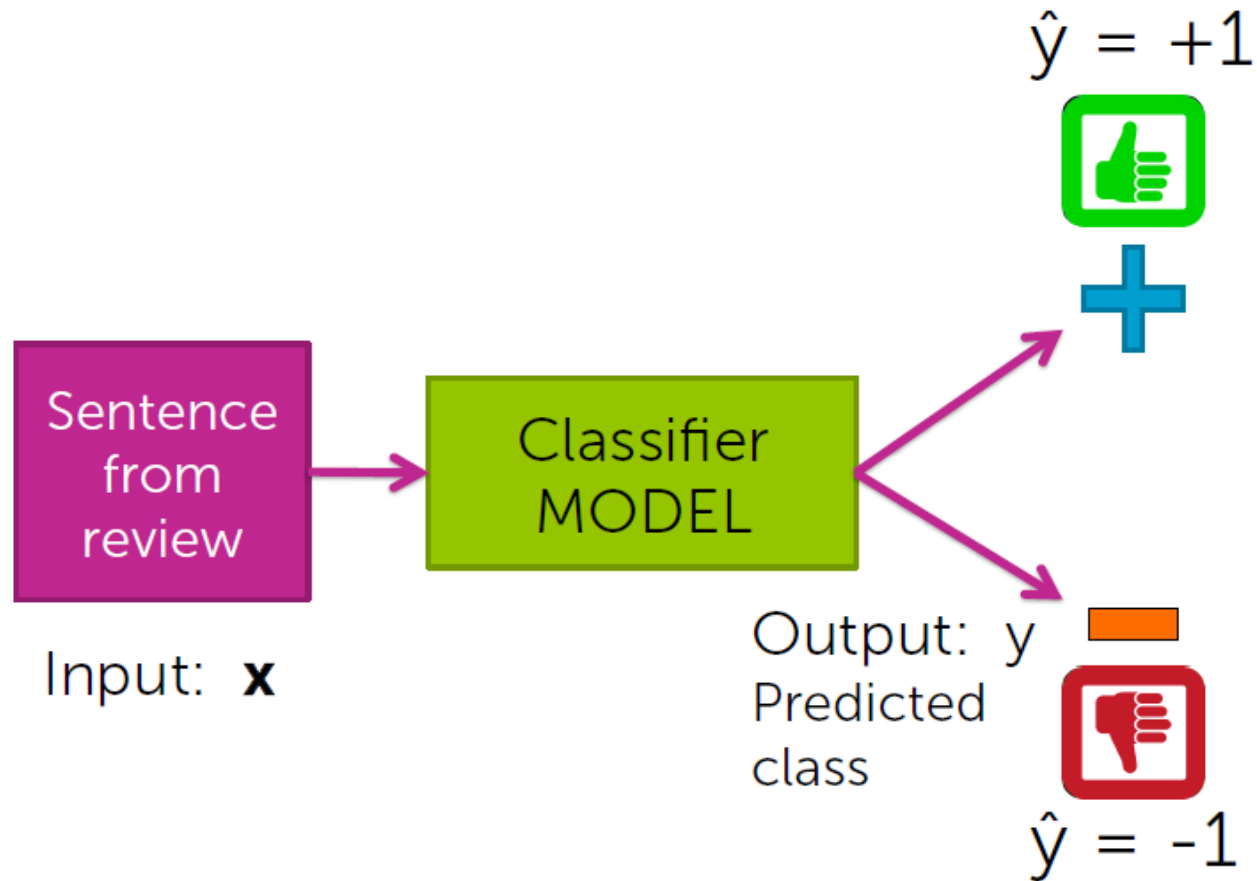


Sentence Sentiment
Classifier



Classifier

8



Note: we'll start talking about 2 classes, and address multiclass later

A (linear) classifier

9

Will use training data to learn a weight for each word

Word	Weight
good	1.0
great	1.5
awesome	2.7
bad	-1.0
terrible	-2.1
awful	-3.3
restaurant, the, we, where, ...	0.0
...	...

Scoring a sentence

10

Word	Coefficient
good	1.0
great	1.2
awesome	1.7
bad	-1.0
terrible	-2.1
awful	-3.3
restaurant, the, we, where, ...	0.0
...	...

Input \mathbf{x}_i :

Sushi was great,
the food was awesome,
but the service was terrible.

$$\begin{aligned} \text{Score}(x_i) &= 1.2 + 1.7 - 2.1 \\ &= 0.8 > 0 \\ \Rightarrow y &= +1 \\ &\text{positive review} \end{aligned}$$

Called a linear classifier, because output is weighted sum of input.

10

Simple linear classifier

11

Word	Coefficient
...	...



Simple linear classifier

$Score(\mathbf{x}) =$ weighted count of words in sentence

If $Score(\mathbf{x}) > 0$:

$$\hat{y} = +1$$

Else:

$$\hat{y} = -1$$

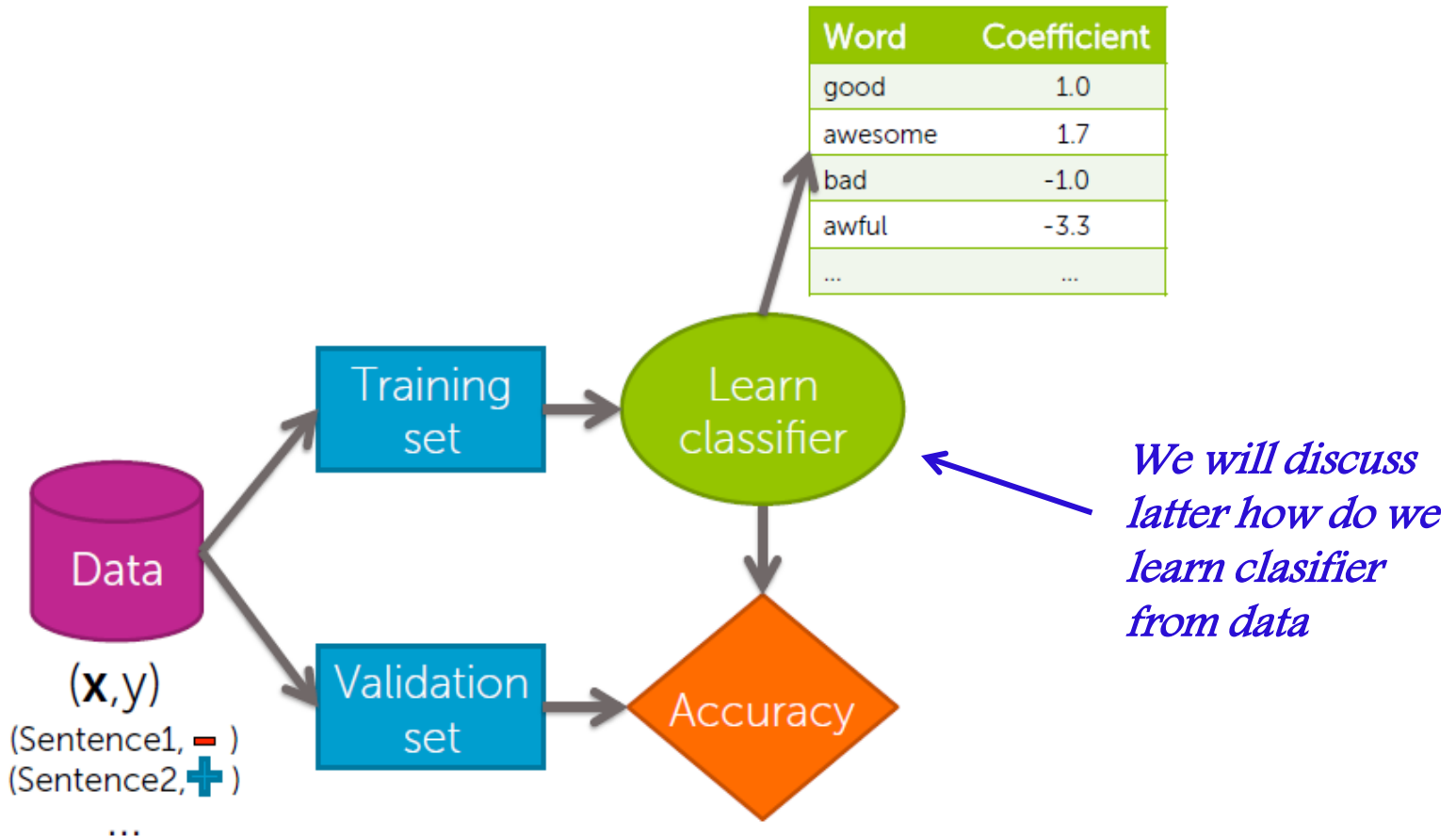
Sentence from review



Input: \mathbf{x}

Training a classifier = Learning the coefficients

12

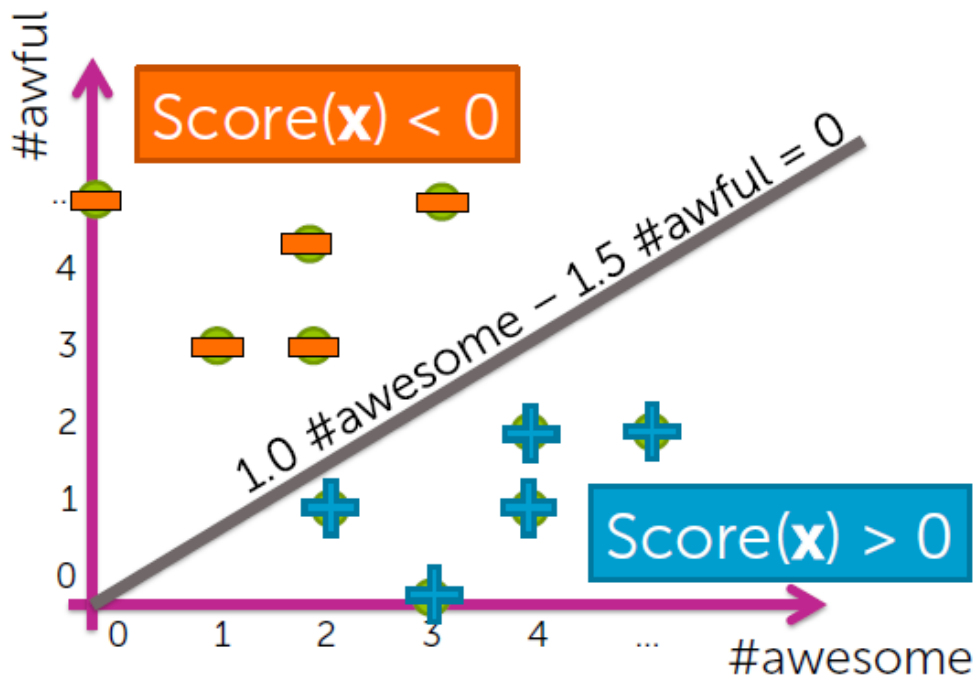


Decision boundary example

13

Word	Coefficient
#awesome	1.0
#awful	-1.5

→ $\text{Score}(x) = 1.0 \#awesome - 1.5 \#awful$

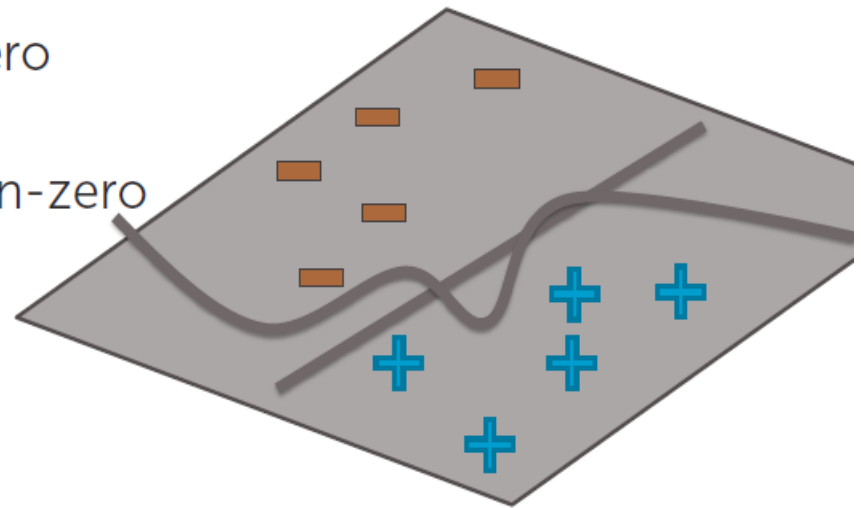


Decision boundary

14

Decision boundary separates positive & negative predictions

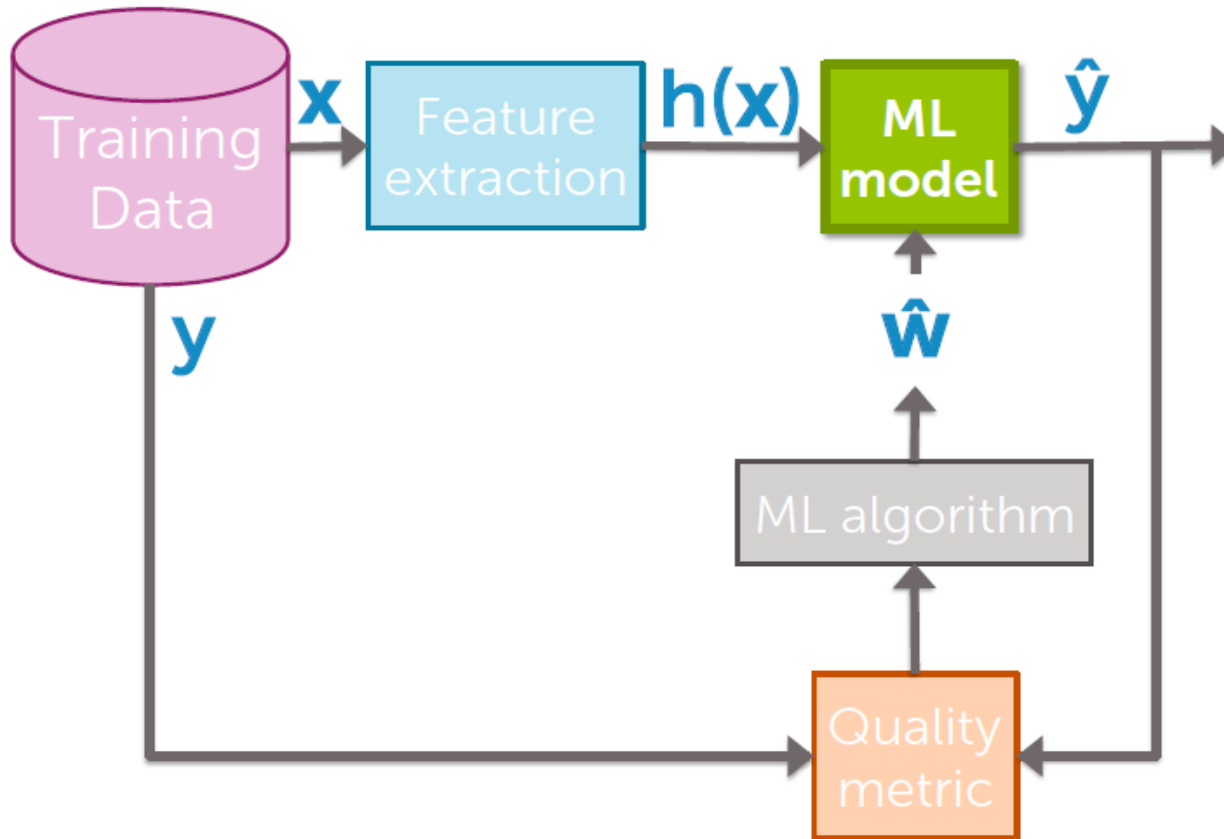
- For linear classifiers:
 - When 2 coefficients are non-zero
→ line
 - When 3 coefficients are non-zero
→ plane
 - When many coefficients are non-zero
→ hyperplane
- For more general classifiers
→ more complicated shapes



Flow chart:

ML
model

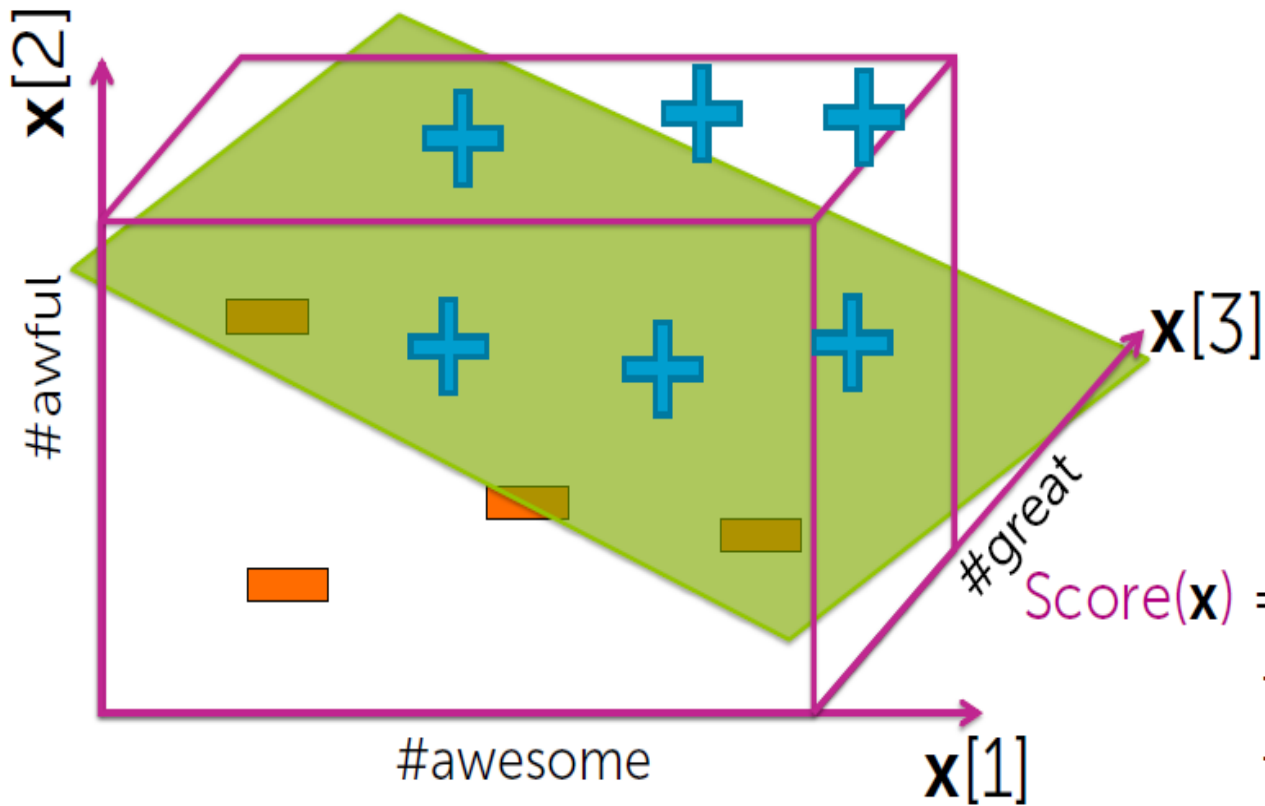
15



21

Coefficients of classifier

16



$$\begin{aligned} \text{Score}(\mathbf{x}) &= w_0 \\ &+ w_1 \#awesome \\ &+ w_2 \#awful \\ &+ w_3 \#great \end{aligned}$$

37

General notation

17

Output: y  $\{-1, +1\}$

Inputs: $\mathbf{x} = (\mathbf{x}[1], \mathbf{x}[2], \dots, \mathbf{x}[d])$

 d -dim vector

Notational conventions:

$\mathbf{x}[j]$ = j^{th} input (*scalar*)

$h_j(\mathbf{x})$ = j^{th} feature (*scalar*)

\mathbf{x}_i = input of i^{th} data point (*vector*)

$\mathbf{x}_i[j]$ = j^{th} input of i^{th} data point (*scalar*)

Simple hyperplane

18

Model: $\hat{y}_i = \text{sign}(\text{Score}(\mathbf{x}_i))$

$$\text{Score}(\mathbf{x}_i) = w_0 + w_1 \mathbf{x}_i[1] + \dots + w_d \mathbf{x}_i[d] = \mathbf{w}^T \mathbf{x}_i$$

feature 1 = 1

feature 2 = $\mathbf{x}[1]$... e.g., #awesome

feature 3 = $\mathbf{x}[2]$... e.g., #awful

...

feature $d+1 = \mathbf{x}[d]$... e.g., #ramen

D-dimensional hyperplane

19

More generic features...

Model: $\hat{y}_i = \text{sign}(\text{Score}(\mathbf{x}_i))$

$\text{Score}(\mathbf{x}_i) = w_0 h_0(\mathbf{x}_i) + w_1 h_1(\mathbf{x}_i) + \dots + w_D h_D(\mathbf{x}_i)$

$$= \sum_{j=0}^D w_j h_j(\mathbf{x}_i) = \mathbf{w}^T \mathbf{h}(\mathbf{x}_i)$$

feature 1 = $h_0(\mathbf{x})$... e.g., 1

feature 2 = $h_1(\mathbf{x})$... e.g., $x[1] = \#awesome$

feature 3 = $h_2(\mathbf{x})$... e.g., $x[2] = \#awful$

or, $\log(x[7]) x[2] = \log(\#bad) \times \#awful$

or, $\text{tf-idf}(\text{"awful"})$

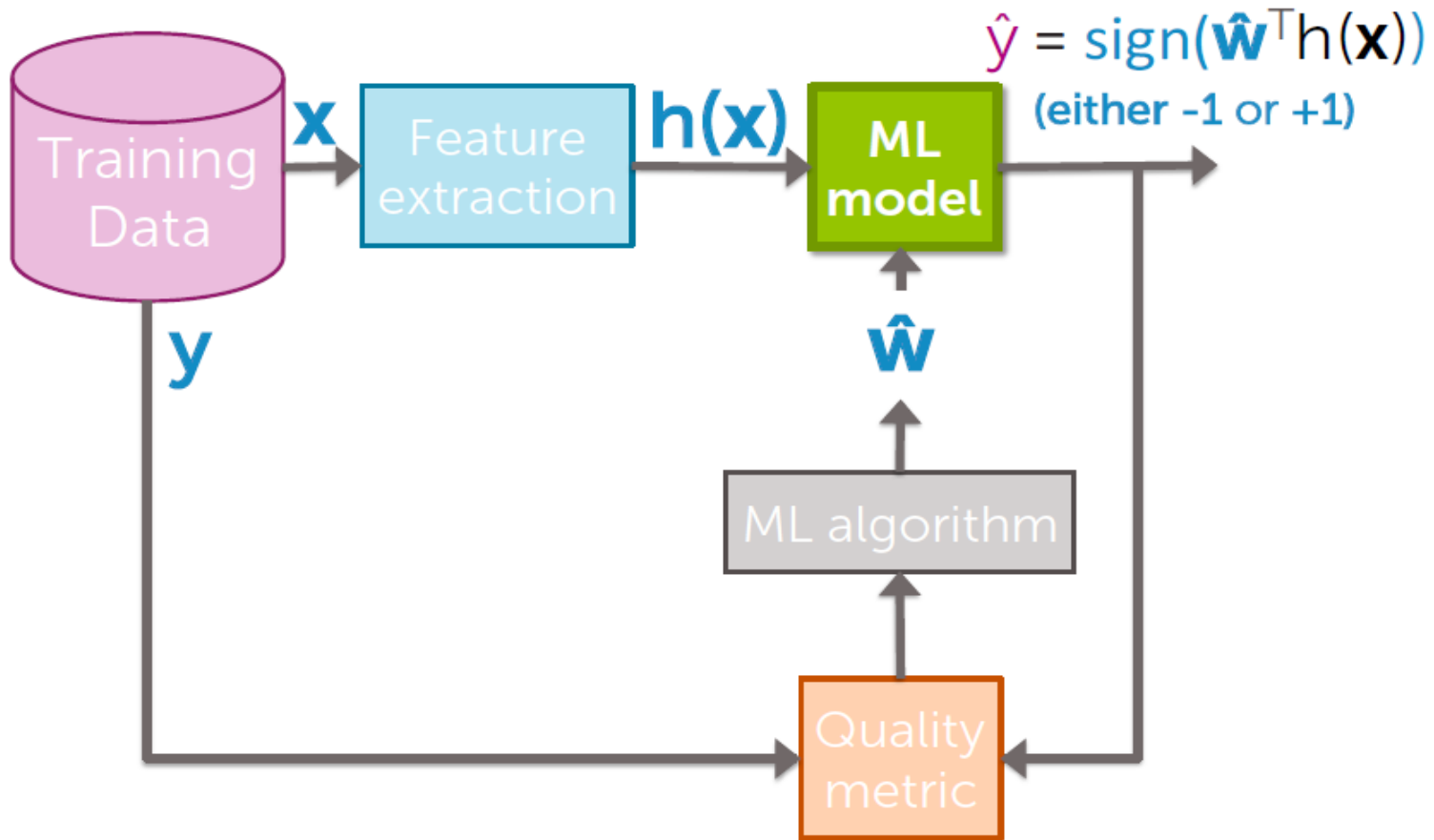
...

feature $D+1 = h_D(\mathbf{x})$... some other function of $x[1], \dots, x[d]$

Flow chart:



20



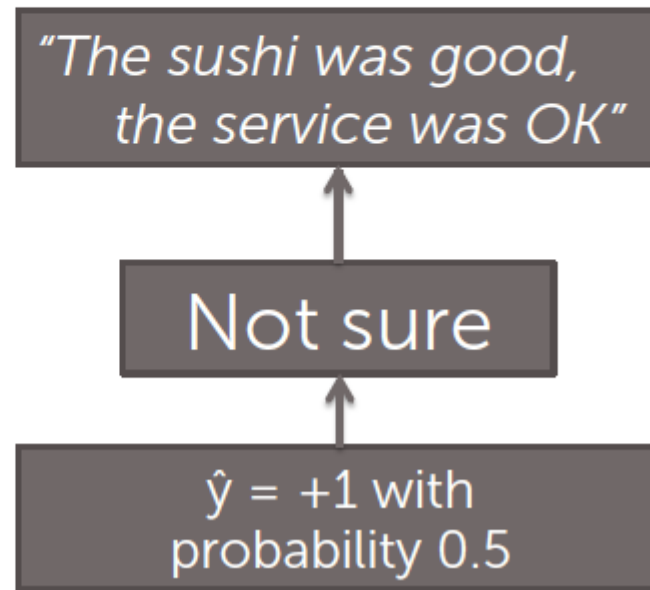
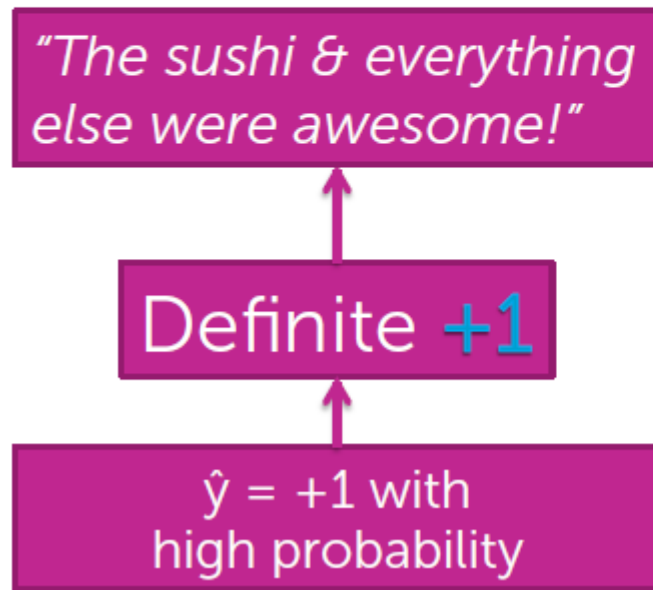
Linear classifier

- ▣ Class probability

How confident is your prediction?

22

- Thus far, we've outputted a prediction **+1** or **-1**
- But, how sure are you about the prediction?



Basics of probabilities

23

Probability a review is positive is 0.7

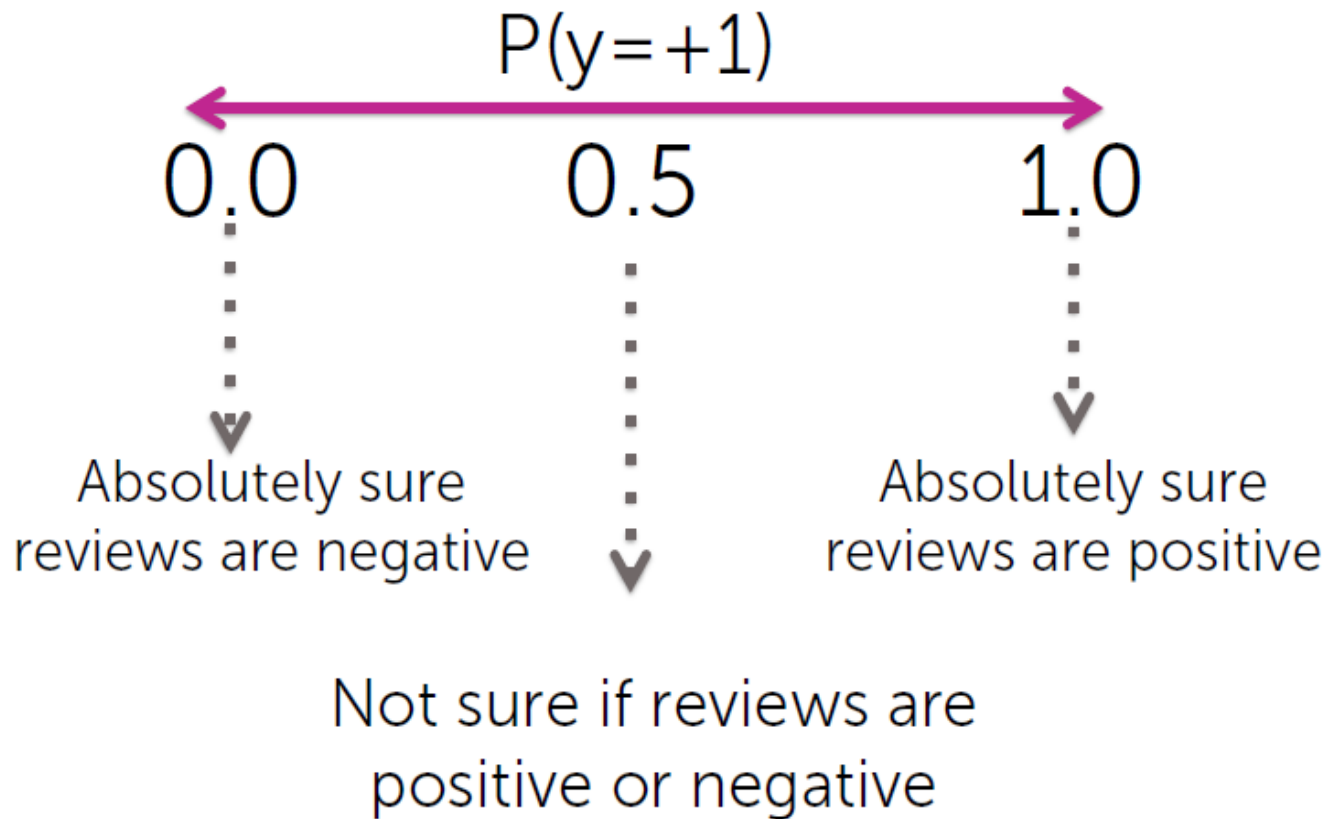


x = review text	y = sentiment
All the sushi was delicious! Easily best sushi in Seattle.	+1
The sushi & everything else were awesome!	+1
My wife tried their ramen, it was pretty forgettable.	-1
The sushi was good, the service was OK	+1
...	...

I expect 70% of rows to have $y = +1$
(Exact number will vary for each specific dataset)

Interpreting probabilities as degrees of belief

24



Conditional probability

25

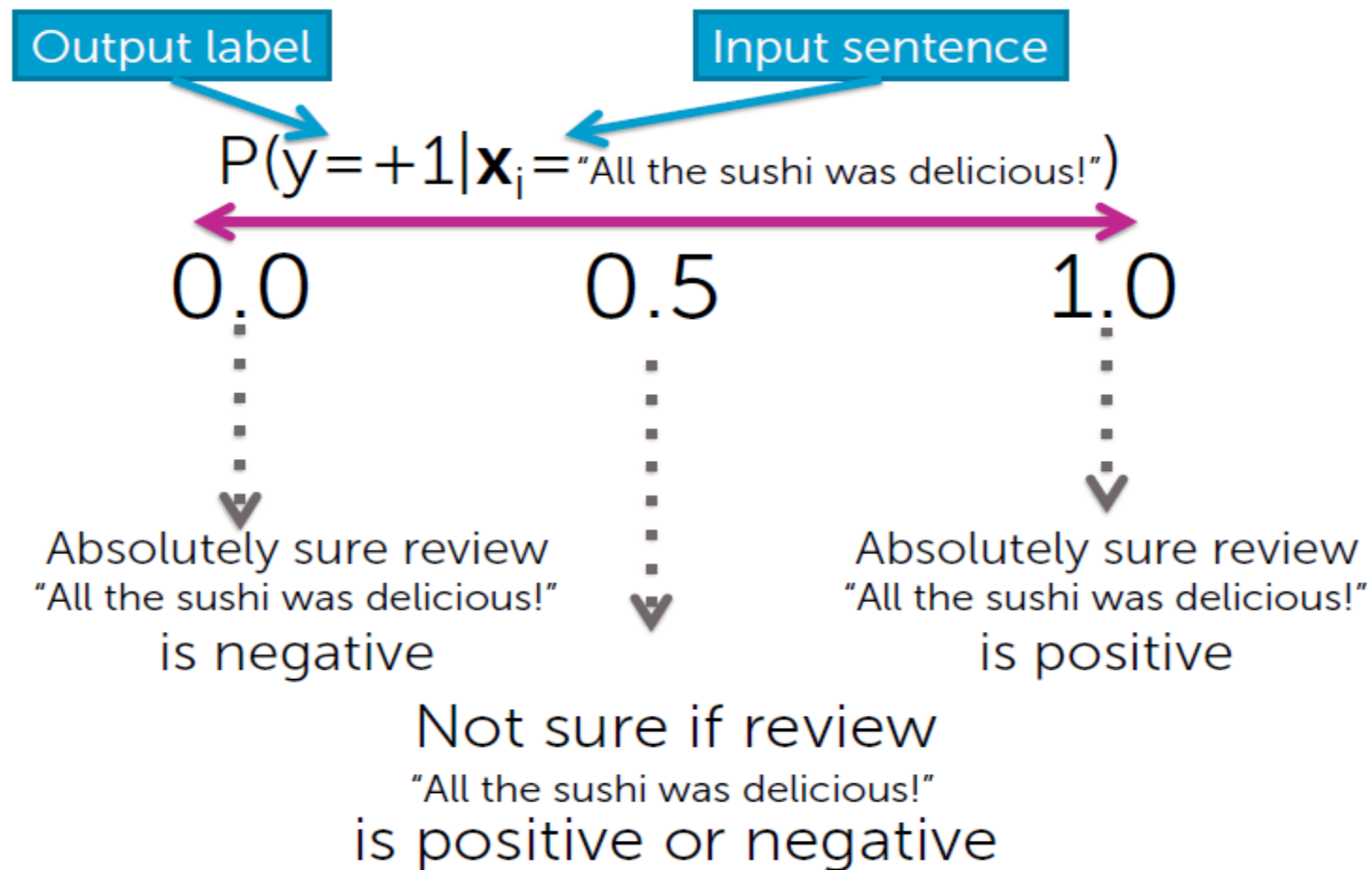
Probability a review with
3 "awesome" and 1 "awful" is positive is 0.9

x = review text	y = sentiment
All the sushi was delicious! Easily best sushi in Seattle.	+1
Sushi was awesome & everything else was awesome ! The service was awful , but overall awesome place!	+1
My wife tried their ramen, it was pretty forgettable.	-1
The sushi was good, the service was OK	+1
...	...
awesome ... awesome ... awful ... awesome	+1
...	...
awesome ... awesome ... awful ... awesome	-1
...	...
...	...
awesome ... awesome ... awful ... awesome	+1

I expect 90% of rows with reviews containing 3 "awesome" & 1 "awful" to have $y = +1$ (Exact number will vary for each specific dataset)

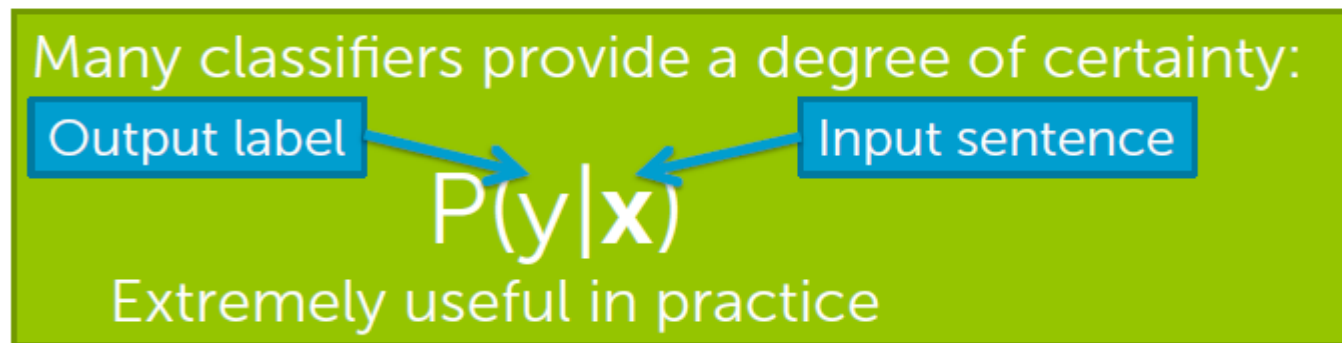
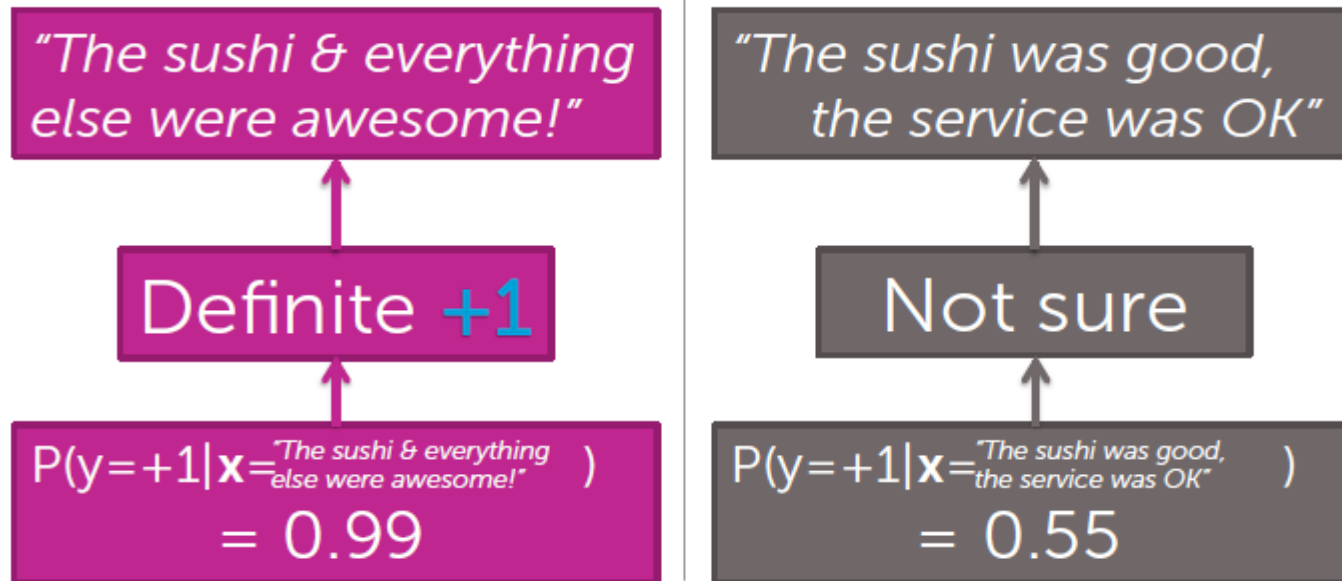
Interpreting conditional probabilities

26



How confident is your prediction?

27



Learn conditional probabilities from data

28

Training data: N observations (\mathbf{x}_i, y_i)

$x[1] = \text{\#awesome}$	$x[2] = \text{\#awful}$	$y = \text{sentiment}$
2	1	+1
0	2	-1
3	3	-1
4	1	+1
...

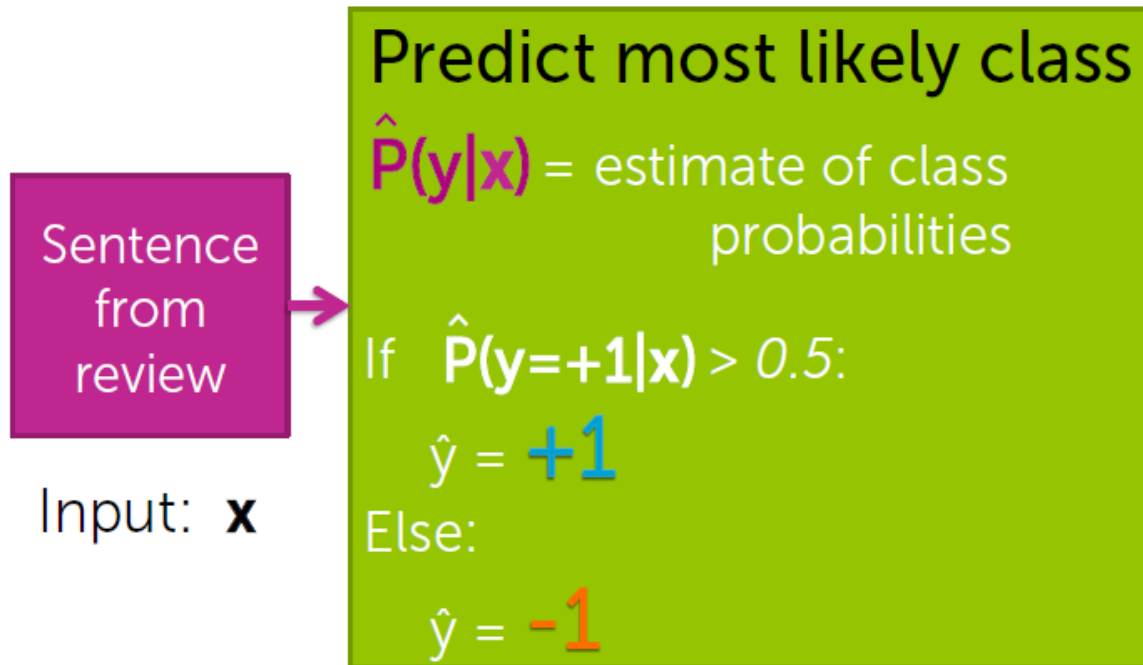
Optimize **quality metric**
on training data

Find best model \hat{P}
by finding best \hat{W}

Useful for
predicting \hat{y}

Predicting class probabilities

29

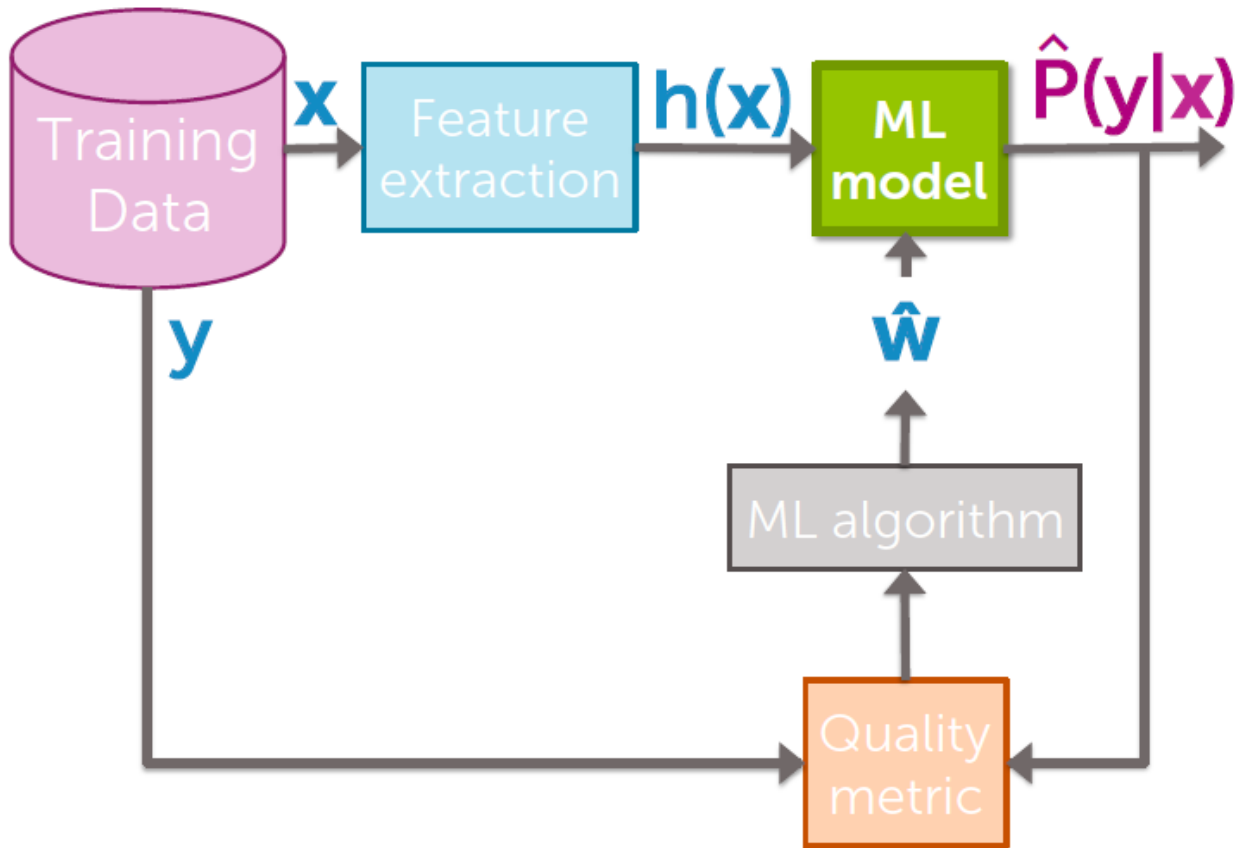


- Estimating $\hat{P}(y|\mathbf{x})$ improves **interpretability**:
 - Predict $\hat{y} = +1$ **and** tell me how sure you are

Flow chart:

ML
model

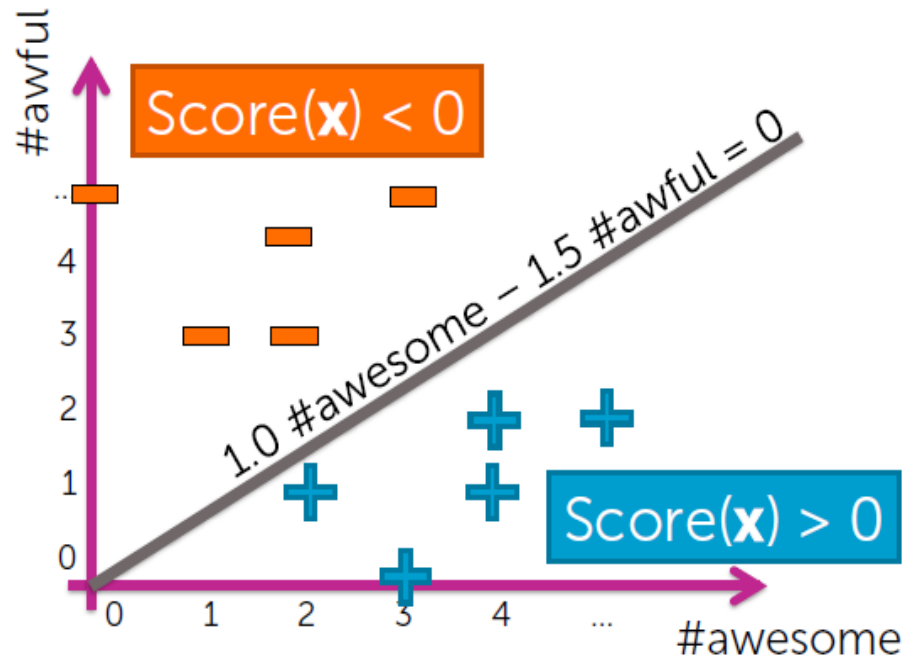
30



Thus far we focused on decision boundaries

31

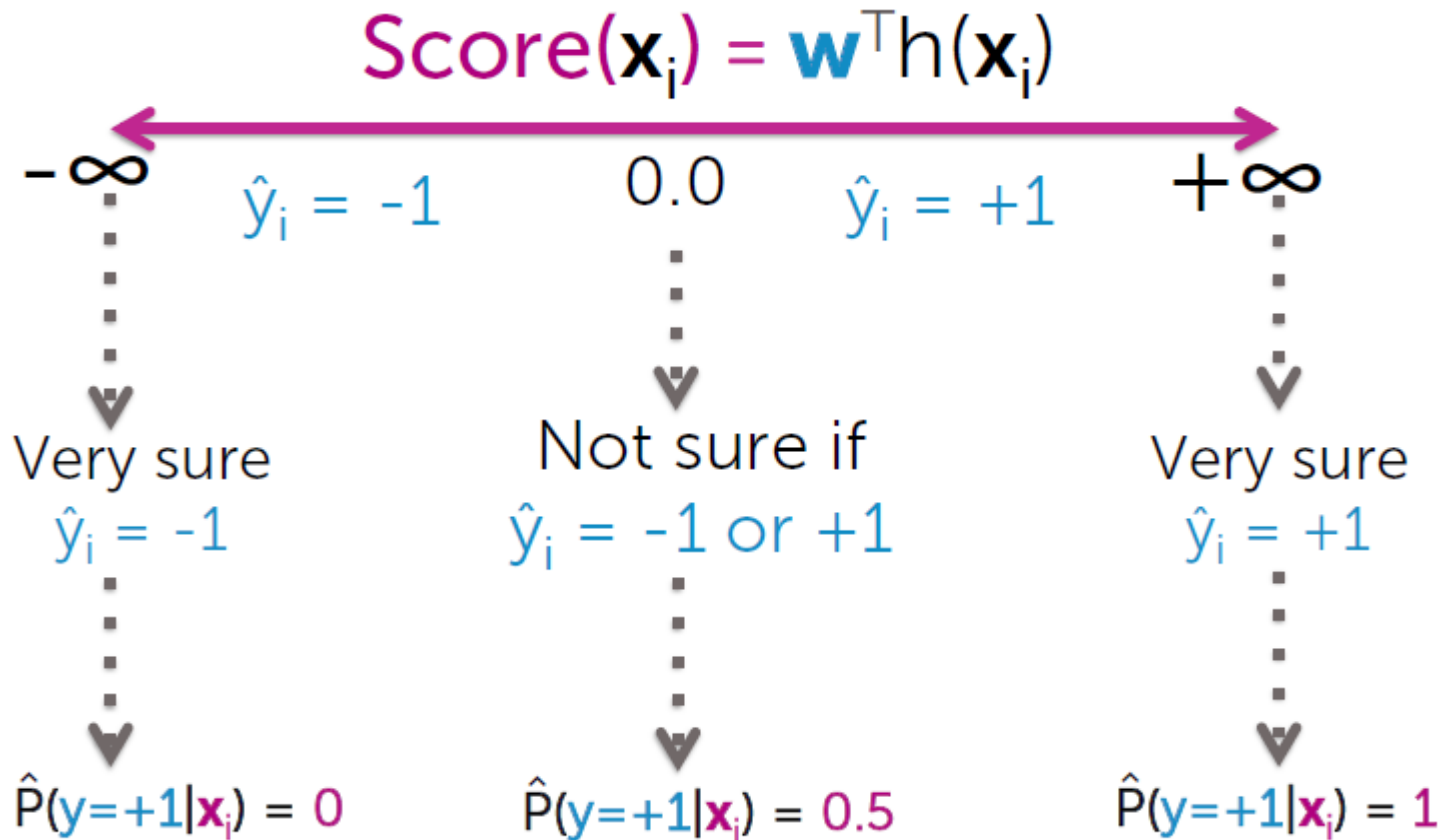
$$\begin{aligned}\text{Score}(\mathbf{x}_i) &= w_0 h_0(\mathbf{x}_i) + w_1 h_1(\mathbf{x}_i) + \dots + w_D h_D(\mathbf{x}_i) \\ &= \mathbf{w}^T \mathbf{h}(\mathbf{x}_i)\end{aligned}$$



How to relate
 $\text{Score}(\mathbf{x}_i)$ to
 $\hat{P}(y=+1|\mathbf{x}, \hat{\mathbf{w}})$?

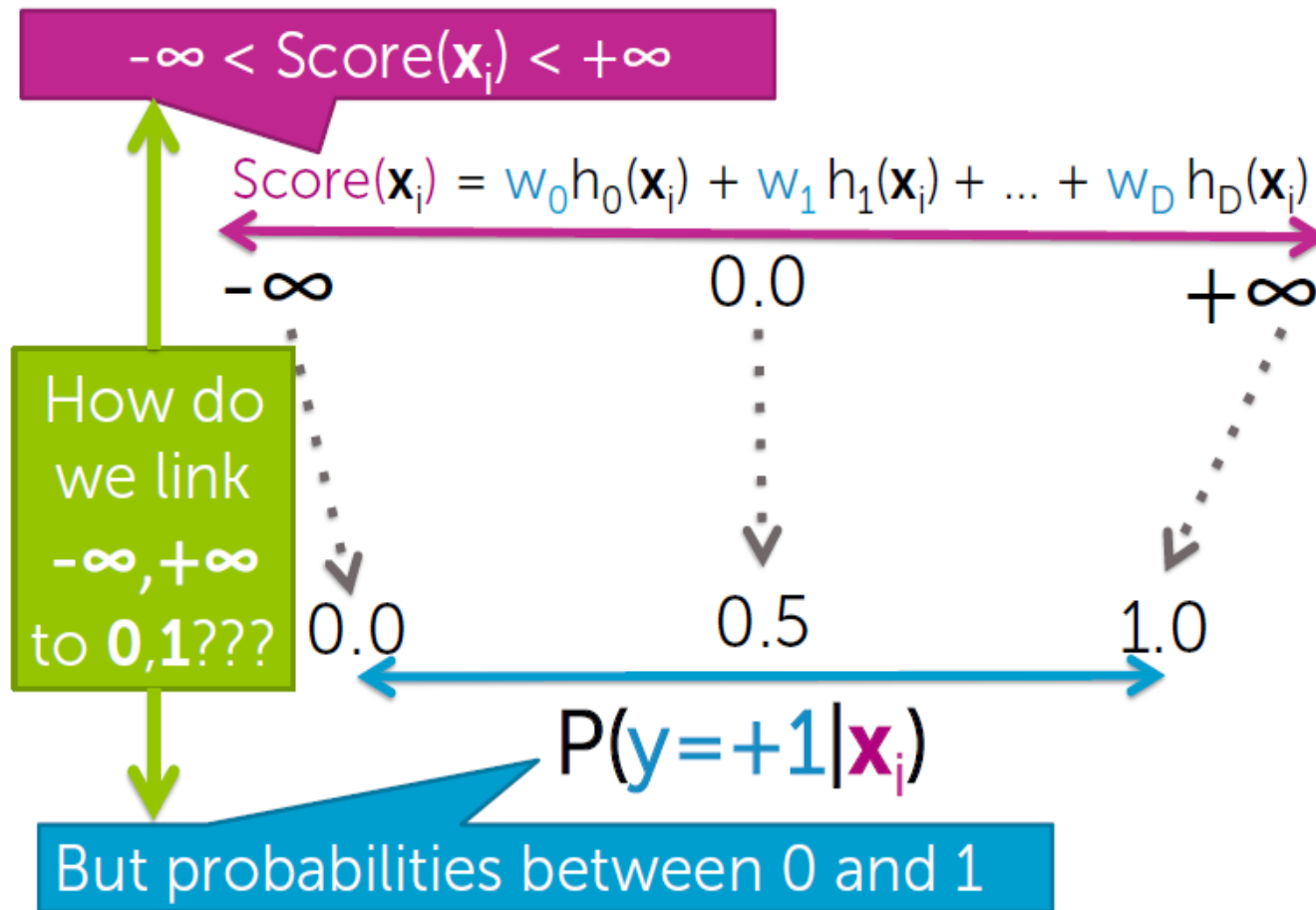
Interpreting Score(\mathbf{x}_i)

32



Why not just use regression to build classifier?

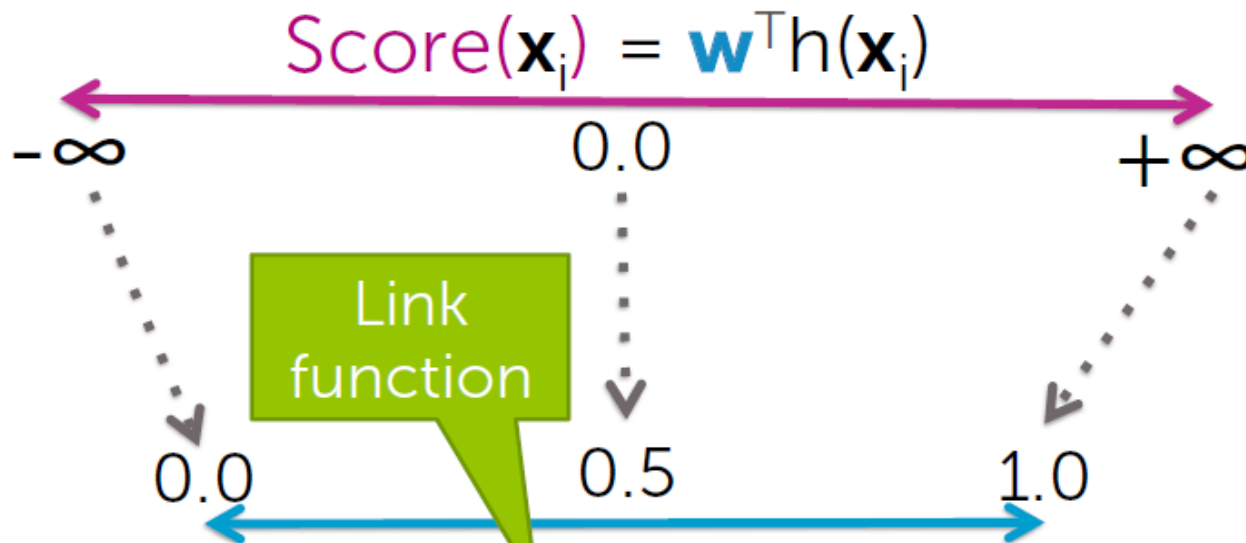
33



Link function

34

Link function: squeeze real line into [0,1]



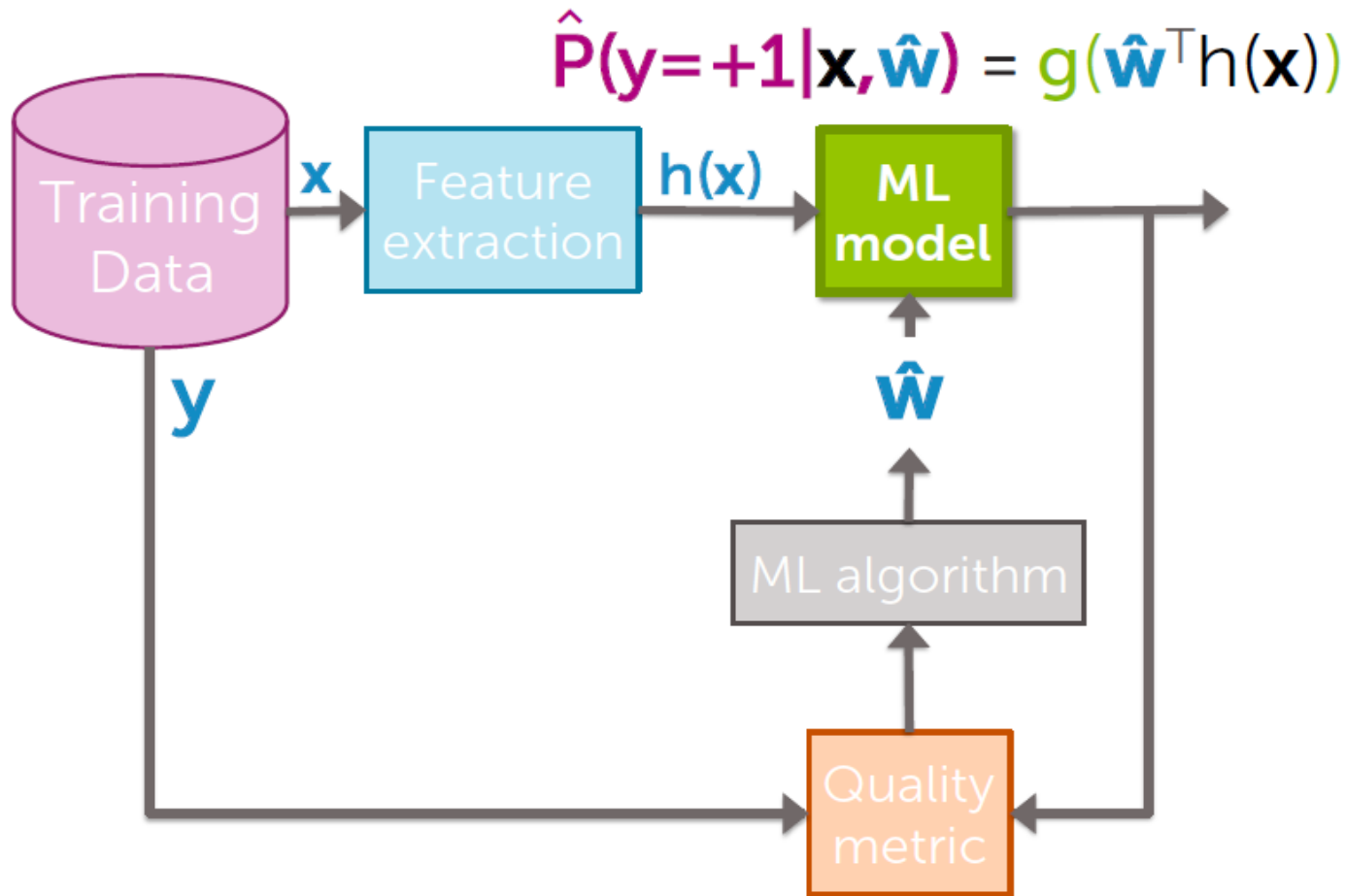
$$\hat{P}(y=+1|\mathbf{x}_i) = g(\mathbf{w}^T \mathbf{h}(\mathbf{x}_i))$$

Generalized linear model

Flow chart:

ML model

35

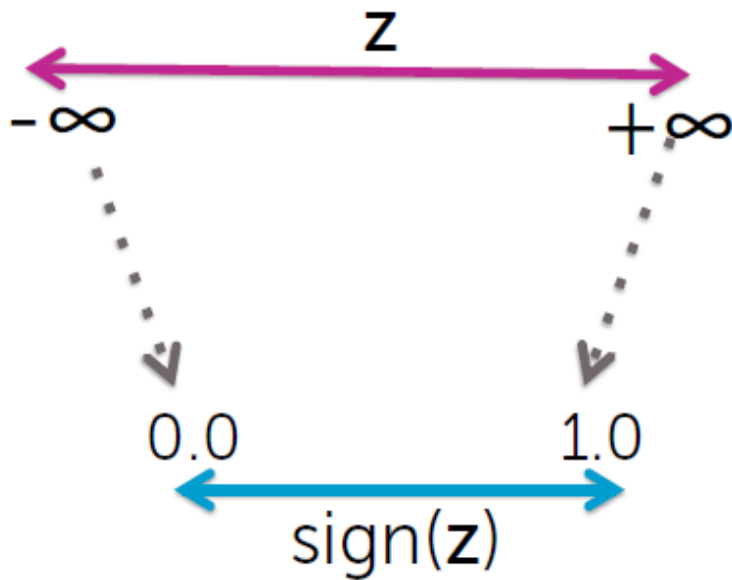


Logistic regression classifier:

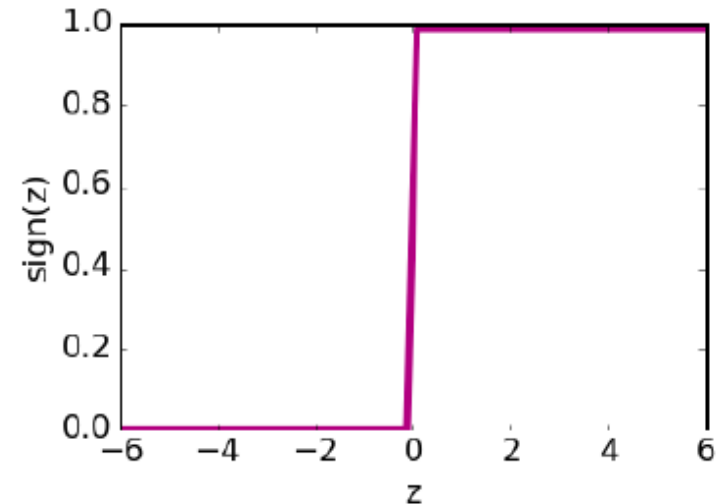
- ▣ linear score with logistic link function

Simplest link function: $\text{sign}(z)$

37



$$\text{sign}(z) = \begin{cases} +1 & \text{if } z \geq 0 \\ -1 & \text{otherwise} \end{cases}$$



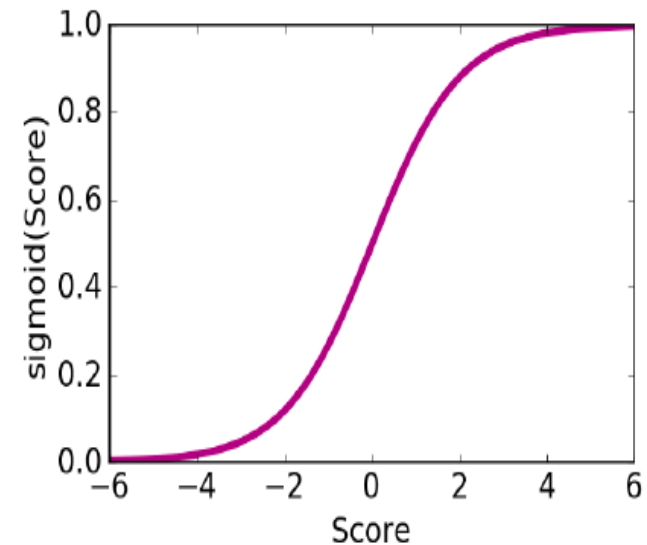
But, $\text{sign}(z)$ only outputs -1 or +1, no probabilities in between

Logistic function (sigmoid, logit)

38

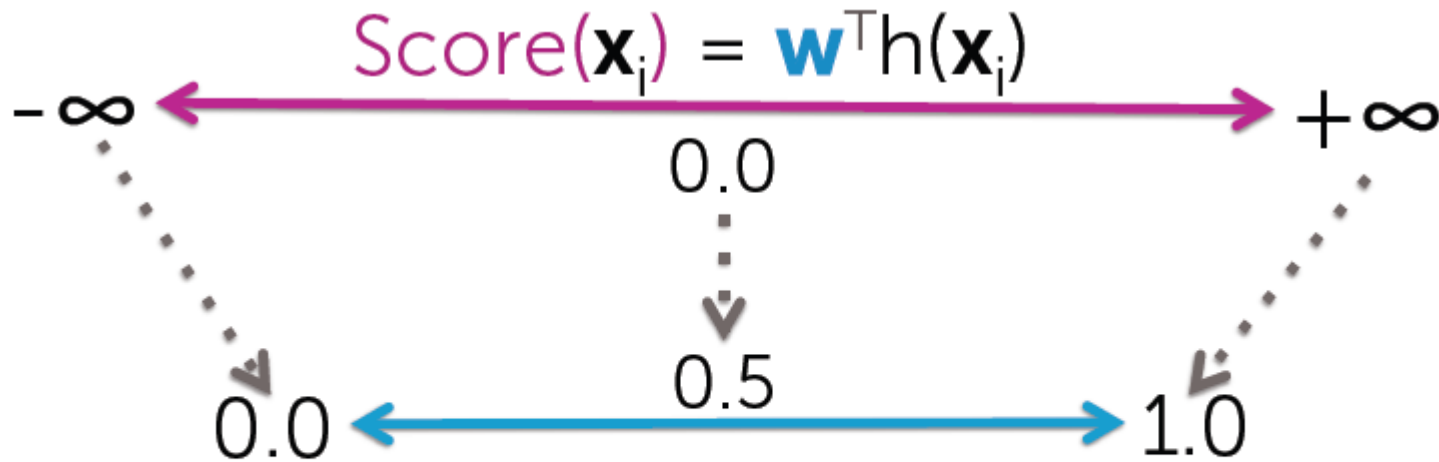
$$\text{sigmoid}(\text{Score}) = \frac{1}{1 + e^{-\text{Score}}}$$

Score	$-\infty$	-2	0.0	+2	$+\infty$
sigmoid(Score)	0.0	0.12	0.5	0.88	1.0



Logistic regression model

39

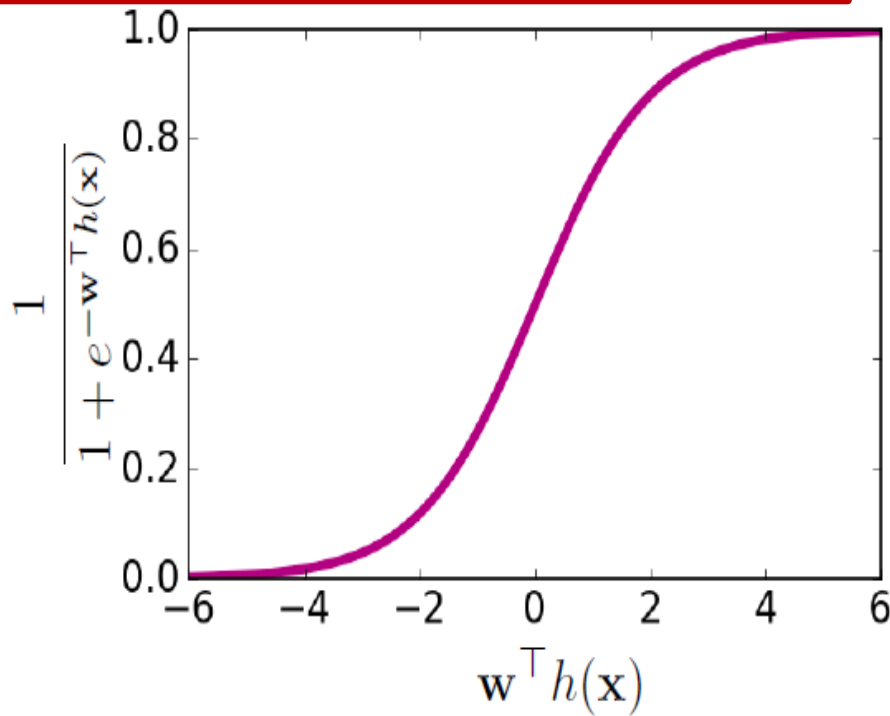


$$P(y=+1|\mathbf{x}_i, \mathbf{w}) = \text{sigmoid}(\text{Score}(\mathbf{x}_i))$$

Understanding the logistic regression model

40

$$P(y = +1 | \mathbf{x}, \mathbf{w}) = \frac{1}{1 + e^{-\mathbf{w}^\top \mathbf{h}(\mathbf{x})}}$$

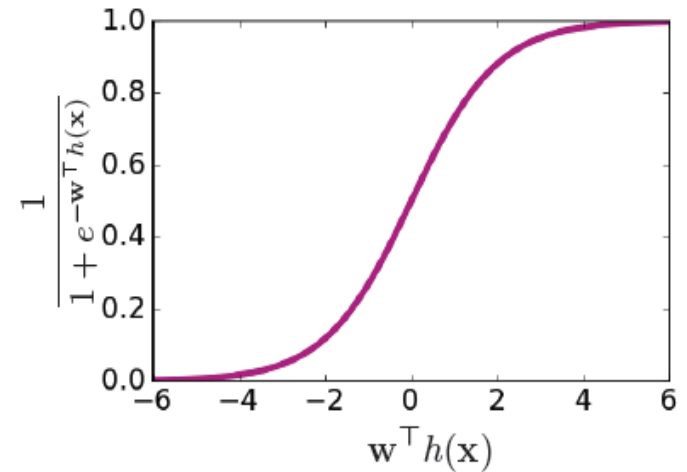
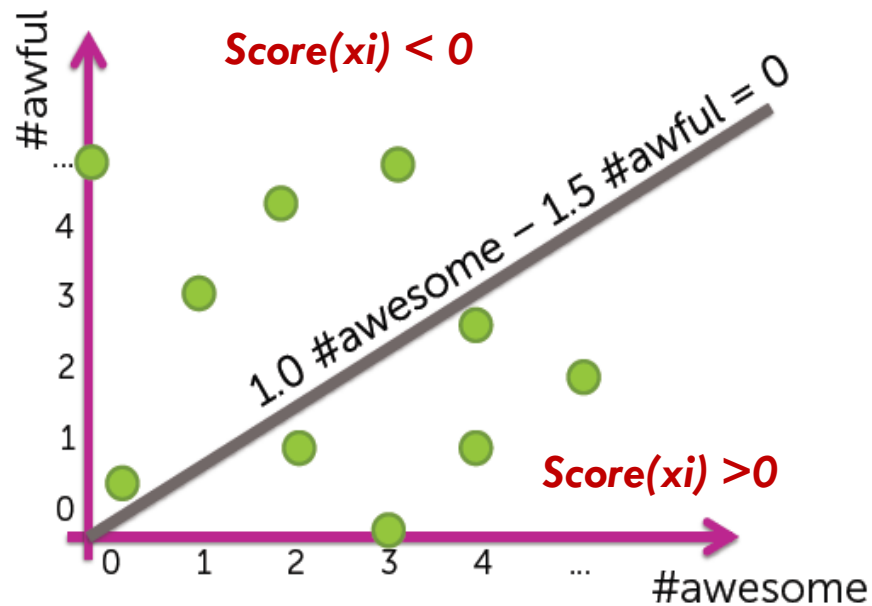


Score(x_i)	$P(y=+1 x_i, \mathbf{w})$
0	0.5
-2	0.12
2	0.88
4	0.98

Logistic regression

41

Logistic regression →
Linear decision boundary

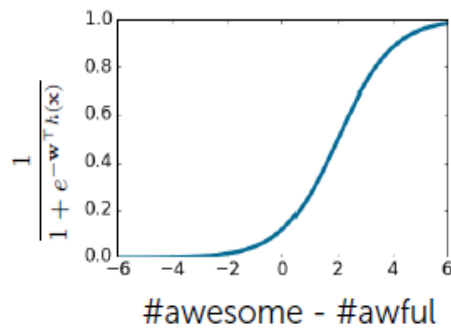


Effect of coefficients

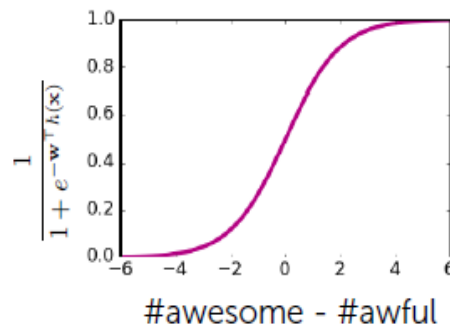
42

Effect of coefficients on logistic regression model

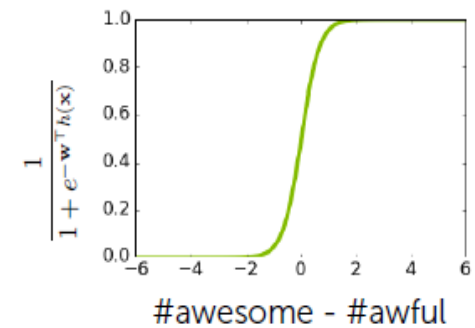
W_0	-2
$W_{\text{\#awesome}}$	+1
$W_{\text{\#awful}}$	-1



W_0	0
$W_{\text{\#awesome}}$	+1
$W_{\text{\#awful}}$	-1



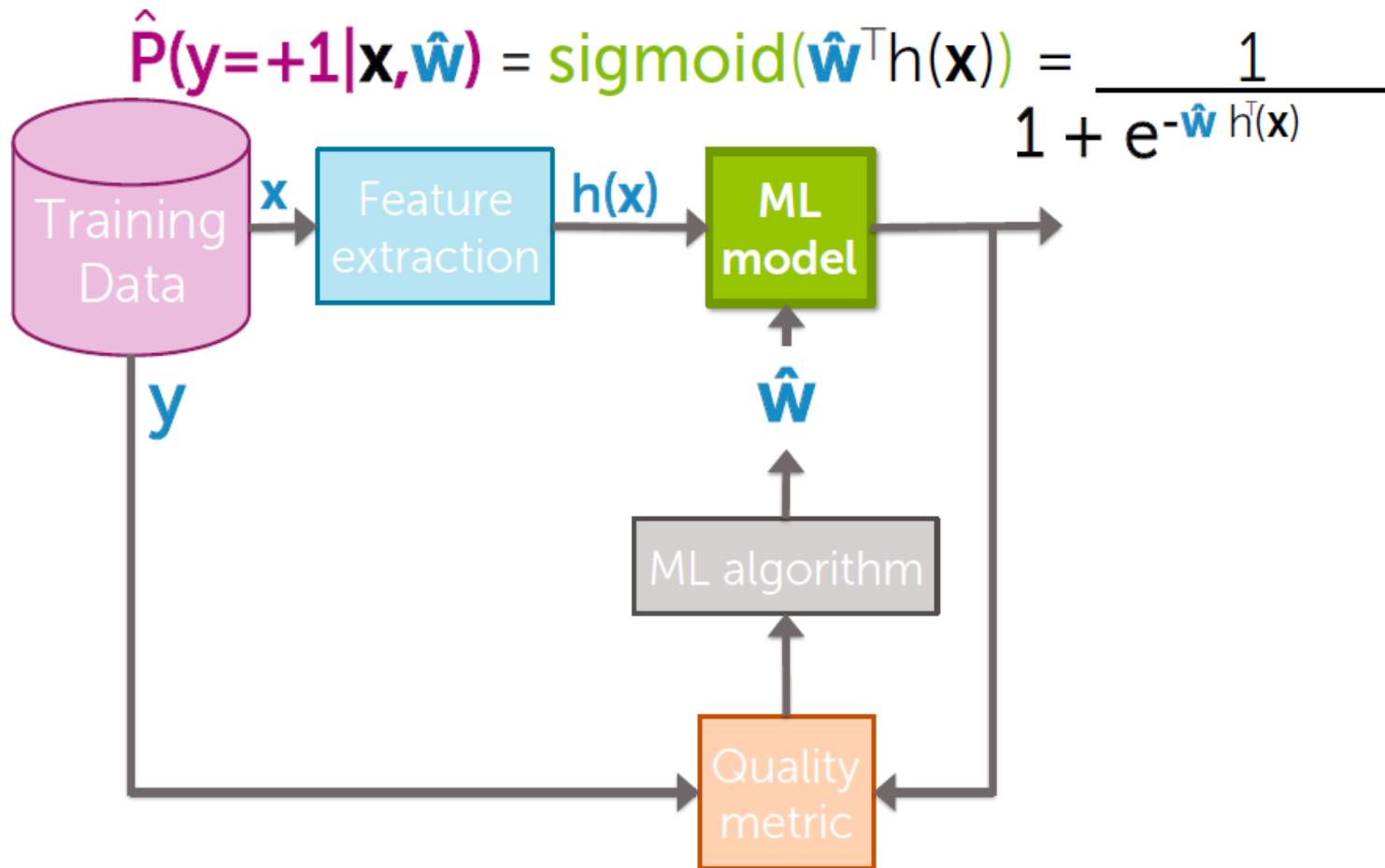
W_0	0
$W_{\text{\#awesome}}$	+3
$W_{\text{\#awful}}$	-3



Flow chart:

ML
model

43



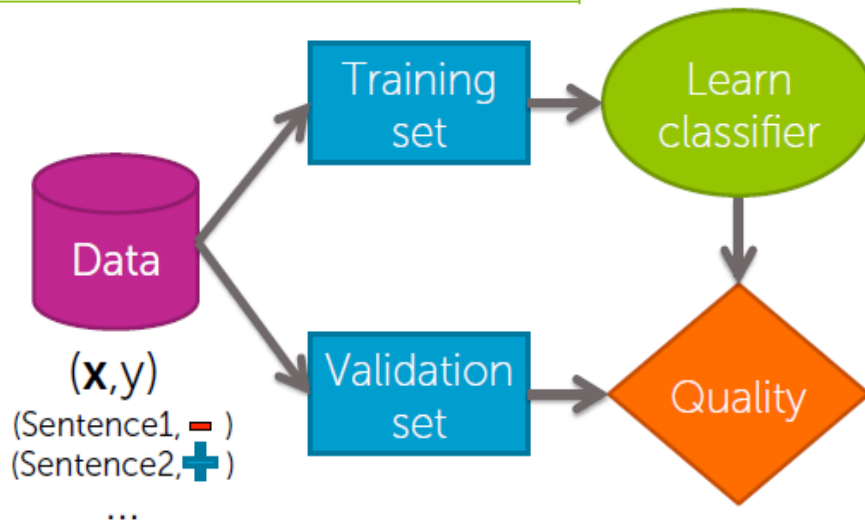
Learning logistic regression model

44

Training a classifier = Learning the coefficients

Word	Coefficient	Value
	\hat{w}_0	-2.0
good	\hat{w}_1	1.0
awesome	\hat{w}_2	1.7
bad	\hat{w}_3	-1.0
awful	\hat{w}_4	-3.3
...

$$\hat{P}(y=+1|\mathbf{x},\hat{\mathbf{w}}) = \frac{1}{1 + e^{-\hat{\mathbf{w}}^T \mathbf{h}(\mathbf{x})}}$$



Categorical inputs

45

- Numeric inputs:
 - #awesome, age, salary,...
 - Intuitive when multiplied by coefficient

- e.g., 1.5 #awesome

Numeric value, but should be interpreted as category
(98195 not about 9x larger than 10005)

- Categorical inputs:



Gender
(Male, Female,...)



Country of birth
(Argentina, Brazil, USA,...)

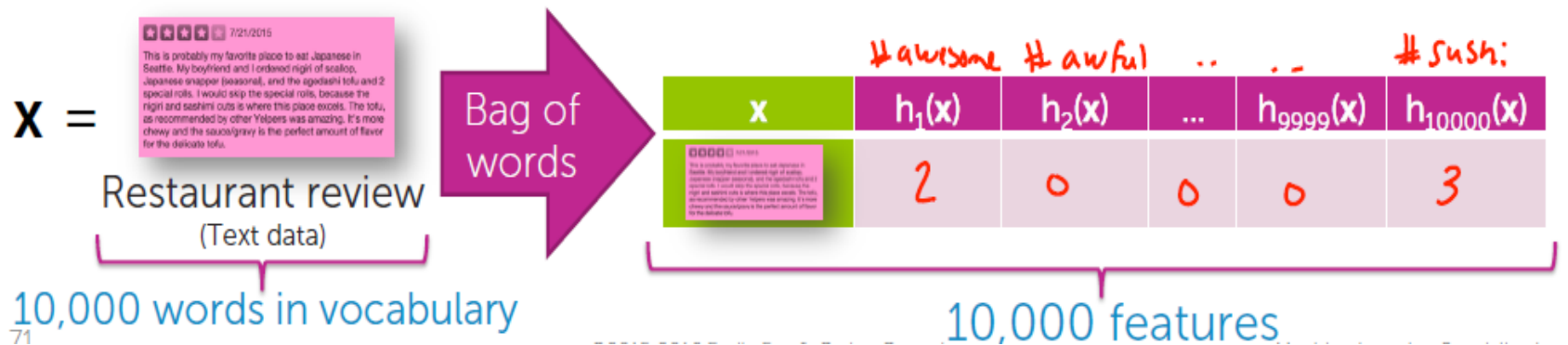
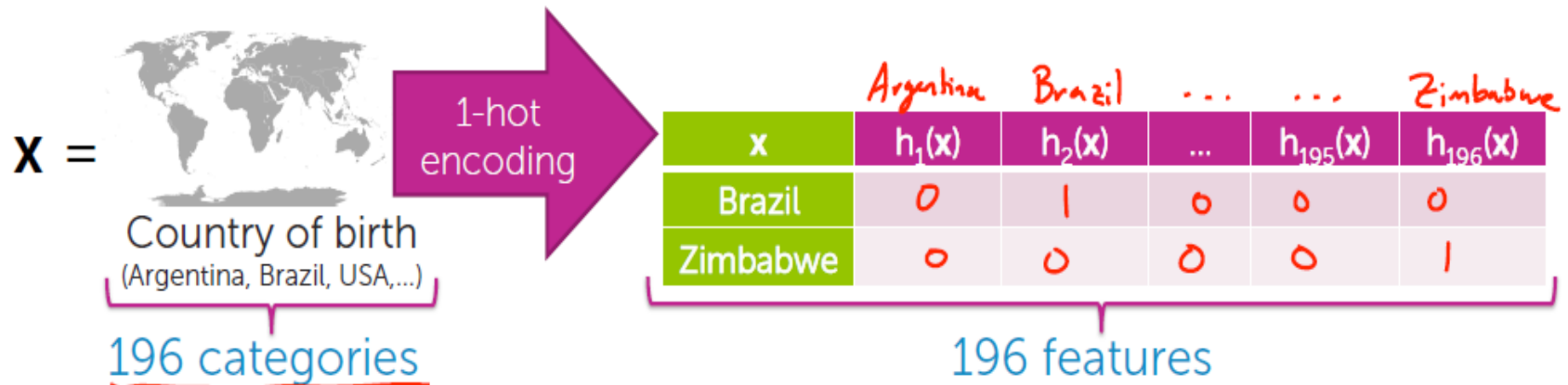


Zipcode
(10005, 98195,...)

How do we multiply category by coefficient??
Must convert categorical inputs into numeric features

Encoding categories as numeric features

46



Multiclass classification

47



Input: x
Image pixels



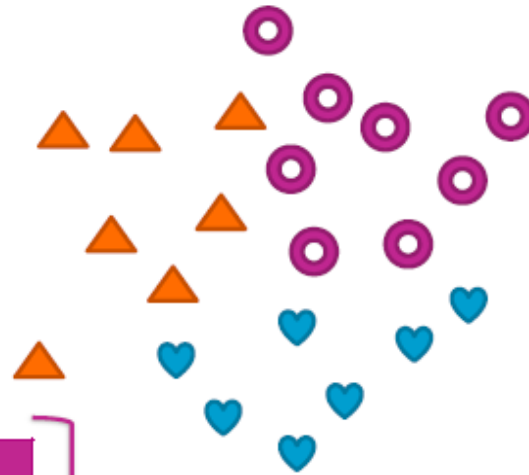
Output: y
Object in image

Multiclass classification

48

- C possible classes:
 - y can be 1, 2, ..., C
- N datapoints:

Data point	x[1]	x[2]	y
\mathbf{x}_1, y_1	2	1	▲
\mathbf{x}_2, y_2	0	2	♥
\mathbf{x}_3, y_3	3	3	○
\mathbf{x}_4, y_4	4	1	○



Learn:

$$\hat{P}(y = \triangle | \mathbf{x})$$

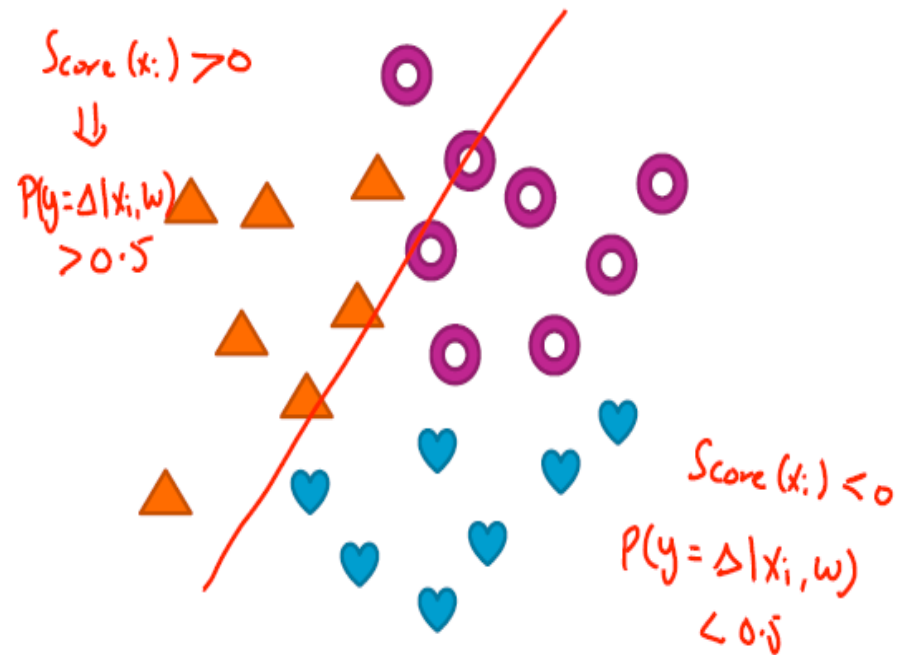
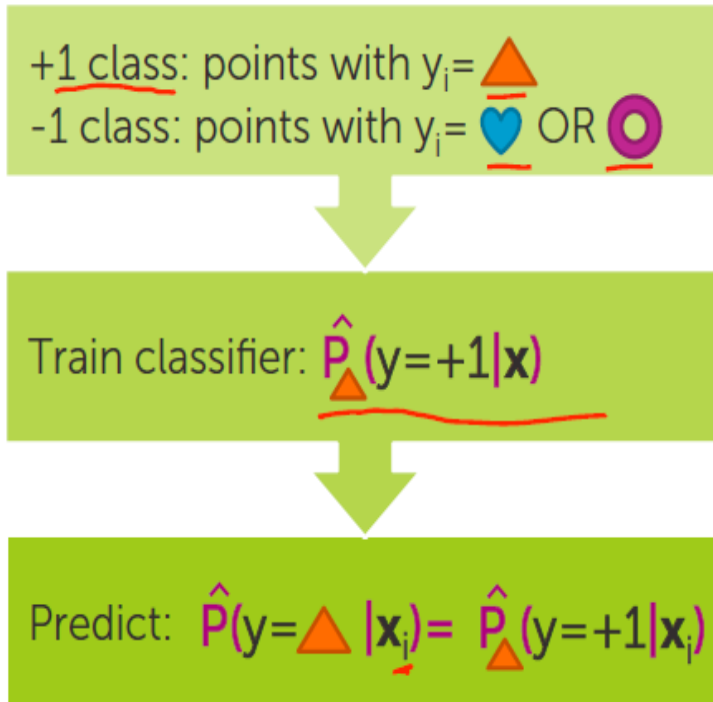
$$\hat{P}(y = \heartsuit | \mathbf{x})$$

$$\hat{P}(y = \circ | \mathbf{x})$$

1 versus all

49

Estimate $\hat{P}(y = \triangle | \mathbf{x})$ using 2-class model

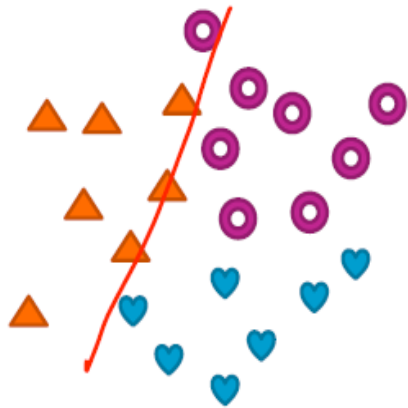


1 versus all

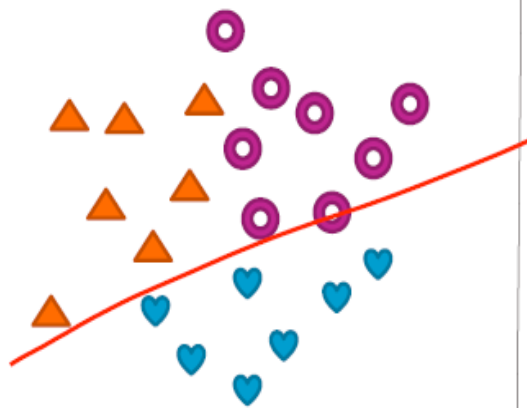
50

1 versus all: simple multiclass classification using C 2-class models

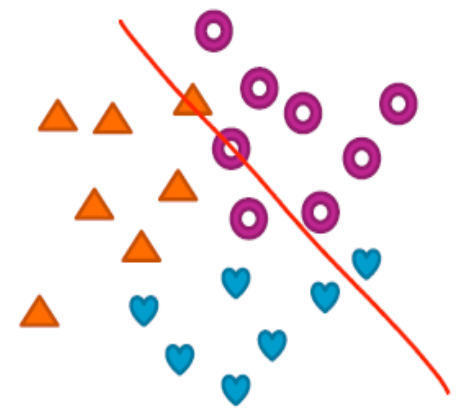
$$\hat{P}(y=\triangle | \mathbf{x}_i) = \hat{P}_{\triangle}(y=+1 | \mathbf{x}_i, \mathbf{w}_{\triangle})$$

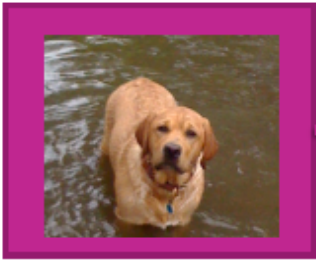


$$\hat{P}(y=\heartsuit | \mathbf{x}_i) = \hat{P}_{\heartsuit}(y=+1 | \mathbf{x}_i, \mathbf{w}_{\heartsuit})$$



$$\hat{P}(y=\circ | \mathbf{x}_i) = \hat{P}_{\circ}(y=+1 | \mathbf{x}_i, \mathbf{w}_{\circ})$$





Input: \mathbf{x}_i

Multiclass training

$\hat{P}_c(y=+1|\mathbf{x})$ = estimate of
1 vs all model for each class

Predict most likely class

max_prob = 0; $\hat{y} = 0$

For $c = 1, \dots, C$:

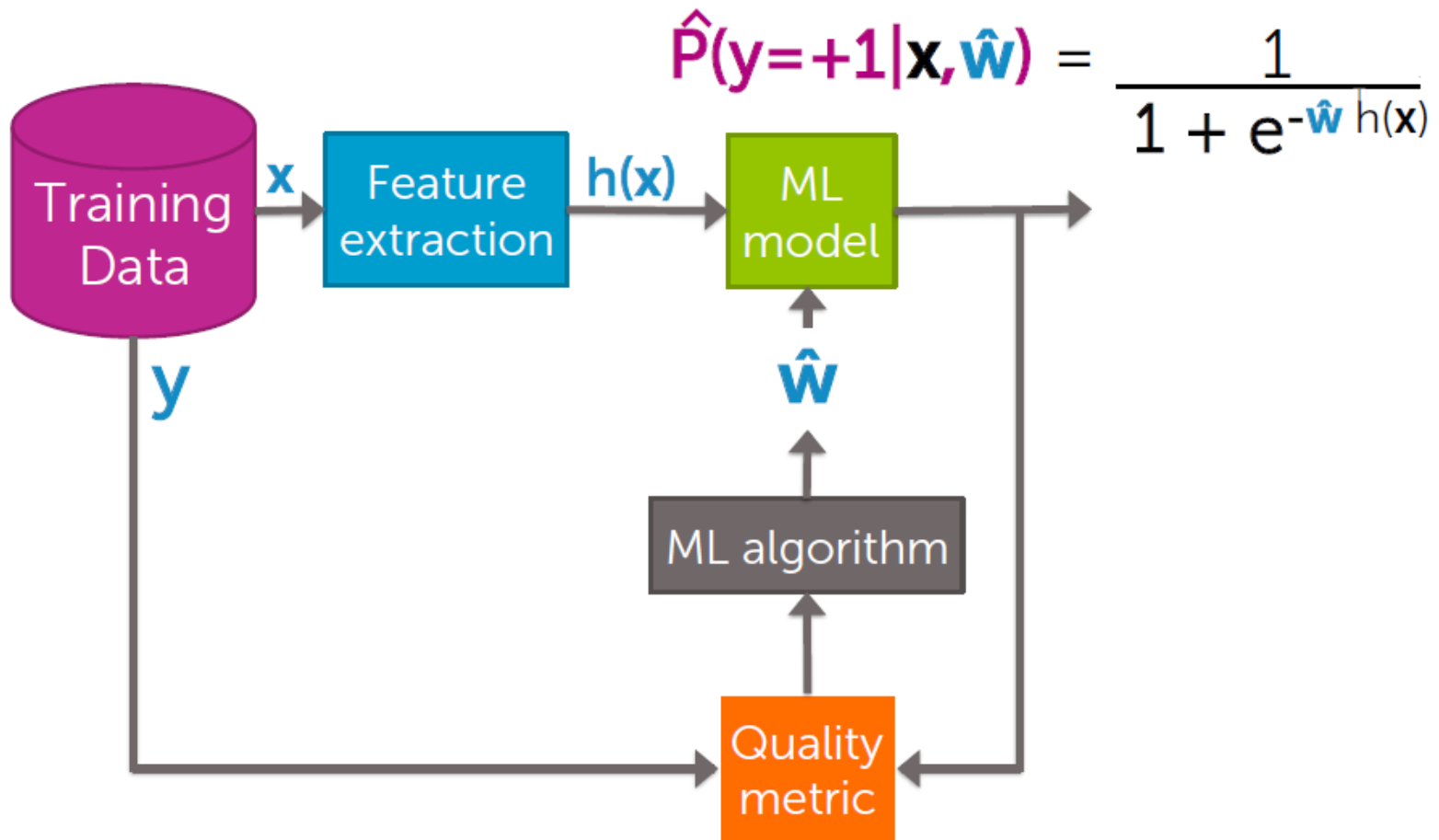
If $\hat{P}_c(y=+1|\mathbf{x}_i) >$ max_prob:

$\hat{y} = c$

max_prob = $\hat{P}_c(y=+1|\mathbf{x}_i)$

Summary: Logistic regression classifier

52



What you can do now...

53

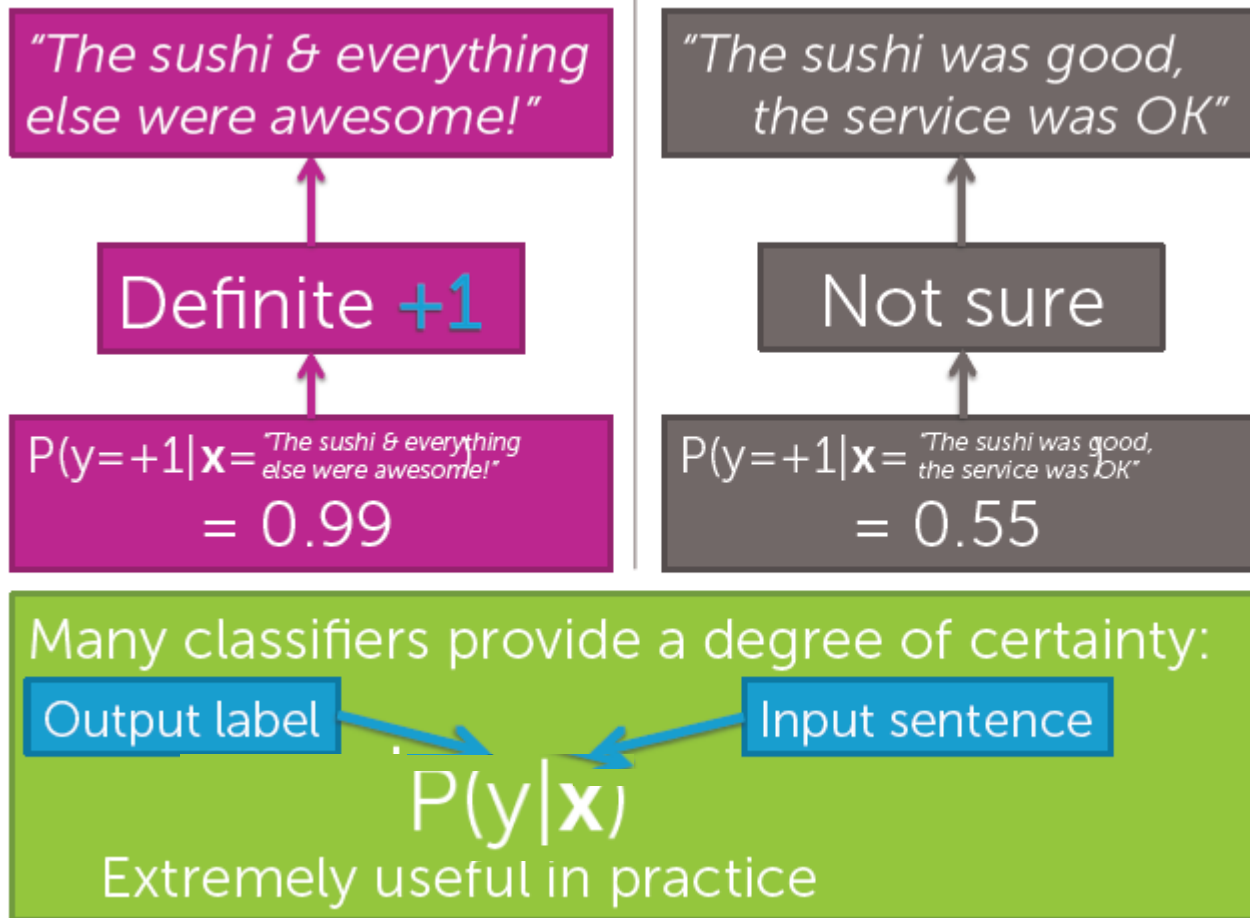
- Describe decision boundaries and linear classifiers
- Use class probability to express degree of confidence in prediction
- Define a logistic regression model
- Interpret logistic regression outputs as class probabilities
- Describe impact of coefficient values on logistic regression output
- Use 1-hot encoding to represent categorical inputs
- Perform multiclass classification using the 1-versus-all approach

Linear classifier

- ▣ Parameters learning

Learn a probabilistic classification model

55



A (linear) classifier

56

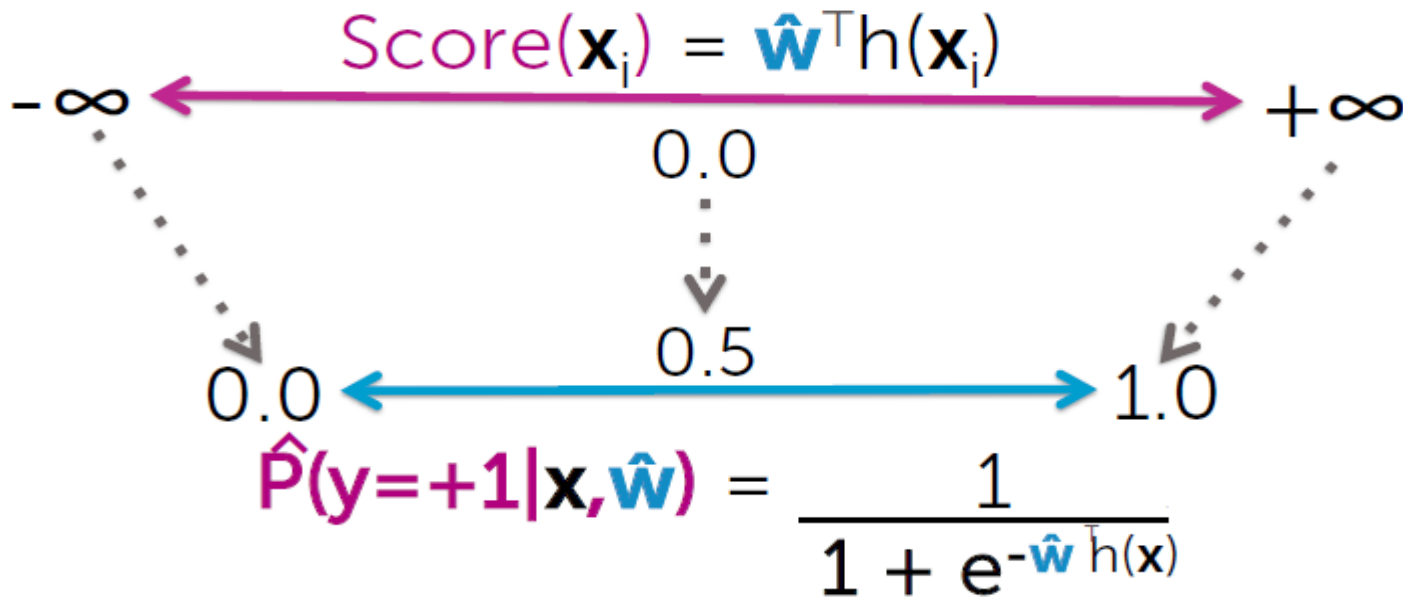
- Will use training data to learn a weight or coefficient for each word

Word	Coefficient	Value
	\hat{W}_0	-2.0
good	\hat{W}_1	1.0
great	\hat{W}_2	1.5
awesome	\hat{W}_3	2.7
bad	\hat{W}_4	-1.0
terrible	\hat{W}_5	-2.1
awful	\hat{W}_6	-3.3
restaurant, the, we, ...	$\hat{W}_7, \hat{W}_8, \hat{W}_9, \dots$	0.0
...		...

Logistic regression

57

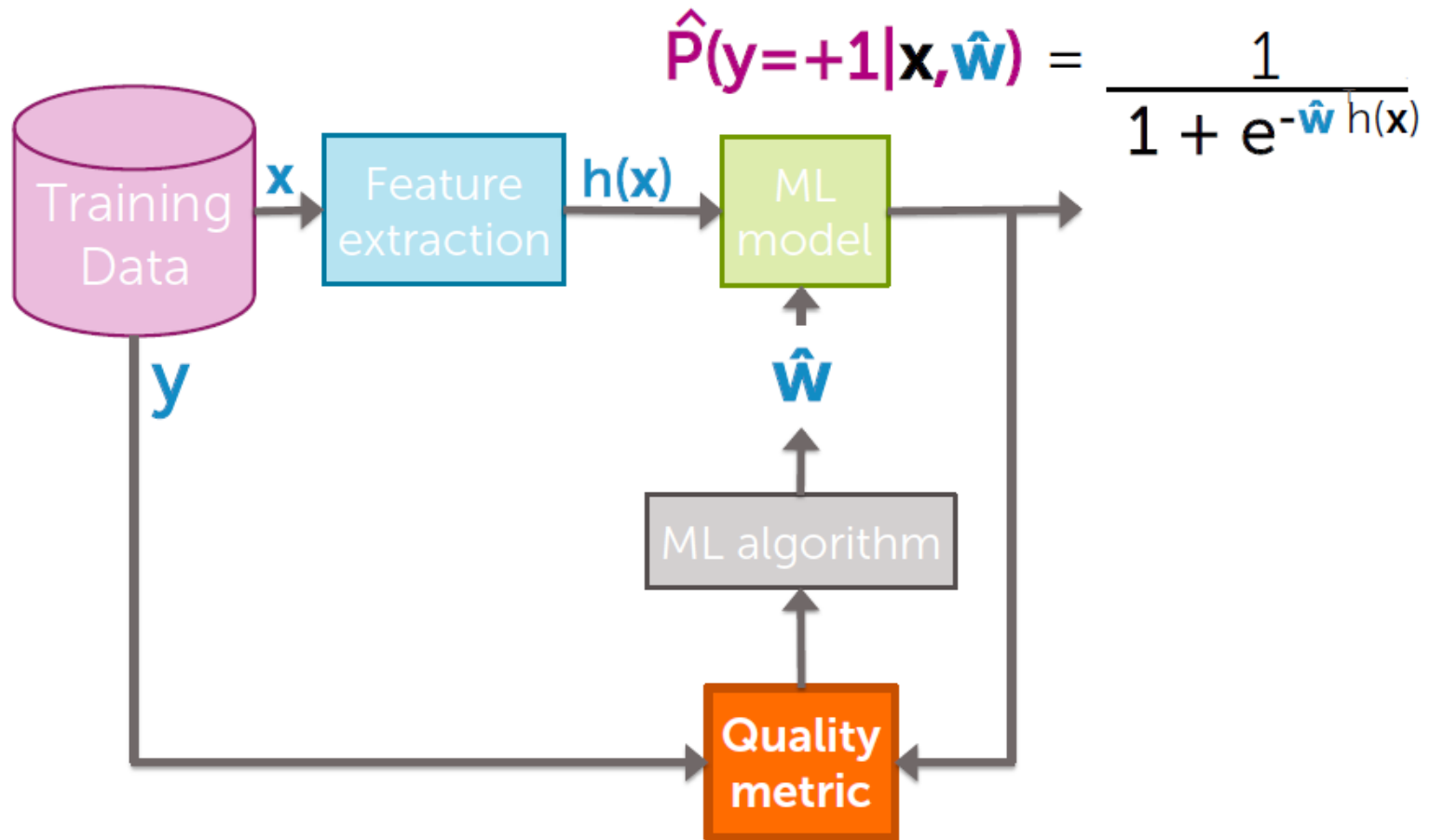
Logistic regression model



Flow chart:

Quality metric

58



Learning problem

59

Training data:

N observations (\mathbf{x}_i, y_i)

$x[1] = \text{\#awesome}$	$x[2] = \text{\#awful}$	$y = \text{sentiment}$
2	1	+1
0	2	-1
3	3	-1
4	1	+1
1	1	+1
2	4	-1
0	3	-1
0	1	-1
2	1	+1



Finding best coefficients

60

x[1] = #awesome	x[2] = #awful	y = sentiment
0	2	-1
3	3	-1
2	4	-1
0	3	-1
0	1	-1

x[1] = #awesome	x[2] = #awful	y = sentiment
2	1	+1
4	1	+1
1	1	+1
2	1	+1

$P(y=+1|x_i, \mathbf{w}) = 0.0$

$P(y=+1|x_i, \mathbf{w}) = 1.0$

Pick $\hat{\mathbf{w}}$ that makes

Quality metric: probability of data

61

$x[1] = \text{\#awesome}$	$x[2] = \text{\#awful}$	$y = \text{sentiment}$
2	1	+1

x_1

y_1

If model good, should predict:

$\hat{y}_1 = +1$

Pick w to maximize:

$$P(y = +1 | x_1, w) = P(y = +1 | x[1]=2, x[2]=1, w)$$

$x[1] = \text{\#awesome}$	$x[2] = \text{\#awful}$	$y = \text{sentiment}$
0	2	-1

x_2

y_2

If model good, should predict:

$\hat{y}_2 = -1$

Pick w to maximize:

$$P(y = -1 | x_2, w)$$

15

Maximizing likelihood (probability of data)

62

Data point	x[1]	x[2]	y	Choose w to maximize
x_1, y_1	2	1	+1	$P(y=+1 x_1, w) = P(y=+1 x_0]=2, x_2]=1, w)$
x_2, y_2	0	2	-1	$P(y=-1 x_2, w)$
x_3, y_3	3	3	<u>-1</u>	$P(y=-1 x_3, w)$
x_4, y_4	4	1	<u>+1</u>	$P(y=+1 x_4, w)$
x_5, y_5	1	1	+1	
x_6, y_6	2	4	-1	
x_7, y_7	0	3	-1	
x_8, y_8	0	1	-1	
x_9, y_9	2	1	+1	

Must combine into single measure of quality ?

Multiply probabilities

$P(y=+1|x_1, w) P(y=-1|x_2, w) P(y=-1|x_3, w) \dots$

Maximum likelihood estimation (MLE)

63

Learn logistic regression model with MLE

Data point	x[1]	x[2]	y	Choose w to maximize
x_1, y_1	2	1	$y_1 = +1$	$P(y = +1 x[1]=2, x[2]=1, w)$
x_2, y_2	0	2	-1	$P(y = -1 x[1]=0, x[2]=2, w)$
x_3, y_3	3	3	-1	$P(y = -1 x[1]=3, x[2]=3, w)$
x_4, y_4	4	1	$+1$	$P(y = +1 x[1]=4, x[2]=1, w)$

No \hat{w} achieves perfect predictions (usually)

Likelihood $\ell(w)$: Measures quality of fit for model with coefficients w

$$\ell(w) = \underbrace{P(y=+1|x[1]=2, x[2]=1, w)}_{P(y_1|x_1, w)} \underbrace{P(y=-1|x[1]=0, x[2]=2, w)}_{P(y_2|x_2, w)} \underbrace{P(y=-1|x[1]=3, x[2]=3, w)}_{P(y_3|x_3, w)} \underbrace{P(y=+1|x[1]=4, x[2]=1, w)}_{P(y_4|x_4, w)}$$

Num. of data points $\rightarrow N$

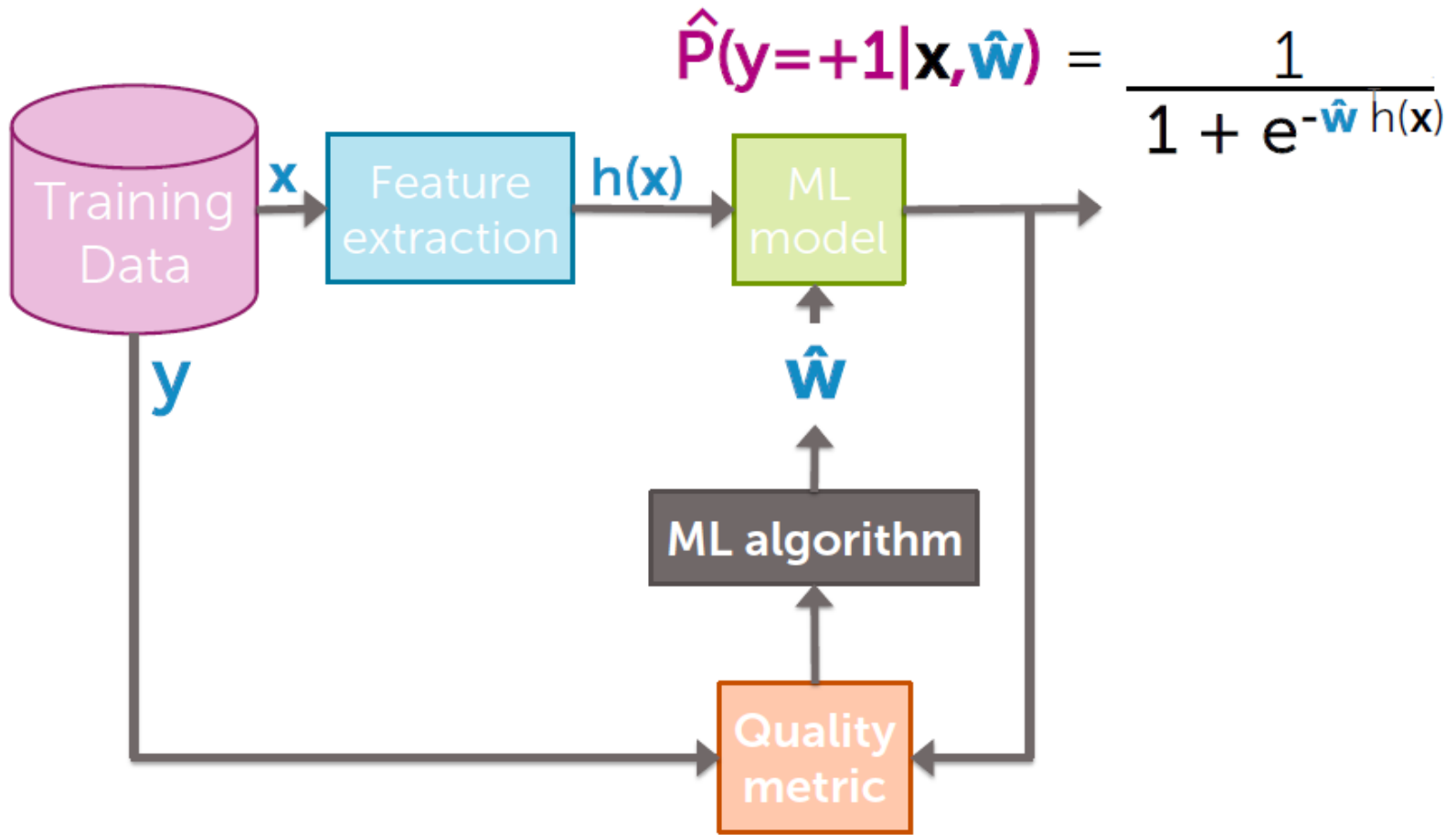
$$\ell(w) = \prod_{i=1}^N P(y_i | x_i, w)$$

pick w to make this fn. as large as possible

Flow chart:

ML algorithm

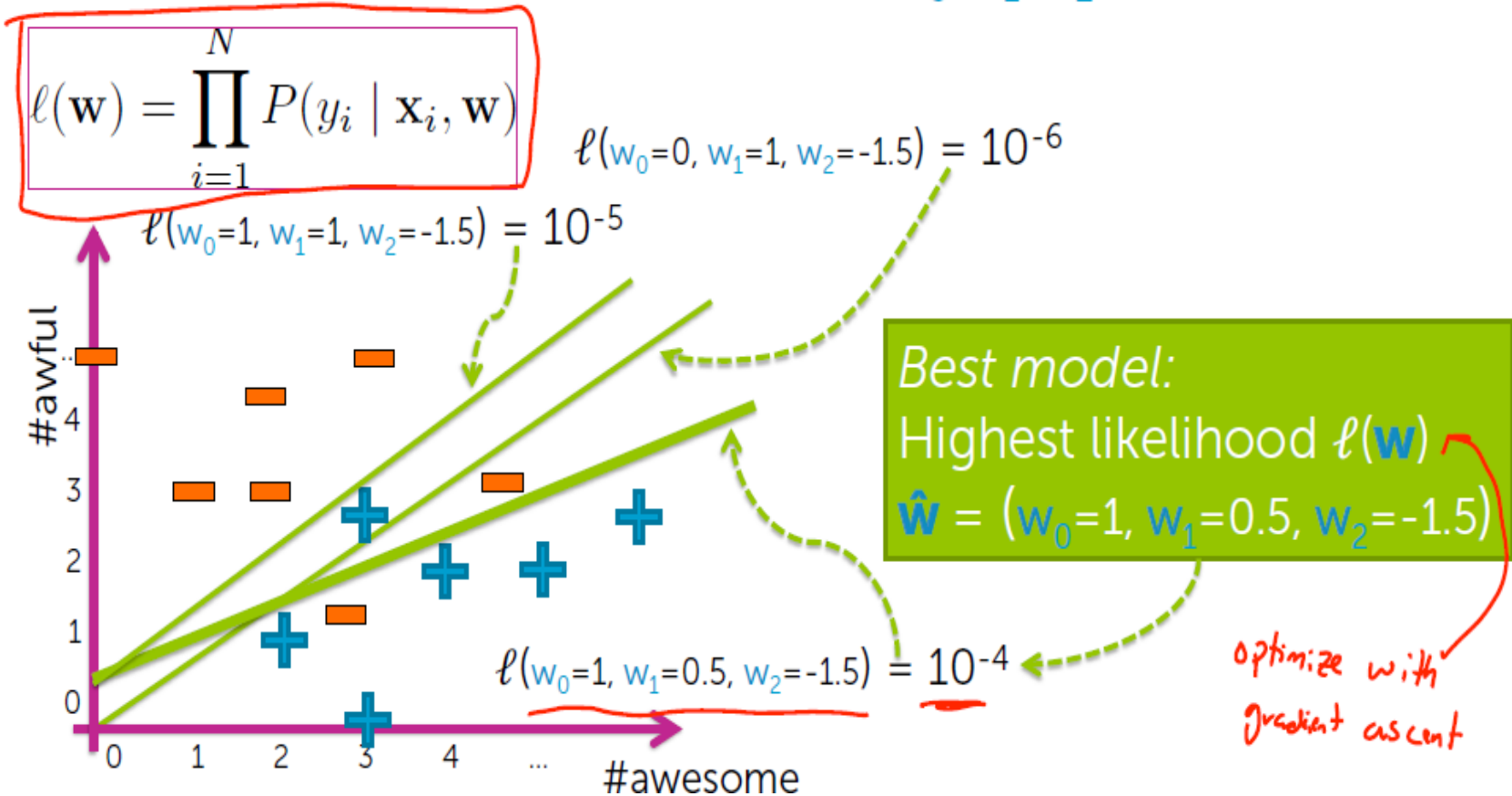
64



Find „best” classifier

65

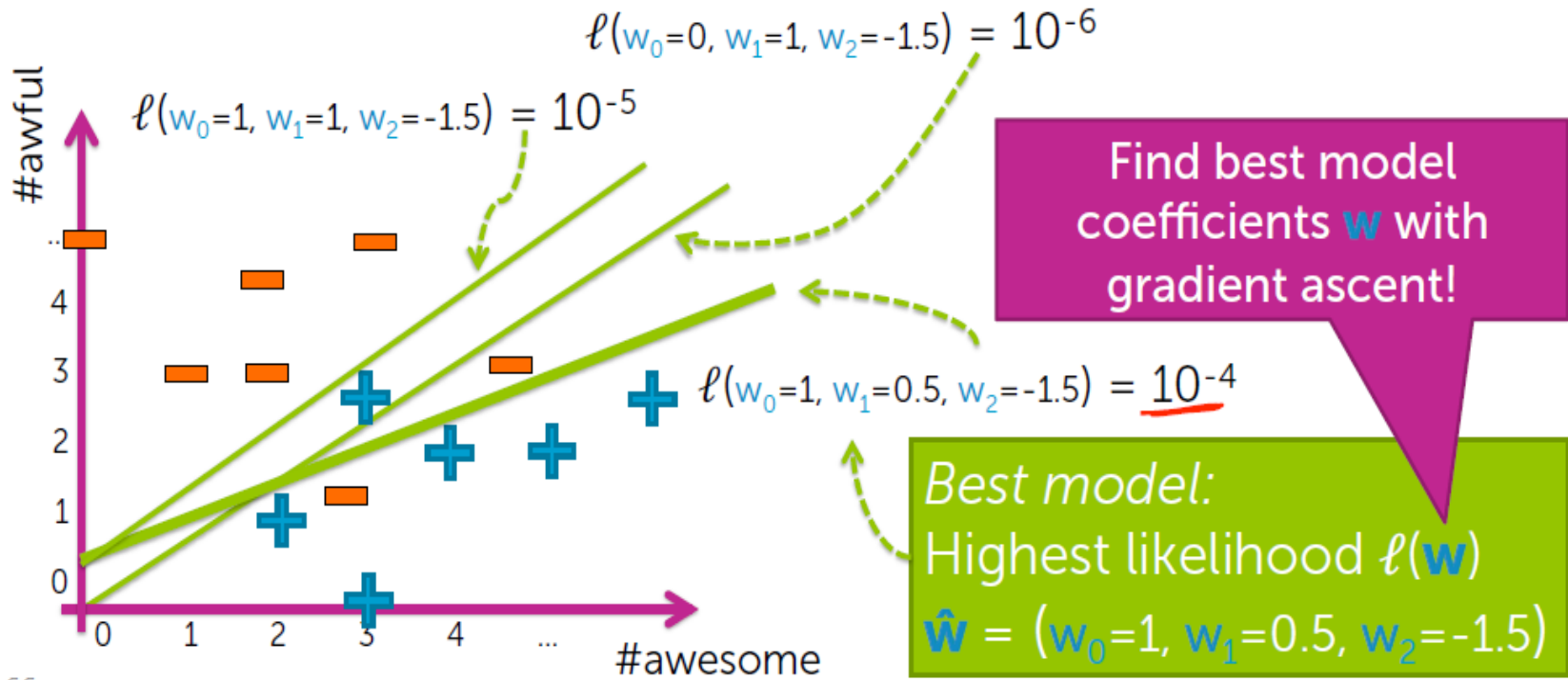
Maximize likelihood over all possible w_0, w_1, w_2



Find best classifier

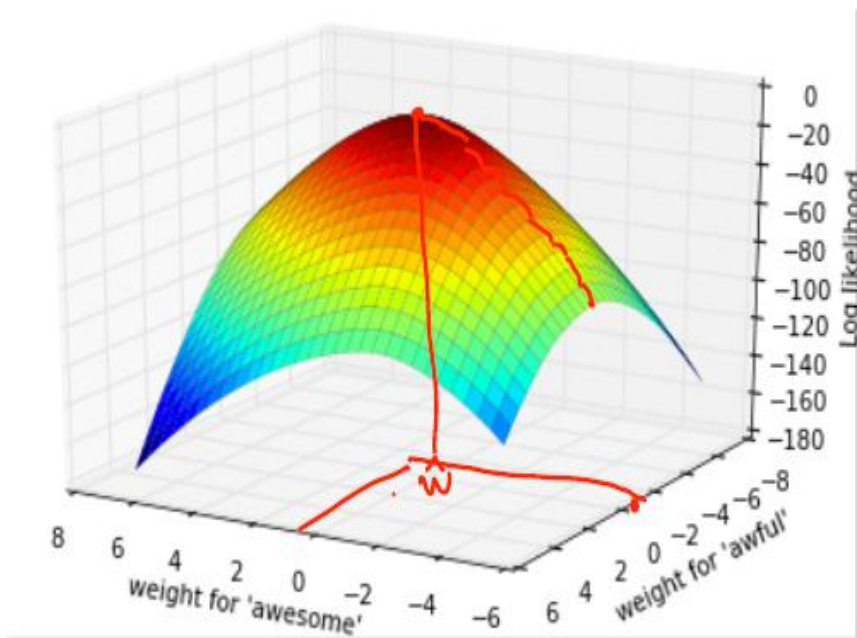
66

Maximize quality metric over all possible w_0, w_1, w_2
Likelihood $\ell(\mathbf{w})$



Maximizing likelihood

67



Maximize function over all possible w_0, w_1, w_2

$$\max_{w_0, w_1, w_2} \prod_{i=1}^N P(y_i | \mathbf{x}_i, \mathbf{w})$$

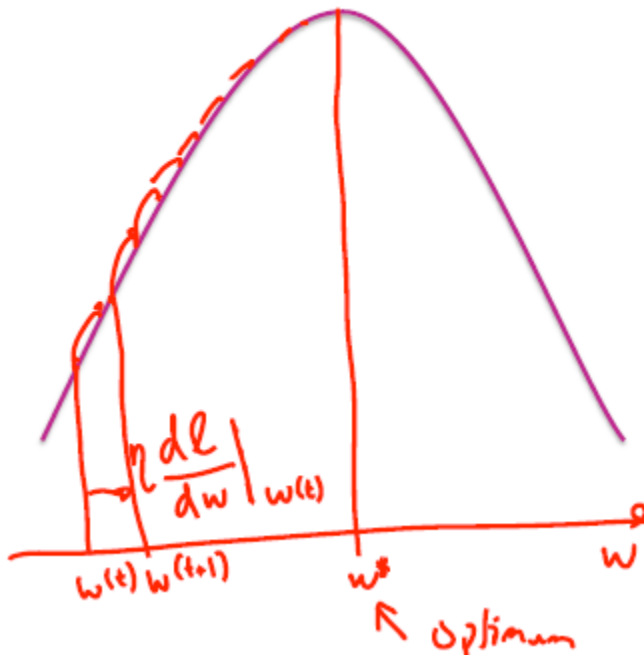
$\ell(w_0, w_1, w_2)$ is a function of 3 variables

No closed-form solution \rightarrow use gradient ascent

Gradient ascent

68

Finding the max via hill climbing



Algorithm:

while not converged

$$w^{(t+1)} \leftarrow w^{(t)} + \eta \frac{d\ell}{dw} \Big|_{w^{(t)}}$$

step size

Gradient ascent

69

Convergence criteria

For convex functions,
optimum occurs when

$$\frac{d\ell}{dw} = 0$$

In practice, stop when

$$\left. \frac{d\ell}{dw} \right|_{w^{(k)}} < \epsilon$$

↑
tolerance



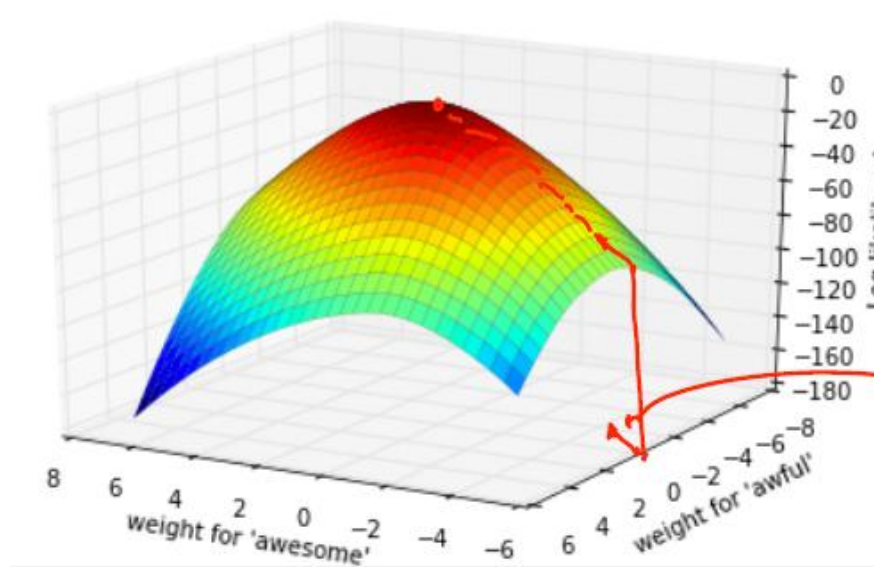
Algorithm:

while not converged
 $w^{(t+1)} \leftarrow w^{(t)} + \eta \left. \frac{d\ell}{dw} \right|_{w^{(t)}}$

Gradient ascent

70

Moving to multiple dimensions: Gradients

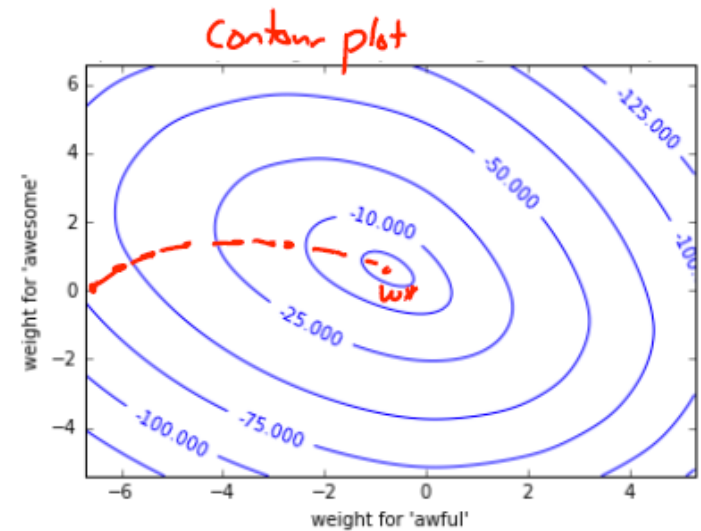
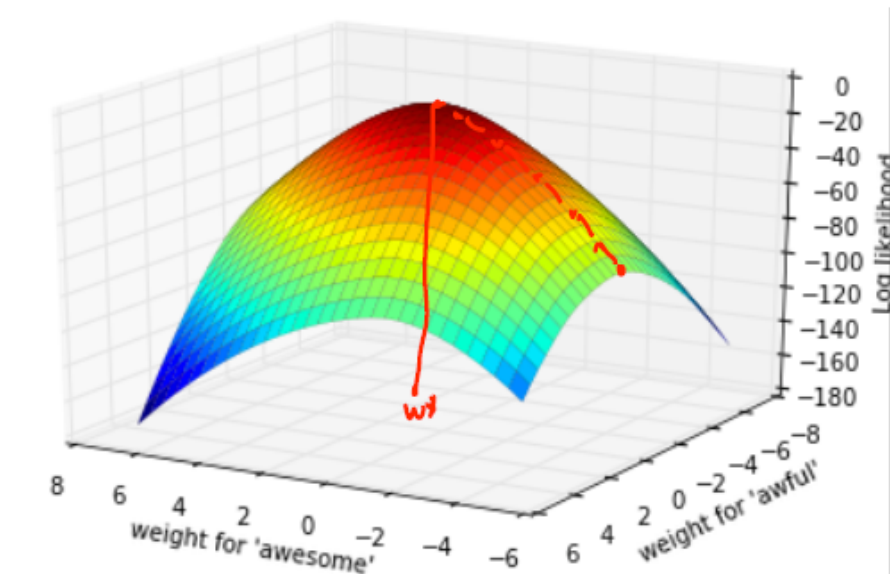


$$\nabla \ell(\mathbf{w}) = \begin{bmatrix} \frac{\partial \ell}{\partial w_0} \\ \frac{\partial \ell}{\partial w_1} \\ \vdots \\ \frac{\partial \ell}{\partial w_D} \end{bmatrix} \leftarrow \begin{array}{l} D+1 \text{ dim} \\ \text{vector} \end{array}$$

Gradient ascent

71

Contour plots

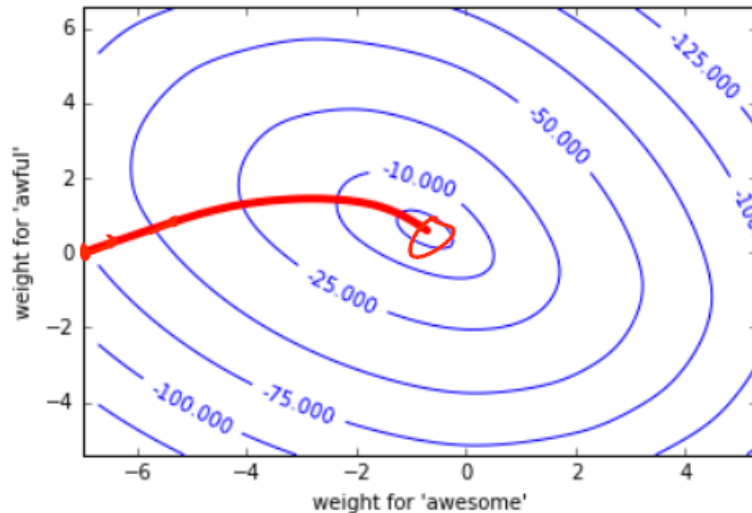


9/11, 16/11, 23/11 2022

Gradient ascent

72

Gradient ascent



Algorithm:

$w^{(0)} = 0$, random, or something smart.

while not converged

$$w^{(t+1)} \leftarrow w^{(t)} + \eta \nabla \ell(w^{(t)})$$

step size

The log trick, often used in ML...

73

- Products become sums:
 $\ln a \cdot b = \ln a + \ln b$ | $\ln \frac{a}{b} = \ln a - \ln b$
- Doesn't change maximum!
 - If $\hat{\mathbf{w}}$ maximizes $f(\mathbf{w})$:

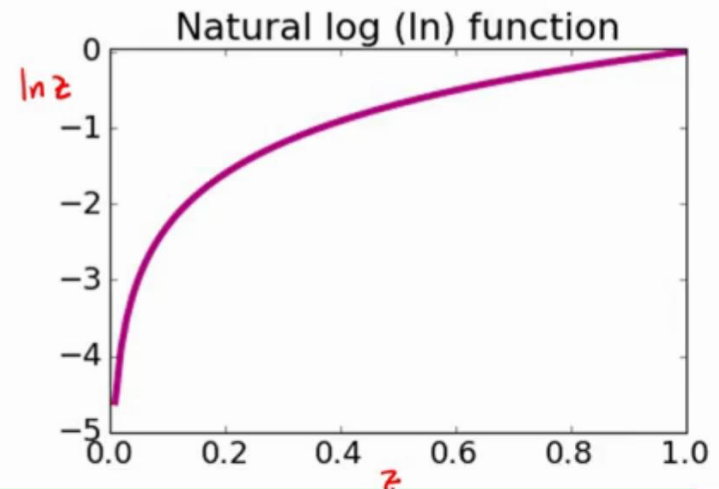
$$\hat{\mathbf{w}} = \underset{\mathbf{w}}{\operatorname{arg\,max}} f(\mathbf{w})$$

the \mathbf{w} that makes $f(\mathbf{w})$ largest

- Then $\hat{\mathbf{w}}_{\ln}$ maximizes $\ln(f(\mathbf{w}))$:

$$\hat{\mathbf{w}}_{\ln} = \underset{\mathbf{w}}{\operatorname{arg\,max}} \ln(f(\mathbf{w}))$$

$$\hat{\mathbf{w}} = \hat{\mathbf{w}}_{\ln}$$



Derivative for logistic regression

74

Derivative of (log-)likelihood

Sum over data points

Feature value

Difference between truth and prediction

$$\frac{\partial \ell(\mathbf{w})}{\partial \mathbf{w}_j} = \sum_{i=1}^N h_j(\mathbf{x}_i) \left(\mathbb{1}[y_i = +1] - \underbrace{P(y = +1 | \mathbf{x}_i, \mathbf{w})}_{\text{predict } x_i \text{ is positive}} \right)$$

Indicator function:

$$\mathbb{1}[y_i = +1] = \begin{cases} 1 & \text{if } y_i = +1 \\ 0 & \text{if } y_i = -1 \end{cases}$$

See slides at the end of this lecture
If you are interested how it is derived.

Derivative for logistic regression

75

Computing derivative

$$\frac{\partial \ell(\mathbf{w}^{(t)})}{\partial w_j} = \sum_{i=1}^N h_j(\mathbf{x}_i) \left(\mathbb{1}[y_i = +1] - P(y = +1 | \mathbf{x}_i, \mathbf{w}^{(t)}) \right)$$

$w^{(t)}$:

$w_0^{(t)}$	0
$w_1^{(t)}$	1
$w_2^{(t)}$	-2

$$\frac{\partial \ell}{\partial w_1}$$

$h_j(\cdot)$ = 4th column

x[1]	x[2]	y	P(y=+1 x,w)	Contribution to derivative for w_1
2	1	+1	0.5	$2(1-0.5) = 1$
0	2	-1	0.02	$0(0-0.02) = 0$
3	3	-1	0.05	$3(0-0.05) = -0.15$
4	1	+1	0.88	$4(1-0.88) = 0.48$

Total derivative:

$$\frac{\partial \ell(\mathbf{w}^{(t)})}{\partial w_1} = 1 + 0 - 0.15 + 0.48 = 1.33$$

$$w_1^{(t+1)} = w_1^{(t)} + \eta \frac{\partial \ell(\mathbf{w}^{(t)})}{\partial w_1} \quad | \quad \eta = 0.1$$

$$= 1 + 0.1 \cdot 1.33 = 1.133$$

Derivative for logistic regression

76

Derivative of (log-)likelihood: Interpretation

Sum over data points Feature value Difference between truth and prediction

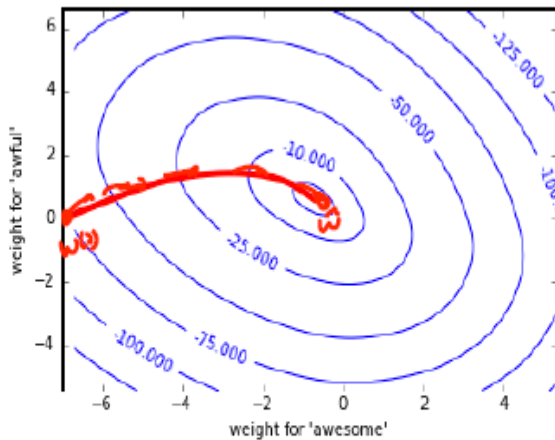
$$\frac{\partial \ell(\mathbf{w})}{\partial w_j} = \sum_{i=1}^N h_j(\mathbf{x}_i) \left(\mathbb{1}[y_i = +1] - P(y = +1 | \mathbf{x}_i, \mathbf{w}) \right)$$

Δ_i

If $h_j(\mathbf{x}_i)=1$:	<u>$P(y=+1 \mathbf{x}_i, \mathbf{w}) \approx 1$</u>	<u>$P(y=+1 \mathbf{x}_i, \mathbf{w}) \approx 0$</u>
<u>$y_i=+1$</u>	$\Delta_i = (1 - 1) \approx 0$ \hookrightarrow don't change anything!	$\Delta_i \approx 1 \Rightarrow$ increase w_j \Rightarrow Score(x_i) becomes larger $\Rightarrow P(y=+1 \mathbf{x}_i, \mathbf{w})$ increases
<u>$y_i=-1$</u>	$\Delta_i = -1 \Rightarrow w_j$ to decrease \Rightarrow Score(x_i) decreases $\Rightarrow P(y=+1 \mathbf{x}_i, \mathbf{w})$ decrease	$\Delta_i \approx 0$ \Rightarrow don't change anything

Gradient ascent for logistic regression

77



init $\mathbf{w}^{(1)} = 0$ (or randomly, or smartly), $t=1$

while $\|\nabla \ell(\mathbf{w}^{(t)})\| > \epsilon$

for $j=0, \dots, D$

$$\text{partial}[j] = \sum_{i=1}^N h_j(\mathbf{x}_i) \left(\mathbf{1}[y_i = +1] - \underbrace{P(y = +1 \mid \mathbf{x}_i, \mathbf{w}^{(t)})}_{\frac{1}{1 + e^{-\mathbf{w}^{(t)T} \mathbf{h}(\mathbf{x}_i)}}} \right)$$

$$w_j^{(t+1)} \leftarrow w_j^{(t)} + \eta \text{partial}[j]$$

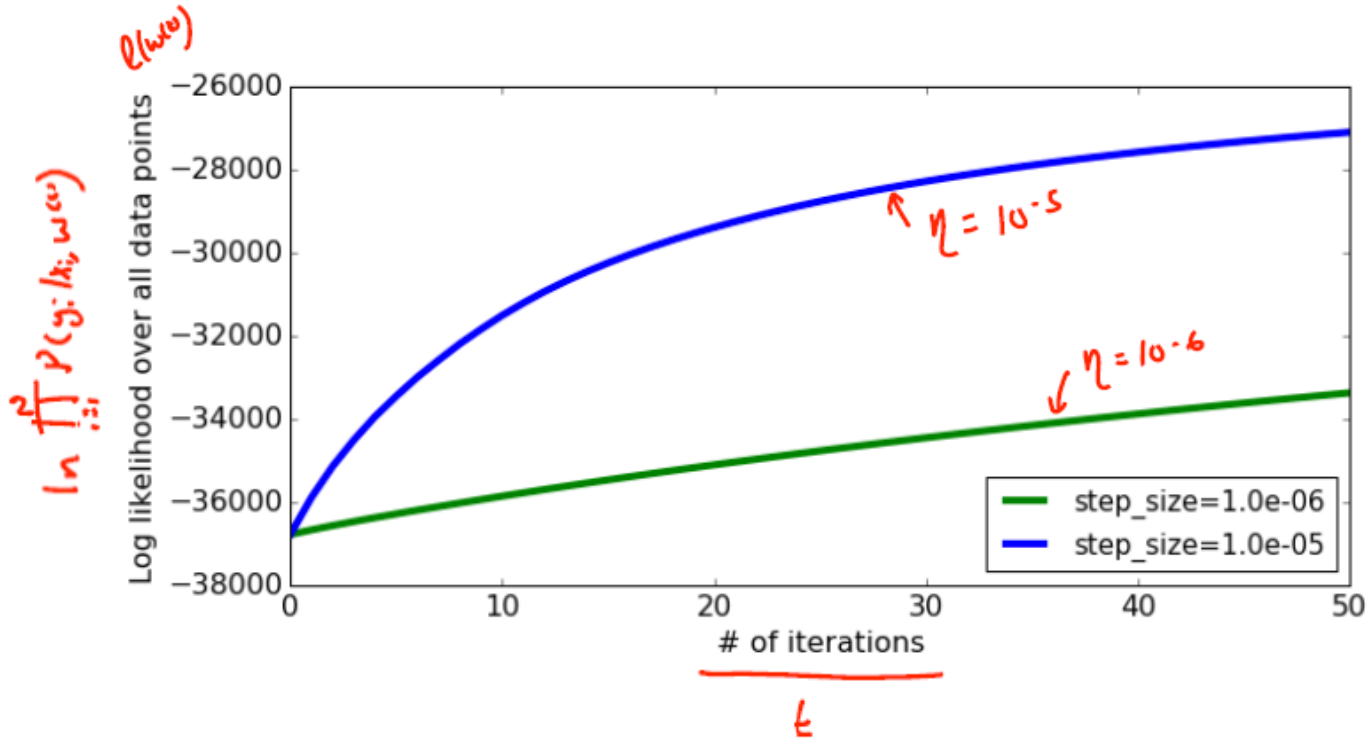
$t \leftarrow t + 1$

step size η $\frac{\partial \ell(\mathbf{w}^{(t)})}{\partial w_j}$

Choosing the step size

78

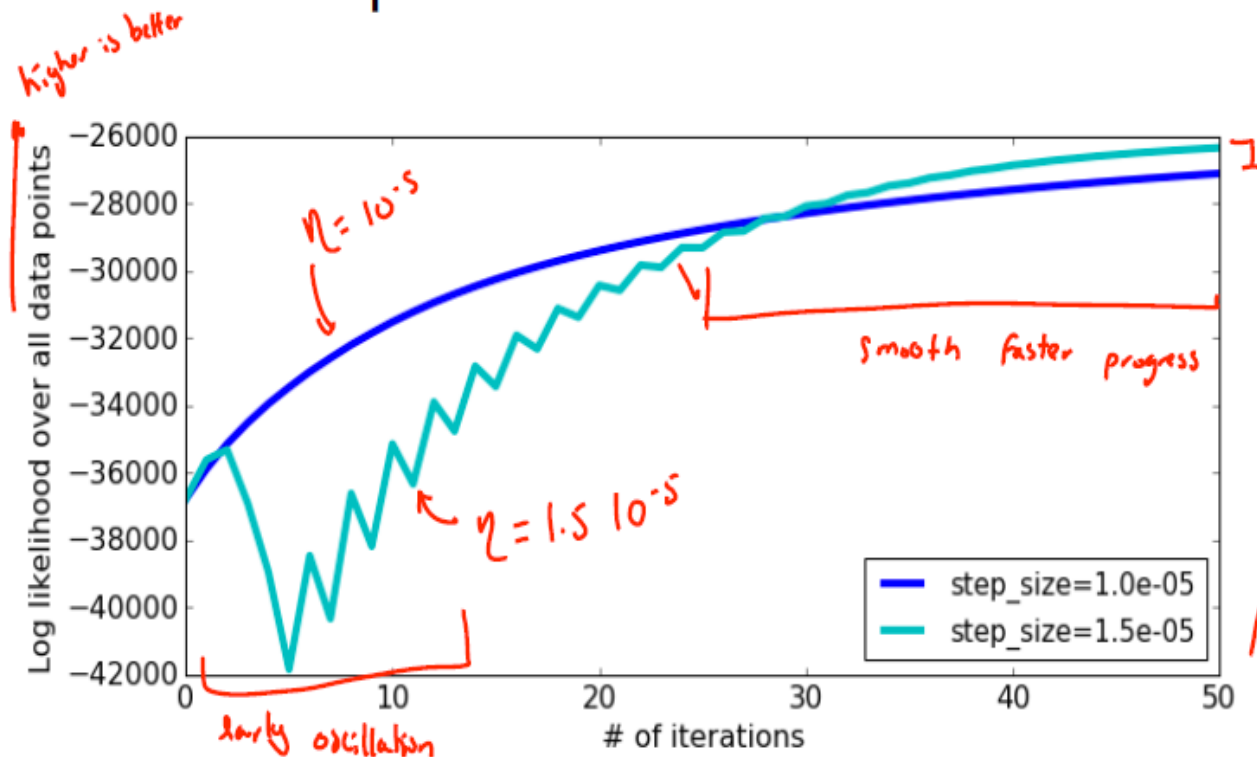
If step size is too small, can take a long time to converge



Choosing the step size

79

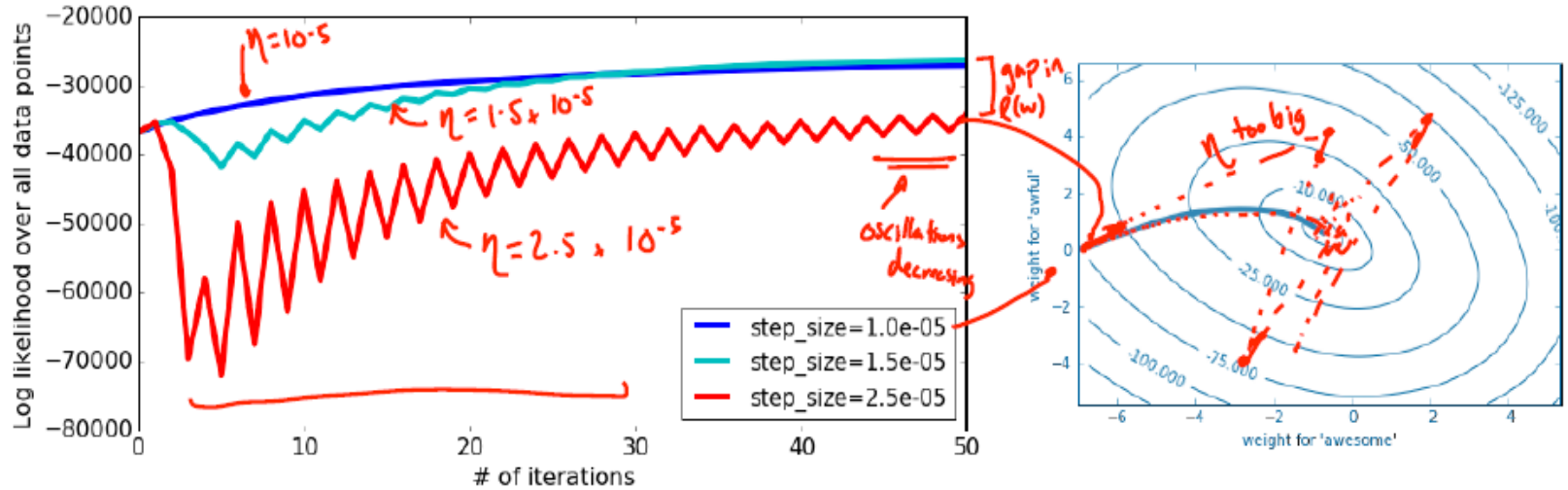
Compare converge with different step sizes



Choosing the step size

80

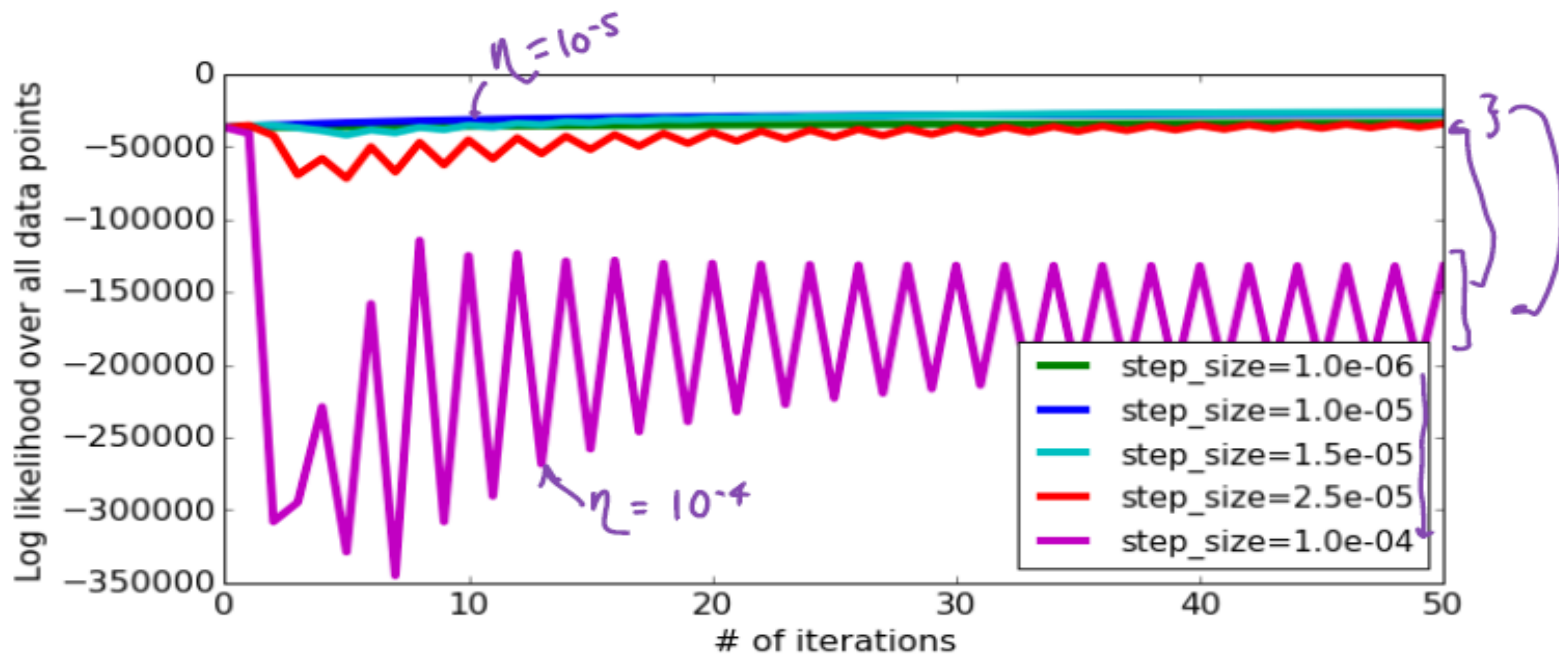
Careful with step sizes that are too large



Choosing the step size

81

Very large step sizes can even cause divergence or wild oscillations



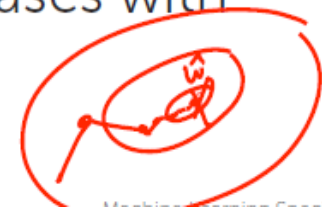
Choosing the step size

82

Simple rule of thumb for picking step size η

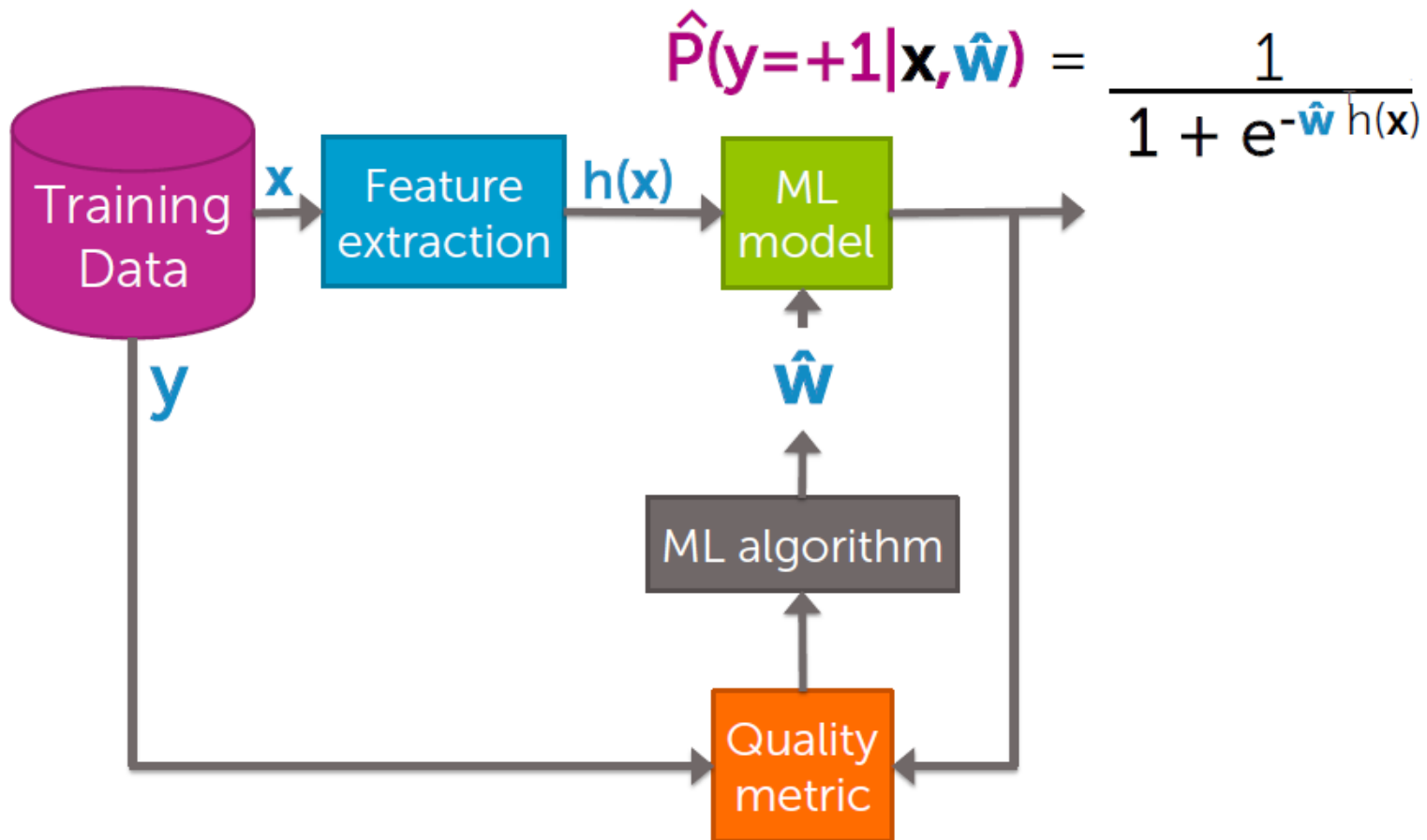
- Unfortunately, picking step size requires a lot of trial and error 😞
- Try a several values, exponentially spaced
 - Goal: plot learning curves to
 - find one η that is too small (smooth but moving too slowly)
 - find one η that is too large (oscillation or divergence)
- Try values in between to find “best” η
 - ↳ exponentially space, pick one that leads best training data likelihood
- Advanced tip: can also try step size that decreases with iterations, e.g.,

$$\eta_t = \frac{\eta_0}{t}$$



Flow chart: final look at it

83



What you can do now

84

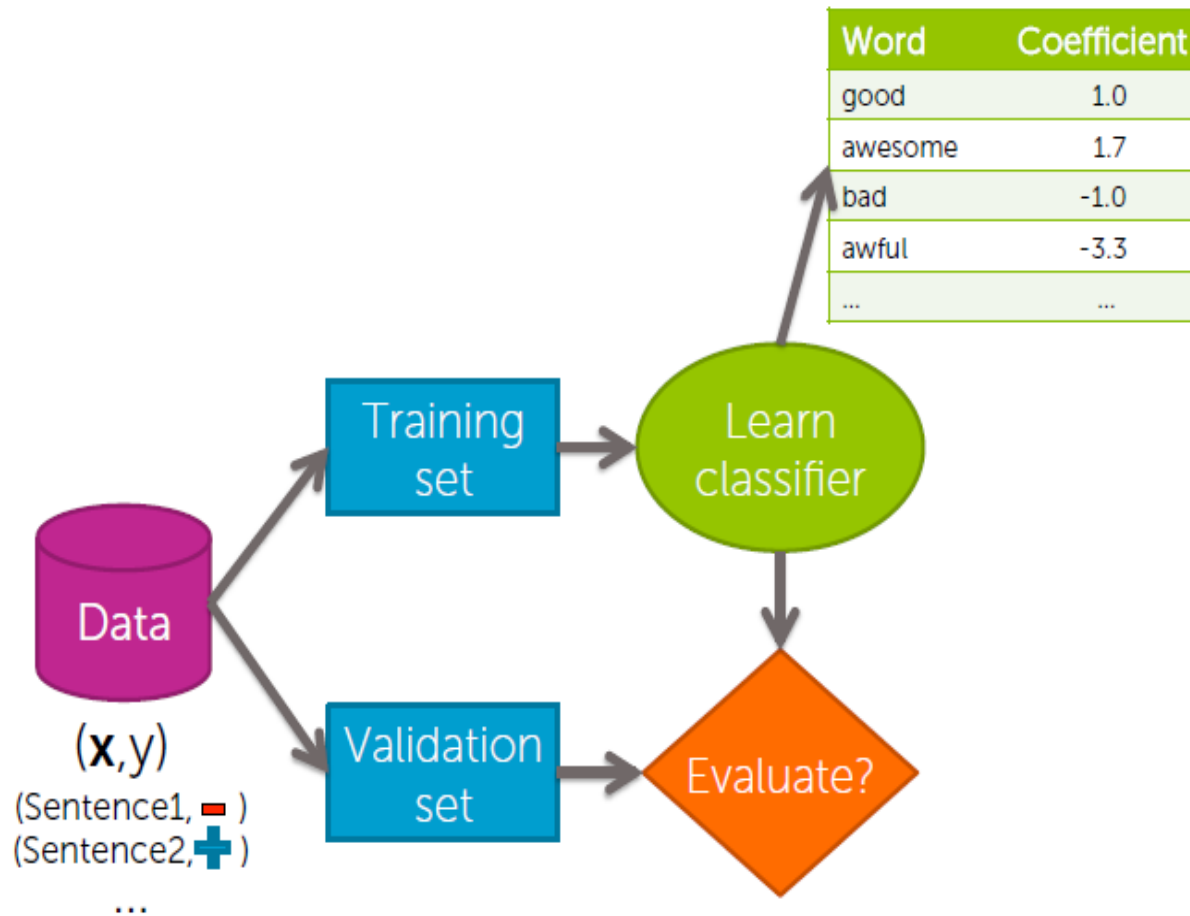
- Measure quality of a classifier using the likelihood function
- Interpret the likelihood function as the probability of the observed data
- Learn a logistic regression model with gradient descent
- (Optional) Derive the gradient descent update rule for logistic regression

Linear classifier

▣ Overfitting & regularization

Training a classifier = Learning the coefficients

86



Classification error & accuracy

87

- Error measures fraction of mistakes

$$\text{error} = \frac{\# \text{ Mistakes}}{\text{Total number of data points}}$$

- Best possible value is 0.0

- Often, measure **accuracy**

- Fraction of correct predictions

$$\text{accuracy} = \frac{\# \text{ Correct}}{\text{Total number of data points}}$$

- Best possible value is 1.0

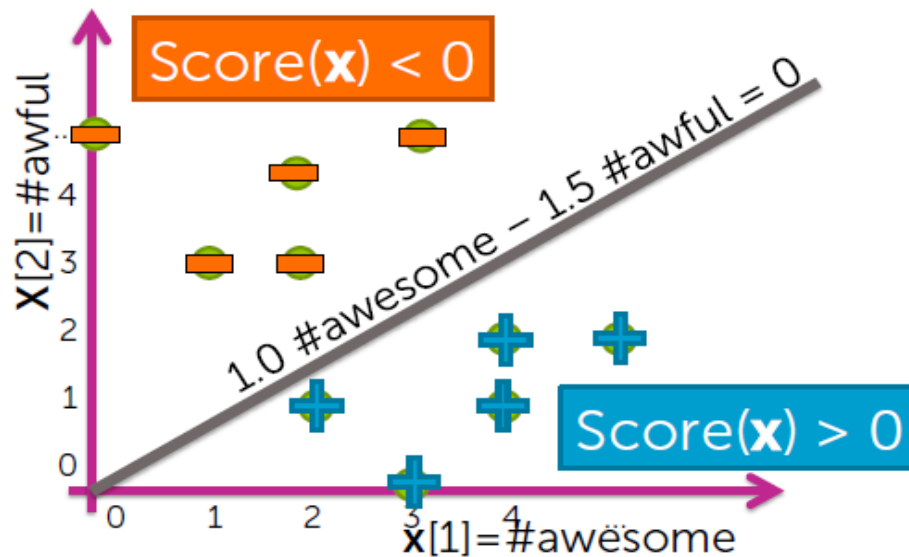
Overfitting in classification

88

Decision boundary example

Word	Coefficient
#awesome	1.0
#awful	-1.5

→ $\text{Score}(\mathbf{x}) = 1.0 \#awesome - 1.5 \#awful$

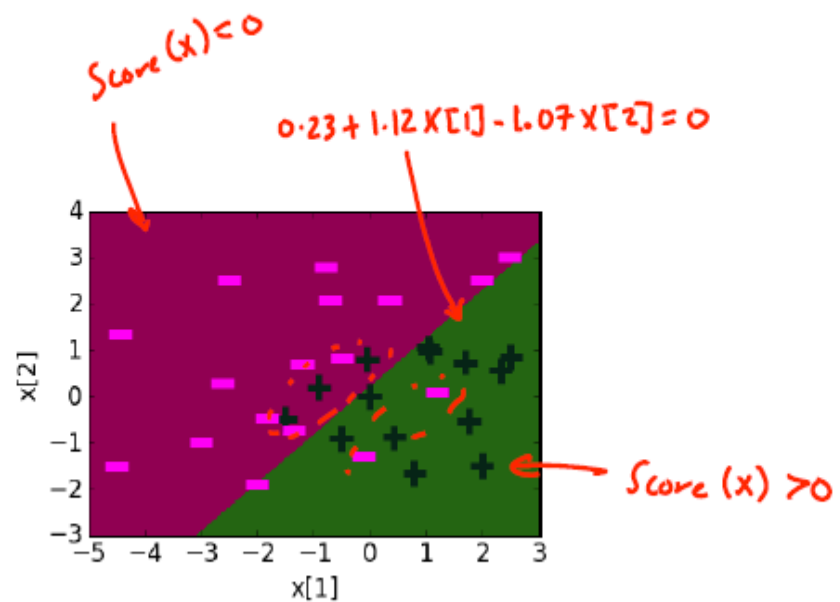
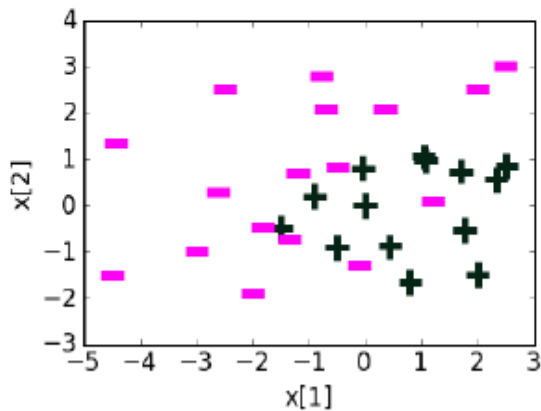


Overfitting in classification

89

Learned decision boundary

Feature	Value	Coefficient learned
$h_0(x)$	$w_0 \cdot 1$	0.23
$h_1(x)$	$w_1 \cdot x[1]$	1.12
$h_2(x)$	$w_2 \cdot x[2]$	-1.07

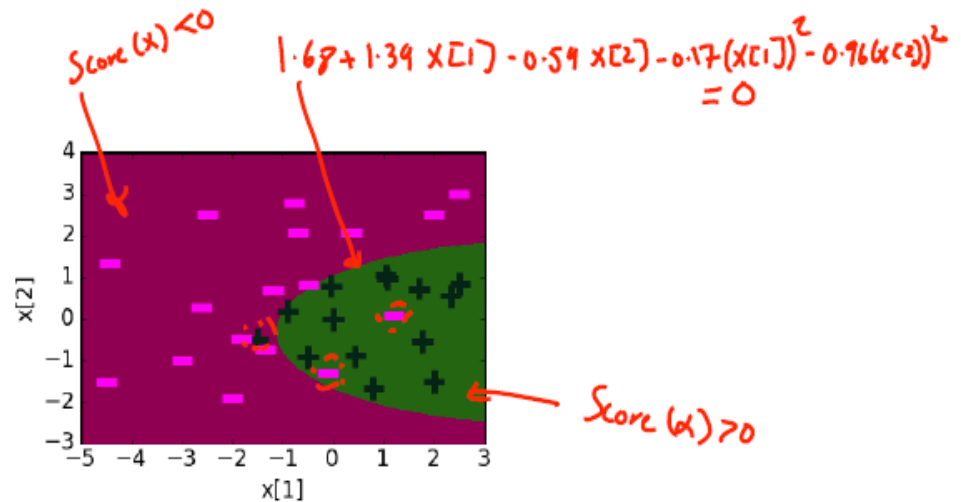
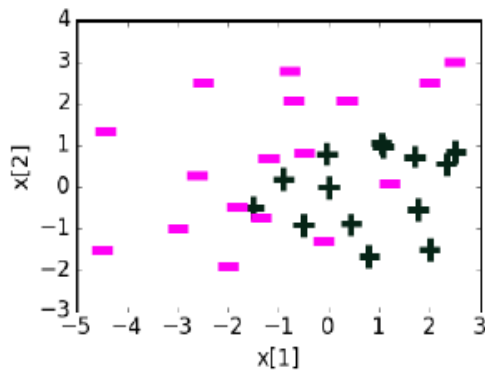


Overfitting in classification

90

Quadratic features (in 2d)

Feature	Value	Coefficient learned
$h_0(x)$	1	1.68
$h_1(x)$	$x[1]$	1.39
$h_2(x)$	$x[2]$	-0.59
$h_3(x)$	$(x[1])^2$	-0.17
$h_4(x)$	$(x[2])^2$	-0.96



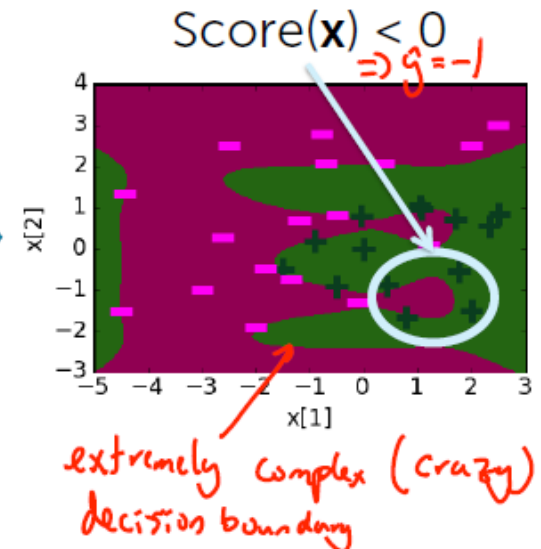
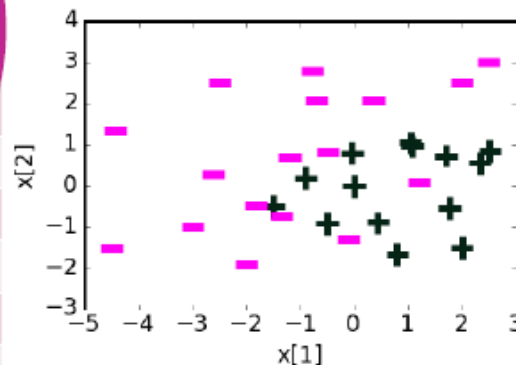
Overfitting in classification

91

Degree 6 features (in 2d)

Feature	Value	Coefficient learned
$h_0(x)$	1	21.6
$h_1(x)$	$x[1]$	5.3
$h_2(x)$	$x[2]$	-42.7
$h_3(x)$	$(x[1])^2$	-15.9
$h_4(x)$	$(x[2])^2$	-48.6
$h_5(x)$	$(x[1])^3$	-11.0
$h_6(x)$	$(x[2])^3$	67.0
$h_7(x)$	$(x[1])^4$	1.5
$h_8(x)$	$(x[2])^4$	48.0
$h_9(x)$	$(x[1])^5$	4.4
$h_{10}(x)$	$(x[2])^5$	-14.2
$h_{11}(x)$	$(x[1])^6$	0.8
$h_{12}(x)$	$(x[2])^6$	-8.6

Coefficient values getting large



18

Overfitting in classification

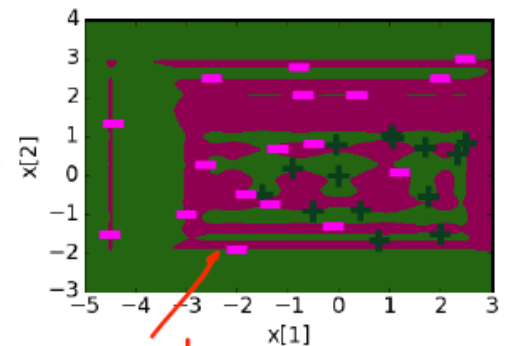
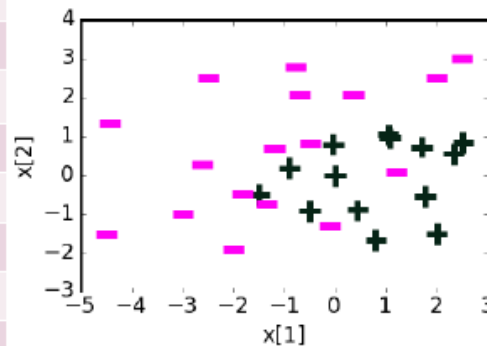
92

Degree 20 features (in 2d)

Feature	Value	Coefficient learned
$h_0(x)$	1	8.7
$h_1(x)$	$x[1]$	5.1
$h_2(x)$	$x[2]$	78.7
...
$h_{11}(x)$	$(x[1])^6$	-7.5
$h_{12}(x)$	$(x[2])^6$	3803
$h_{13}(x)$	$(x[1])^7$	21.1
$h_{14}(x)$	$(x[2])^7$	-2406
...
$h_{37}(x)$	$(x[1])^{19}$	$-2 \cdot 10^{-6}$
$h_{38}(x)$	$(x[2])^{19}$	-0.15
$h_{39}(x)$	$(x[1])^{20}$	$-2 \cdot 10^{-8}$
$h_{40}(x)$	$(x[2])^{20}$	0.03

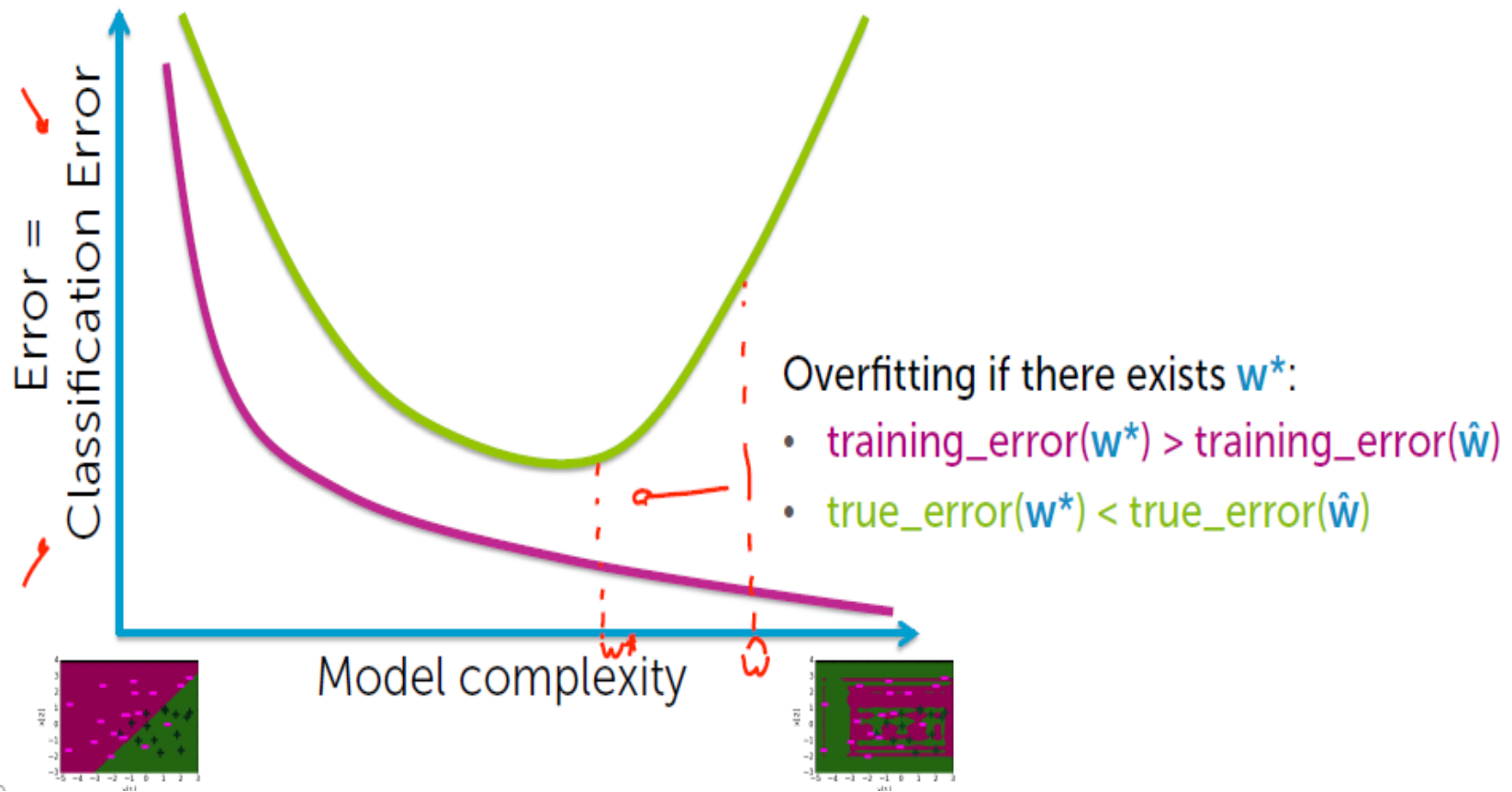
10

Often, overfitting associated with very large estimated coefficients \hat{w}



Overfitting in classification

93



Overfitting in logistic regression

94

The subtle (negative) consequence of overfitting in logistic regression

Overfitting → Large coefficient values

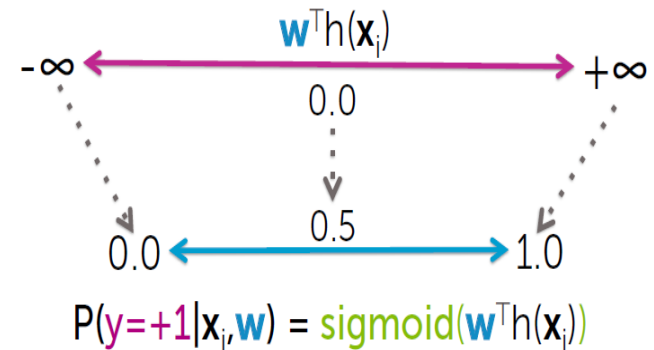


$\hat{w}^T h(\mathbf{x}_i)$ is very positive (or very negative)
→ $\text{sigmoid}(\hat{w}^T h(\mathbf{x}_i))$ goes to 1 (or to 0)



Model becomes extremely overconfident of predictions

Logistic regression model



Remember about this probability interpretation

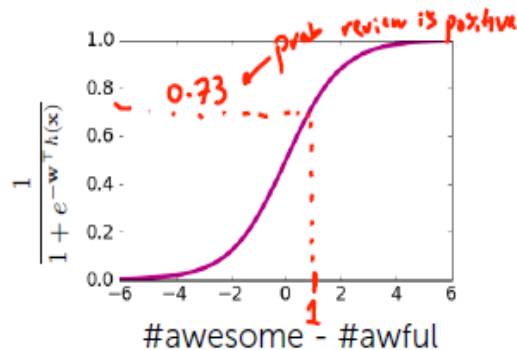
Effect of coefficients on logistic regression model

95

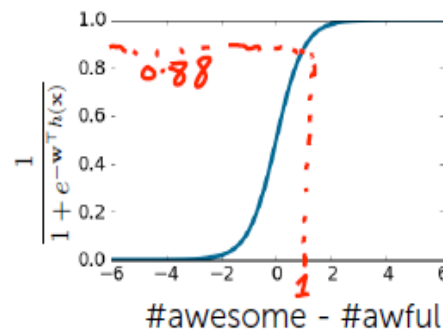
With increasing coefficients model becomes overconfident on predictions

Input x : #awesome=2, #awful=1

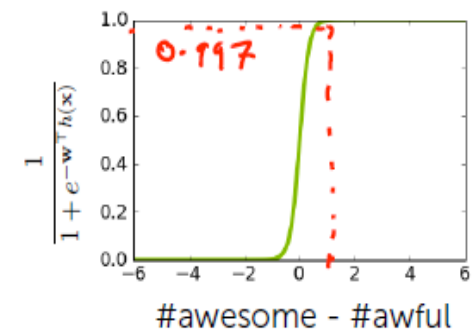
w_0	0
$w_{\text{\#awesome}}$	+1
$w_{\text{\#awful}}$	-1



w_0	0
$w_{\text{\#awesome}}$	+2
$w_{\text{\#awful}}$	-2



w_0	0
$w_{\text{\#awesome}}$	+6
$w_{\text{\#awful}}$	-6



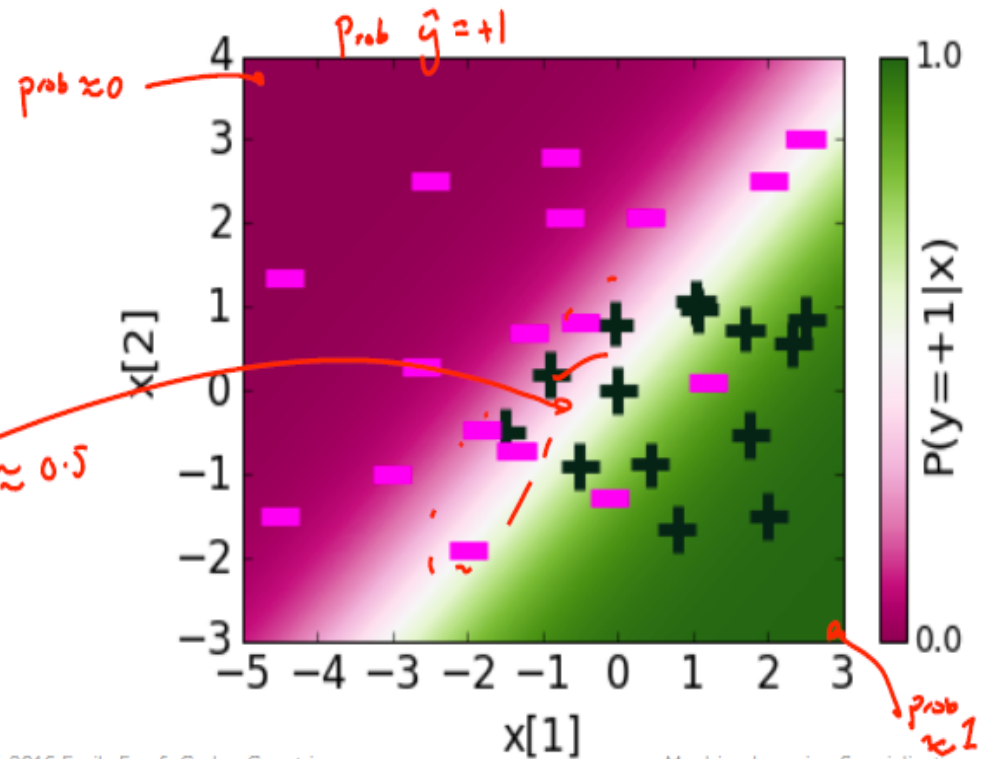
Learned probabilities

96

Feature	Value	Coefficient learned
$h_0(x)$	1	0.23
$h_1(x)$	$x[1]$	1.12
$h_2(x)$	$x[2]$	-1.07

$$P(y = +1 | \mathbf{x}, \mathbf{w}) = \frac{1}{1 + e^{-\mathbf{w}^\top \mathbf{h}(\mathbf{x})}}$$

Make sense
wide region
of uncertainty



27

Quadratic features: learned probabilities

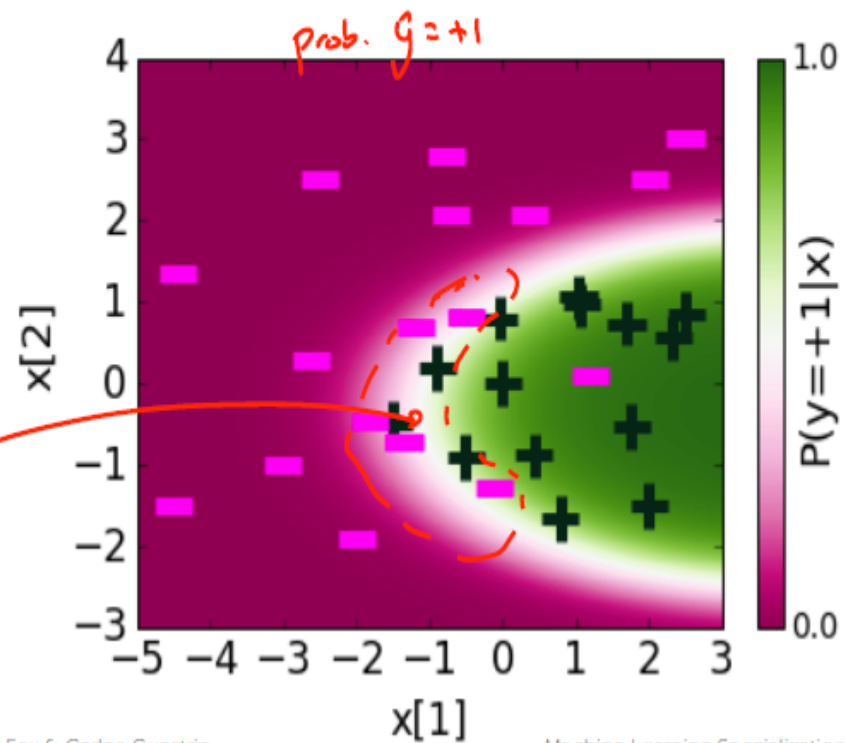
97

Feature	Value	Coefficient learned
$h_0(\mathbf{x})$	1	1.68
$h_1(\mathbf{x})$	$x[1]$	1.39
$h_2(\mathbf{x})$	$x[2]$	-0.58
$h_3(\mathbf{x})$	$(x[1])^2$	-0.17
$h_4(\mathbf{x})$	$(x[2])^2$	-0.96

better fit to data

$$P(y = +1 | \mathbf{x}, \mathbf{w}) = \frac{1}{1 + e^{-\mathbf{w}^\top \mathbf{h}(\mathbf{x})}}$$

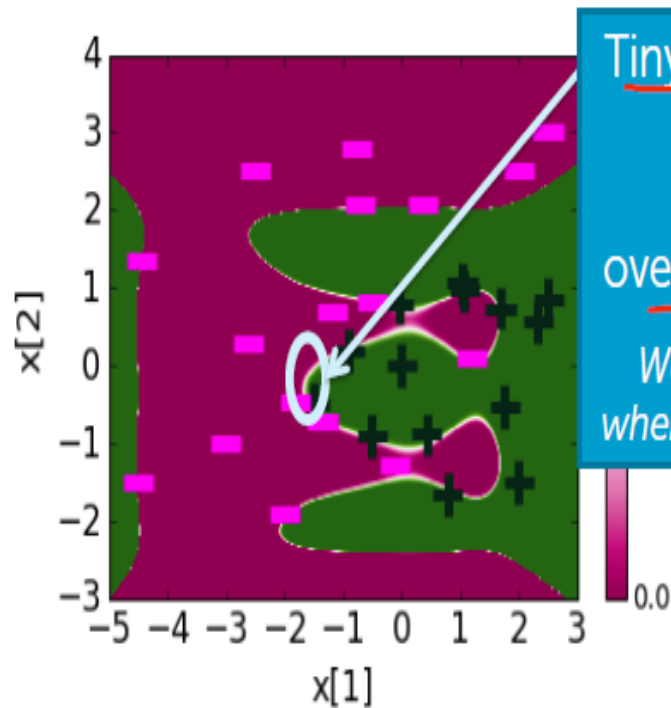
Uncertainty region narrower



Overfitting → overconfident predictions

98

Degree 6: Learned probabilities



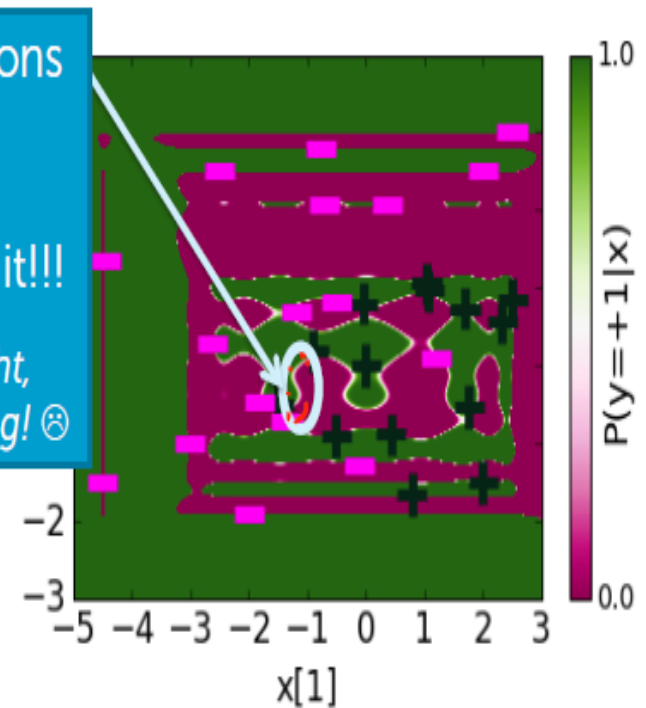
Tiny uncertainty regions



Overfitting & overconfident about it!!!

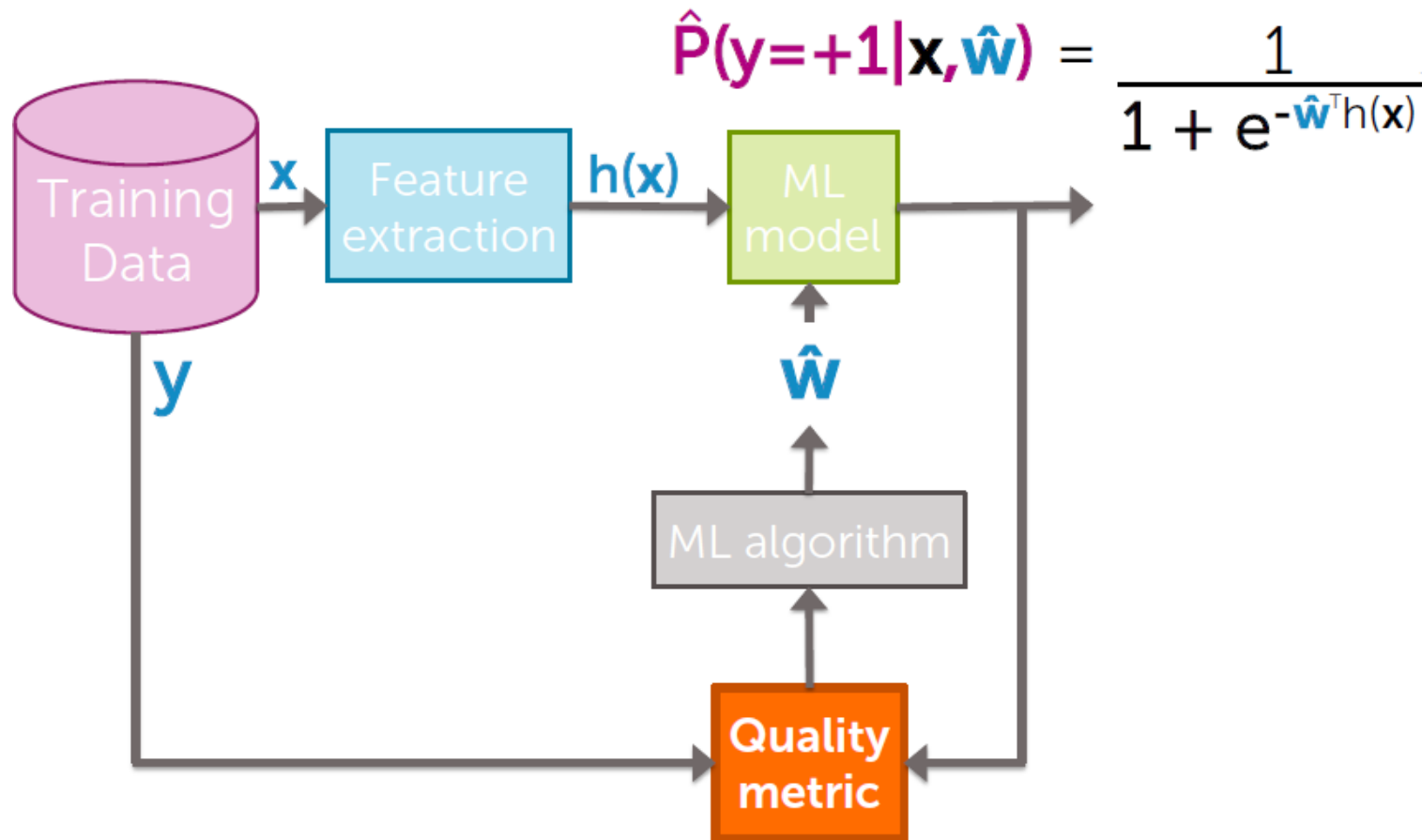
We are sure we are right, when we are surely wrong! ☹️

Degree 20: Learned probabilities



Quality metric \rightarrow penalizing large coefficients

99

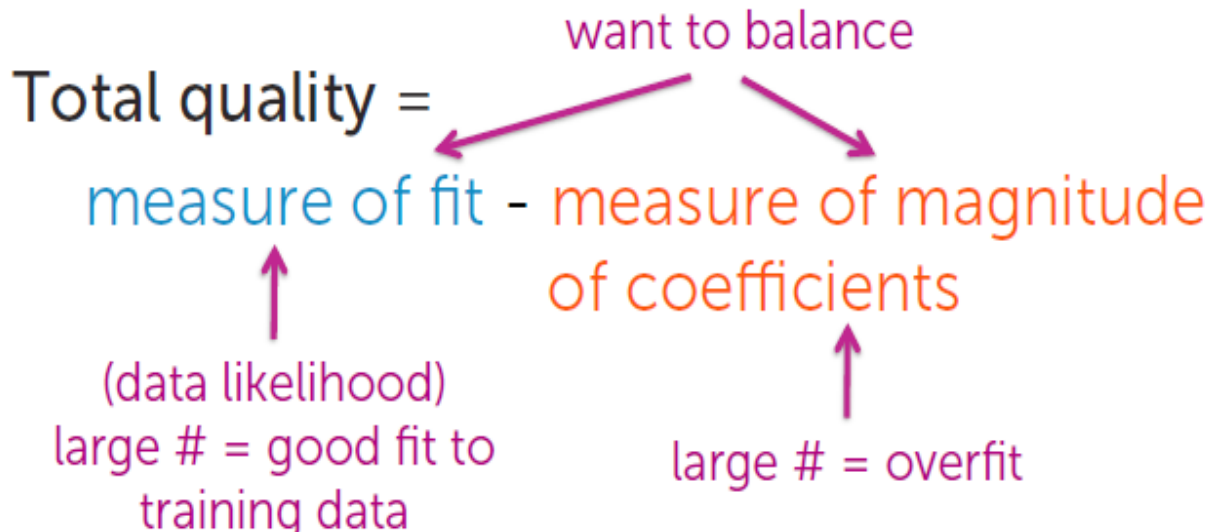


Desired total cost format

100

Want to balance:

- i. How well function fits data
- ii. Magnitude of coefficients



Maximum likelihood estimation (MLE)

101

□ Measure of fit = Data likelihood

- Choose coefficients \mathbf{w} that maximize likelihood:

$$\prod_{i=1}^N P(y_i | \mathbf{x}_i, \mathbf{w})$$

- Typically, we use the log of likelihood function (simplifies math and has better convergence properties) ← !!!

$$\ell(\mathbf{w}) = \ln \prod_{i=1}^N P(y_i | \mathbf{x}_i, \mathbf{w})$$

Measure of magnitude of logistic regression coefficients

102

What summary # is indicative of size of logistic regression coefficients?

- Sum of squares (L_2 norm)

$$\|w\|_2^2 = w_0^2 + w_1^2 + w_2^2 + \dots + w_D^2$$

Penalize large coefficients

- Sum of absolute value (L_1 norm)

$$\|w\|_1 = |w_0| + |w_1| + |w_2| + \dots + |w_D|$$

Sparse solution

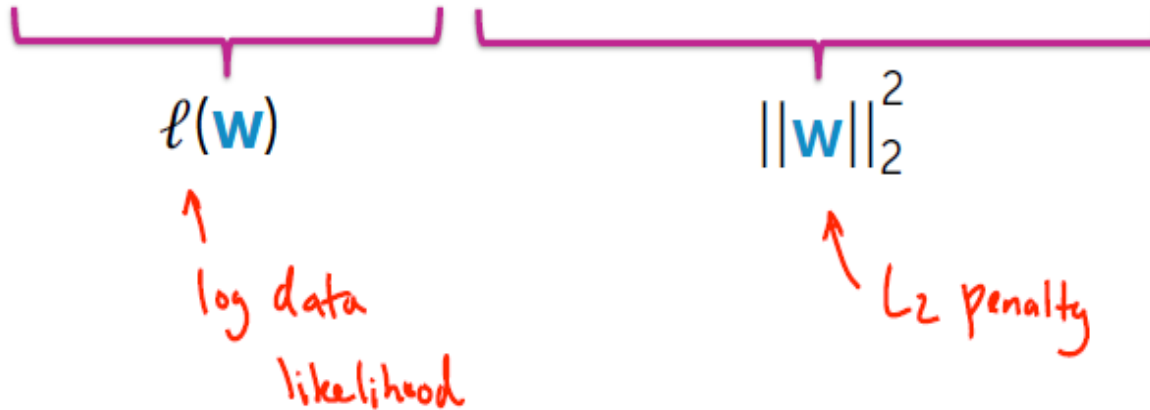
Consider specific total cost

103

max
w

Total quality =

measure of fit - measure of magnitude
of coefficients



Consider resulting objectives

104

What if $\hat{\mathbf{w}}$ selected to minimize

$$\ell(\mathbf{w}) - \lambda \|\mathbf{w}\|_2^2$$

↑ tuning parameter = balance of fit and magnitude

If $\lambda=0$:

→ Reduces $\max_{\mathbf{w}} \ell(\mathbf{w}) \rightarrow$ Standard (unpenalized) MLE solution

If $\lambda=\infty$:

→ $\max_{\mathbf{w}} \ell(\mathbf{w}) - \infty \|\mathbf{w}\|_2^2 \rightarrow$ only care about penalizing \mathbf{w} , large coefficients $\rightarrow \mathbf{w}=0$

If λ in between:

→ Balance data fit against the magnitude of the coefficients

45

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Machine Learning Foundations

9/11, 16/11, 23/11 2022

Consider resulting objectives

105

What if $\hat{\mathbf{w}}$ selected to minimize

$$\ell(\mathbf{w}) - \lambda \|\mathbf{w}\|_2^2$$

 tuning parameter = balance of fit and magnitude

L_2 regularized
logistic regression

Pick λ using:

- Validation set (for large datasets)
- Cross-validation (for smaller datasets)

Bias-variance tradeoff

106

Large λ :

high bias, low variance

(e.g., $\hat{\mathbf{w}} = 0$ for $\lambda = \infty$)

In essence, λ
controls model
complexity

Small λ :

low bias, high variance

(e.g., maximum likelihood (MLE) fit of
high-order polynomial for $\lambda = 0$)

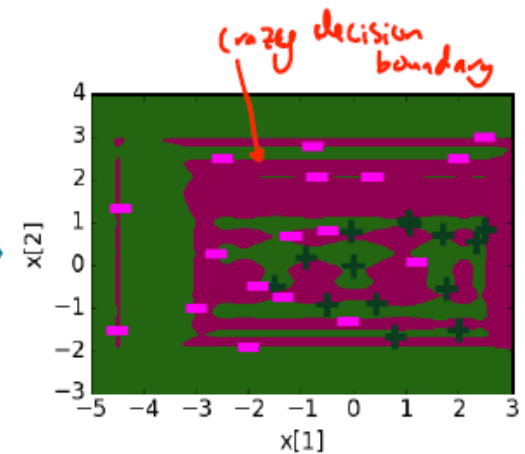
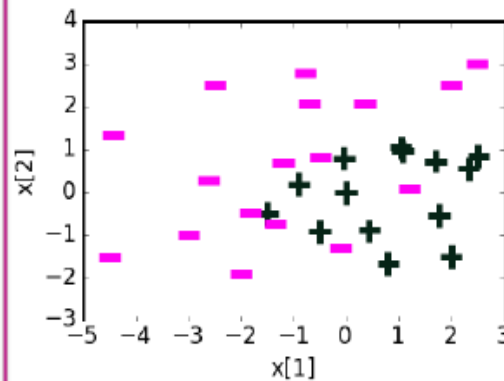
Visualizing effect of regularisation

107

Degree 20 features, $\lambda=0$

Feature	Value	Coefficient learned
$h_0(\mathbf{x})$	1	8.7
$h_1(\mathbf{x})$	$x[1]$	5.1
$h_2(\mathbf{x})$	$x[2]$	78.7
...
$h_{11}(\mathbf{x})$	$(x[1])^6$	-7.5
$h_{12}(\mathbf{x})$	$(x[2])^6$	3803
$h_{13}(\mathbf{x})$	$(x[1])^7$	21.1
$h_{14}(\mathbf{x})$	$(x[2])^7$	-2406
...
$h_{37}(\mathbf{x})$	$(x[1])^{19}$	$-2 \cdot 10^{-6}$
$h_{38}(\mathbf{x})$	$(x[2])^{19}$	-0.15
$h_{39}(\mathbf{x})$	$(x[1])^{20}$	$-2 \cdot 10^{-8}$
$h_{40}(\mathbf{x})$	$(x[2])^{20}$	0.03

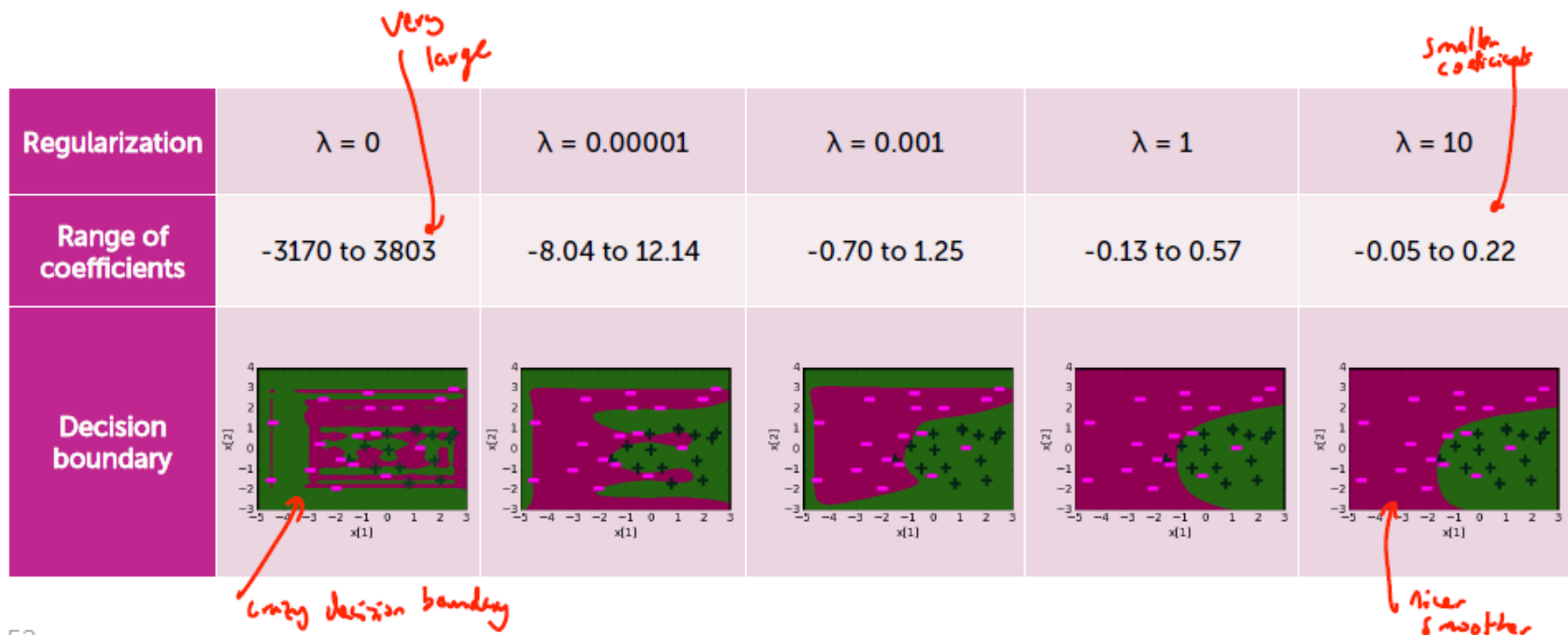
Coefficients range from -3170 to 3803



Visualizing effect of regularisation

108

Degree 20 features,
effect of regularization penalty λ

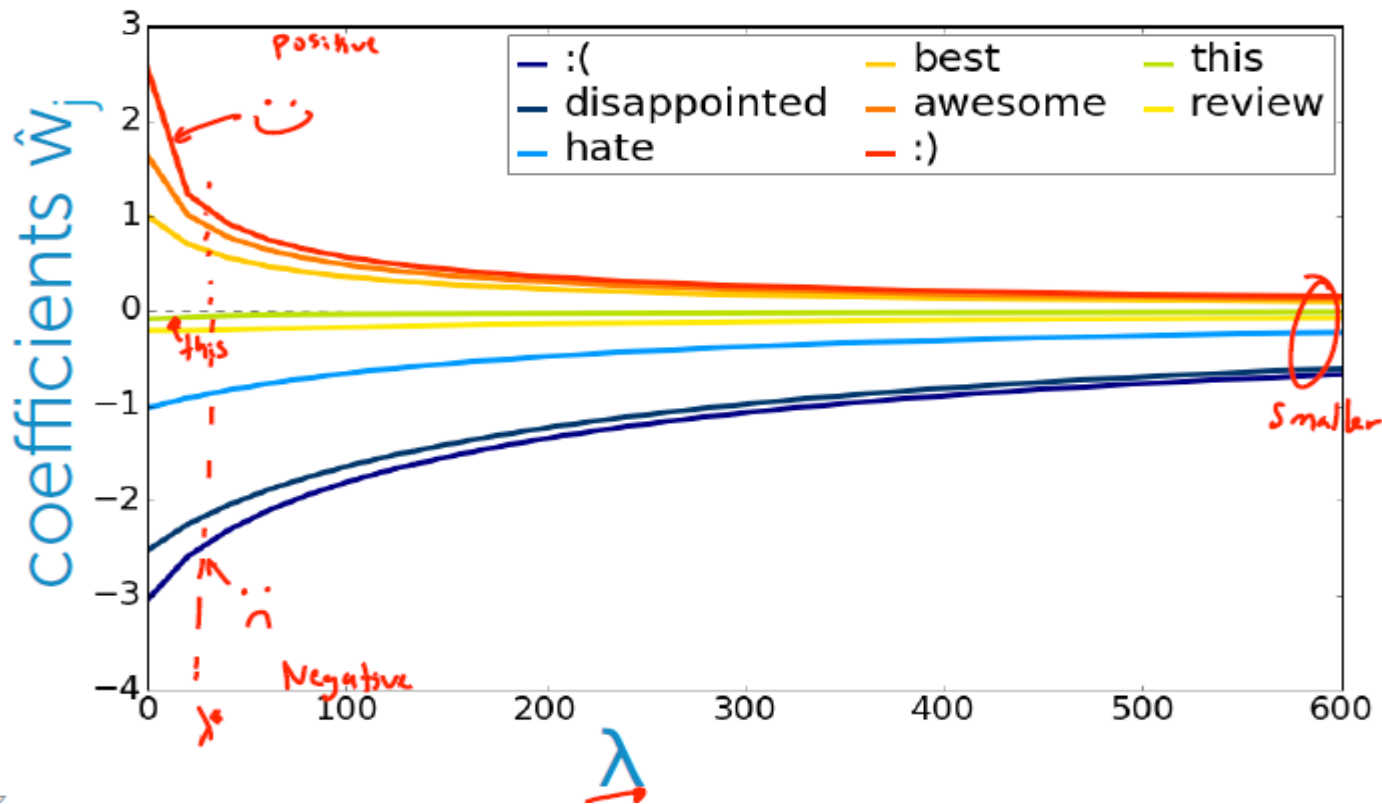


59

Effect of regularisation

109

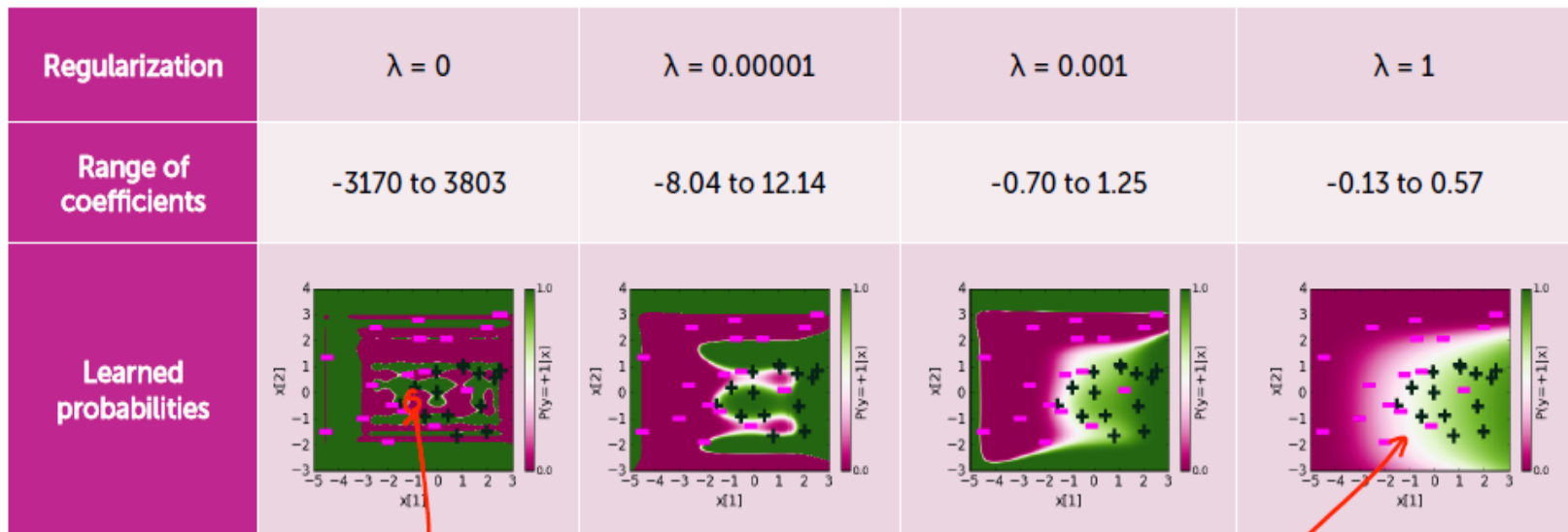
Coefficient path



Visualizing effect of regularisation

110

Degree 20 features:
regularization reduces "overconfidence"



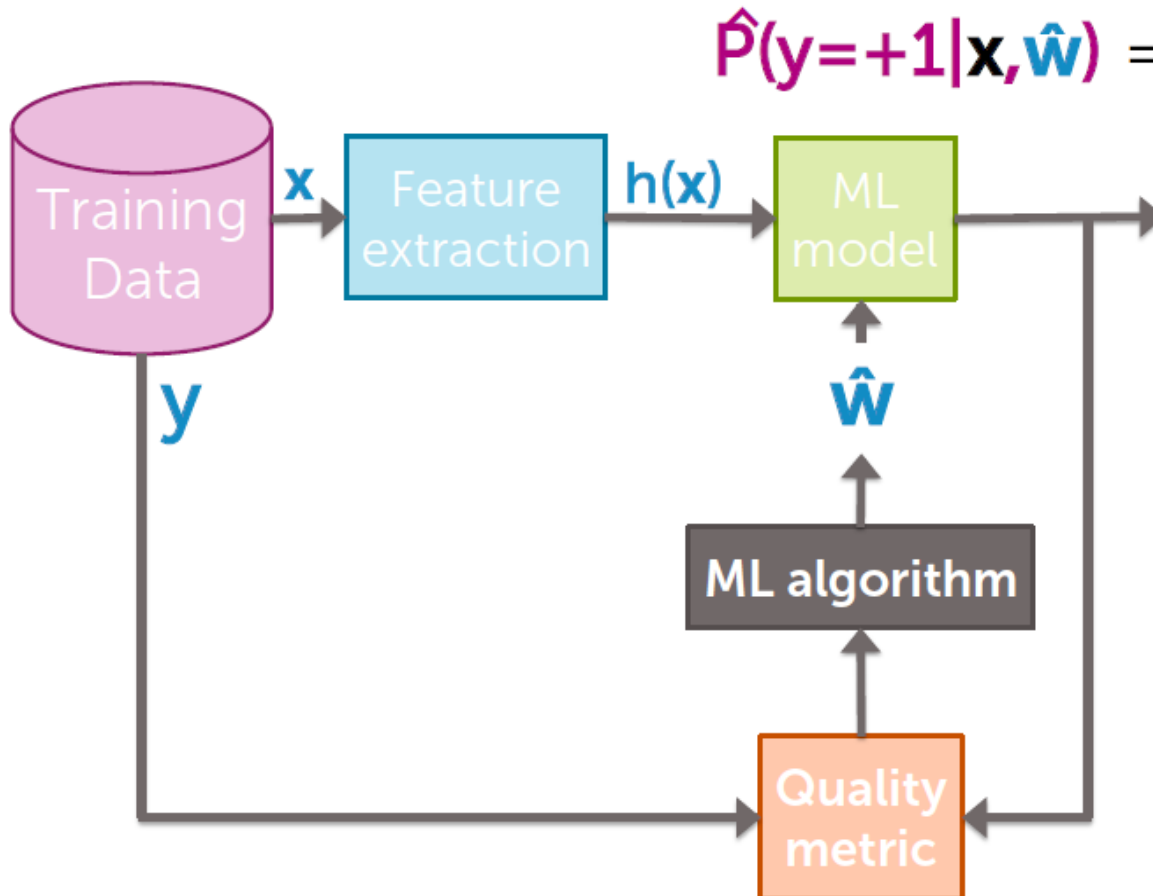
highly over confident

very natural uncertainty region

Flow chart:

ML algorithm

111

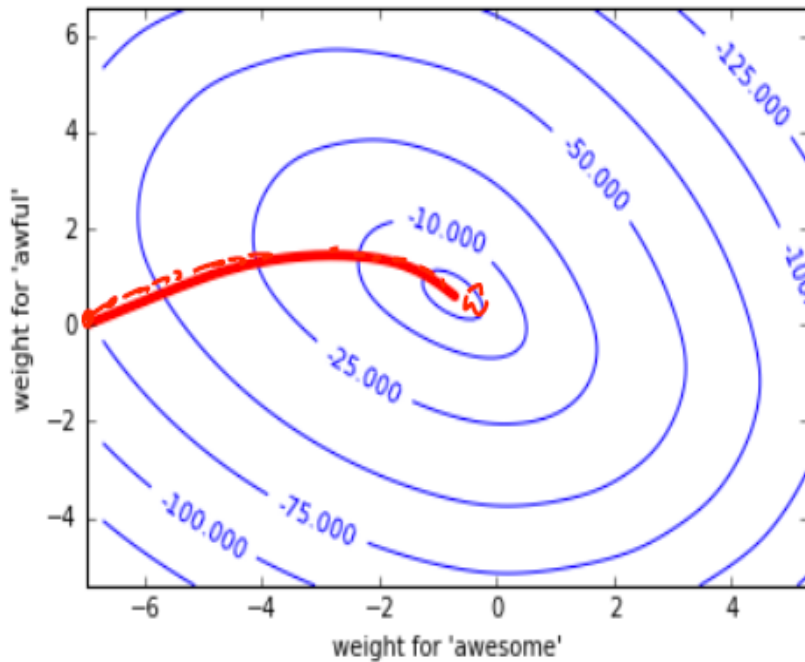


$$\hat{P}(y=+1|\mathbf{x}, \hat{\mathbf{w}}) = \frac{1}{1 + e^{-\hat{\mathbf{w}}^T h(\mathbf{x})}}$$

*Lets discuss now
finding best
L2-regularized
linear classifier
with gradient ascent*

Gradient ascent

112



Algorithm:

while not converged

$$\underline{w}^{(t+1)} \leftarrow \underline{w}^{(t)} + \eta \nabla \ell(\underline{w}^{(t)})$$

*need the gradient of
regularized log likelihood*

Gradient of L2 regularized log-likelihood

113

Total quality =
measure of fit - measure of magnitude
of coefficients

A diagram showing two terms under purple brackets. The first term is $\ell(\mathbf{w})$ and the second term is $\lambda \|\mathbf{w}\|_2^2$. Purple arrows point from each term down to the corresponding terms in the derivative equation below.

$$\text{Total derivative} = \frac{\partial \ell(\mathbf{w})}{\partial \mathbf{w}_j} - \lambda \frac{\partial \|\mathbf{w}\|_2^2}{\partial \mathbf{w}_j}$$

Gradient of L2 regularized log-likelihood

114

Derivative of (log-)likelihood

$$\frac{\partial \ell(\mathbf{w})}{\partial \mathbf{w}_j} = \sum_{i=1}^N h_j(\mathbf{x}_i) \left(\mathbb{1}[y_i = +1] - P(y = +1 | \mathbf{x}_i, \mathbf{w}) \right)$$

Derivative of L₂ penalty

$$\frac{\partial \|\mathbf{w}\|_2^2}{\partial \mathbf{w}_j} = \frac{\partial}{\partial w_j} [w_0^2 + w_1^2 + w_2^2 + \dots + w_j^2 + \dots + w_D^2] = 2w_j$$

Gradient of L2 regularized log-likelihood

115

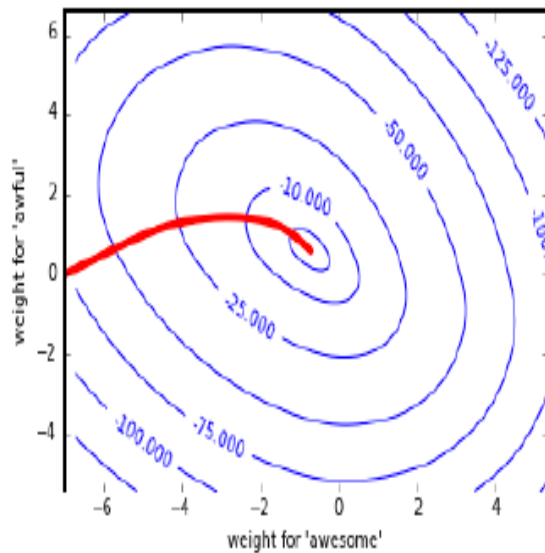
Understanding contribution of L_2 regularization

$$\frac{\partial \ell(\mathbf{w})}{\partial w_j} \quad \overbrace{- 2\lambda w_j}^{\text{Term from } L_2 \text{ penalty}}$$

	$- 2 \lambda w_j$	Impact on w_j
$w_j > 0$	< 0	decreases $w_j \Rightarrow w_j$ becomes closer to 0
$w_j < 0$	> 0	increases $w_j \Rightarrow w_j$ becomes closer to 0

Gradient ascent with L2 regularization

116



init $\mathbf{w}^{(1)} = 0$ (or randomly, or smartly), $t=1$

while not converged:

for $j=0, \dots, D$

$$\text{partial}[j] = \sum_{i=1}^N h_j(\mathbf{x}_i) \left(1[y_i = +1] - P(y = +1 | \mathbf{x}_i, \mathbf{w}^{(t)}) \right)$$

$$\mathbf{w}_j^{(t+1)} \leftarrow \mathbf{w}_j^{(t)} + \eta \left(\text{partial}[j] - 2\lambda \mathbf{w}_j^{(t)} \right)$$

$$t \leftarrow t + 1$$

step size

$\frac{\partial \mathcal{L}(\mathbf{w})}{\partial w_j}$

only change w_j !!

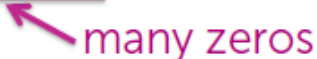
Logistic regression with L1 regularization

117

Recall **sparsity** (many $\hat{w}_j=0$)
gives efficiency and interpretability

Efficiency:

- If $\text{size}(\mathbf{w}) = 100\text{B}$, each prediction is expensive
- If $\hat{\mathbf{w}}$ **sparse**, computation only depends on # of non-zeros



$$\hat{y}_i = \text{sign} \left(\sum_{\hat{w}_j \neq 0} \hat{w}_j h_j(\mathbf{x}_i) \right)$$

Interpretability:

- Which features are relevant for prediction?

Sparse logistic regression

118

Total quality =

measure of fit - measure of magnitude
of coefficients

The diagram shows two horizontal curly braces. The first brace is under the text 'measure of fit' and is labeled with the mathematical expression $\ell(\mathbf{w})$. The second brace is under the text 'measure of magnitude of coefficients' and is labeled with the mathematical expression $\|\mathbf{w}\|_1 = |w_0| + \dots + |w_D|$.

L_1 regularized
logistic regression

Leads to
sparse
solutions!

L1 regularised logistic regression

119

Just like L2 regularization, solution is governed by a continuous parameter λ

$$\ell(\mathbf{w}) - \lambda \|\mathbf{w}\|_1$$

tuning parameter =
balance of fit and sparsity

If $\lambda=0$:

→ No regularization → standard MLE solution

If $\lambda=\infty$:

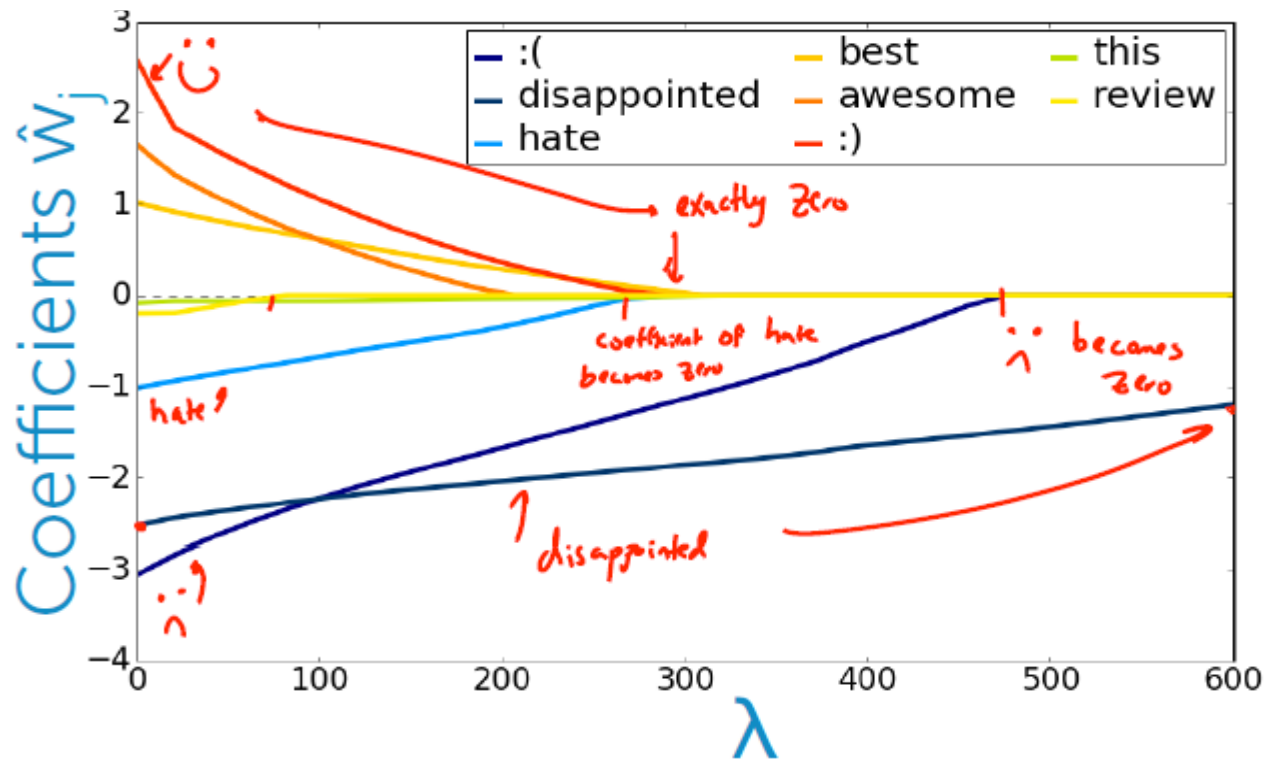
→ all weight is on regularization → $\hat{\mathbf{w}} = \mathbf{0}$

If λ in between:

→ Sparse solutions: some $\hat{w}_i \neq 0$, many other $\hat{w}_i = 0$

L1 regularised logistic regression

120



What you can do now...

121

- Identify when overfitting is happening
- Relate large learned coefficients to overfitting
- Describe the impact of overfitting on decision boundaries and predicted probabilities of linear classifiers
- Motivate the form of L_2 regularized logistic regression quality metric
- Describe what happens to estimated coefficients as tuning parameter λ is varied
- Interpret coefficient path plot
- Estimate L_2 regularized logistic regression coefficients using gradient ascent
- Describe the use of L_1 regularization to obtain sparse logistic regression solutions

Decision trees

What makes a loan risky?

123

I want a to buy a new house!

A sample loan application form with various fields and text, including a header that reads 'New Homeowner's Loan Application'.

Loan Application



Credit
★★★★

Income
★★★

Term
★★★★★

Personal Info
★★★

Credit history explained

124

Did I pay previous loans on time?



Example: excellent, good, or fair

Credit History



Income



Term



Personal Info

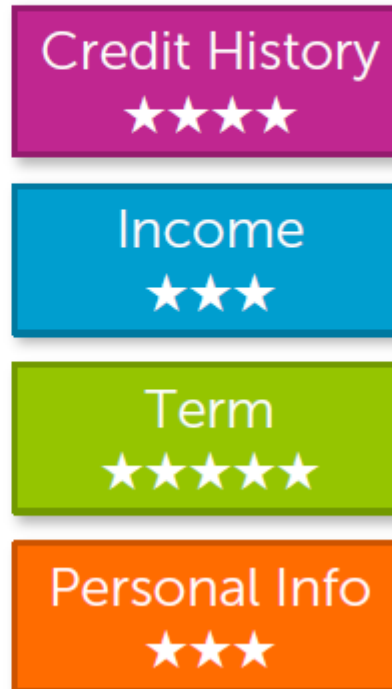


Income

125

What's my income?

Example:
\$80K per year



Loan terms

126

How soon do I need to pay the loan?

Example: 3 years,
5 years,...



Credit History
★★★★★

Income
★★★

Term
★★★★★

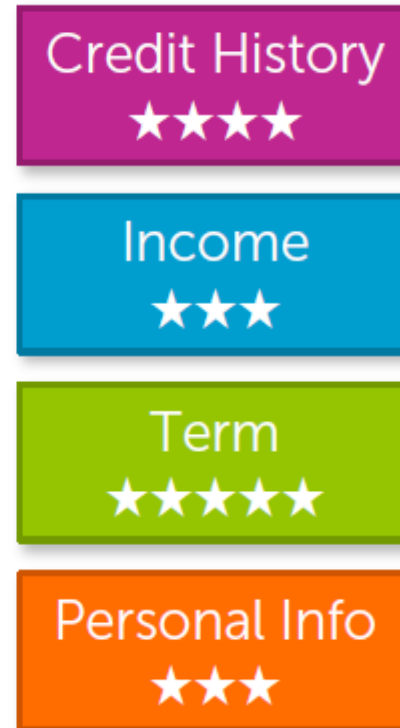
Personal Info
★★★

Personal information

127

Age, reason for the loan, marital status,...

Example: Home loan for a married couple



Intelligent application

128

Loan Applications



Intelligent loan application review system

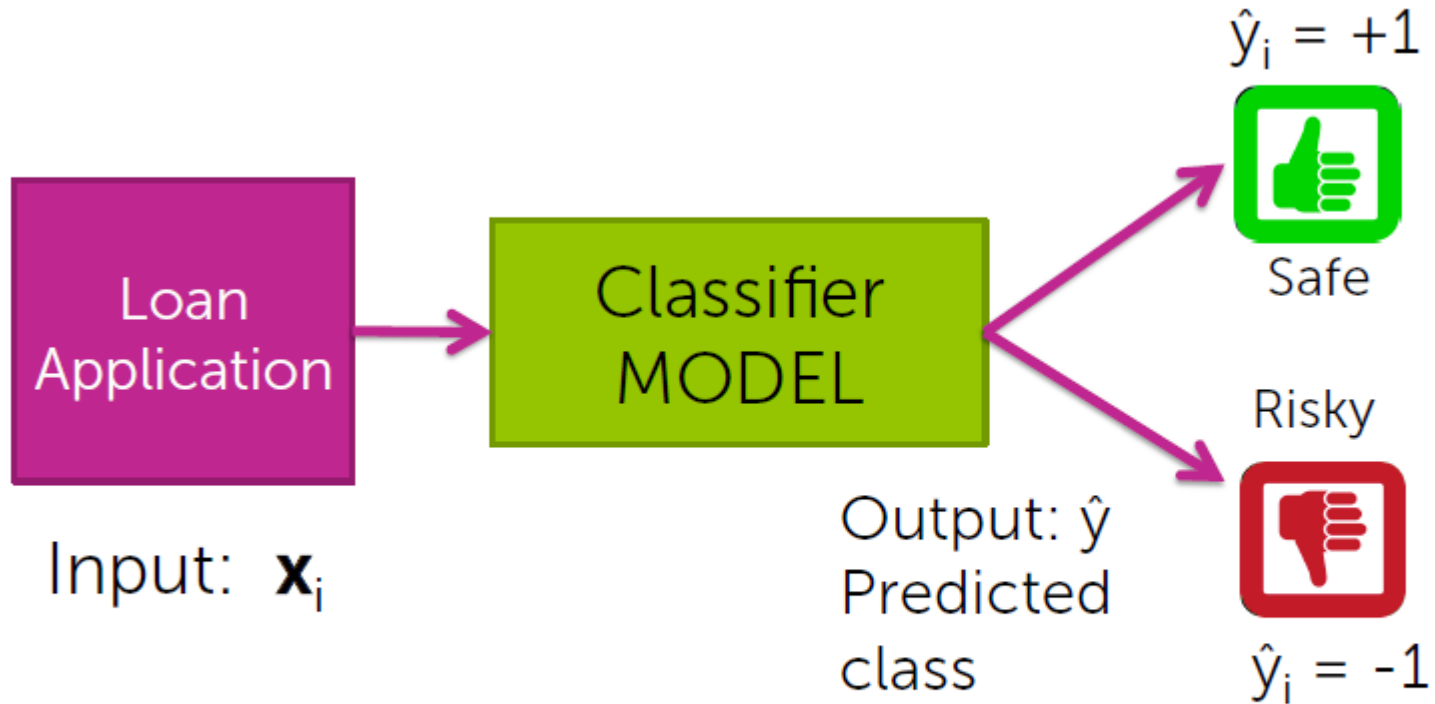
Safe
✓

Risky
✗

Risky
✗

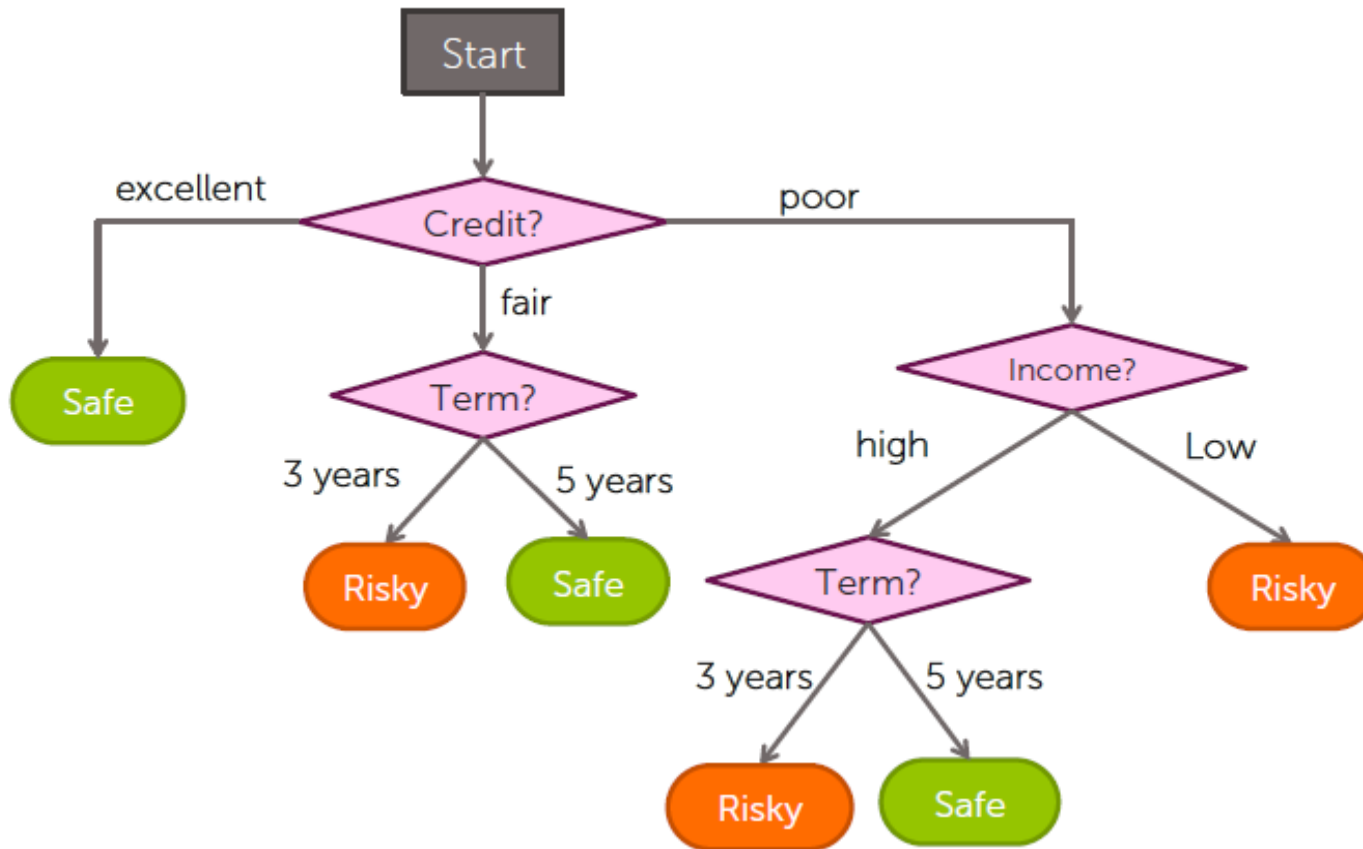
Classifier: review type

129



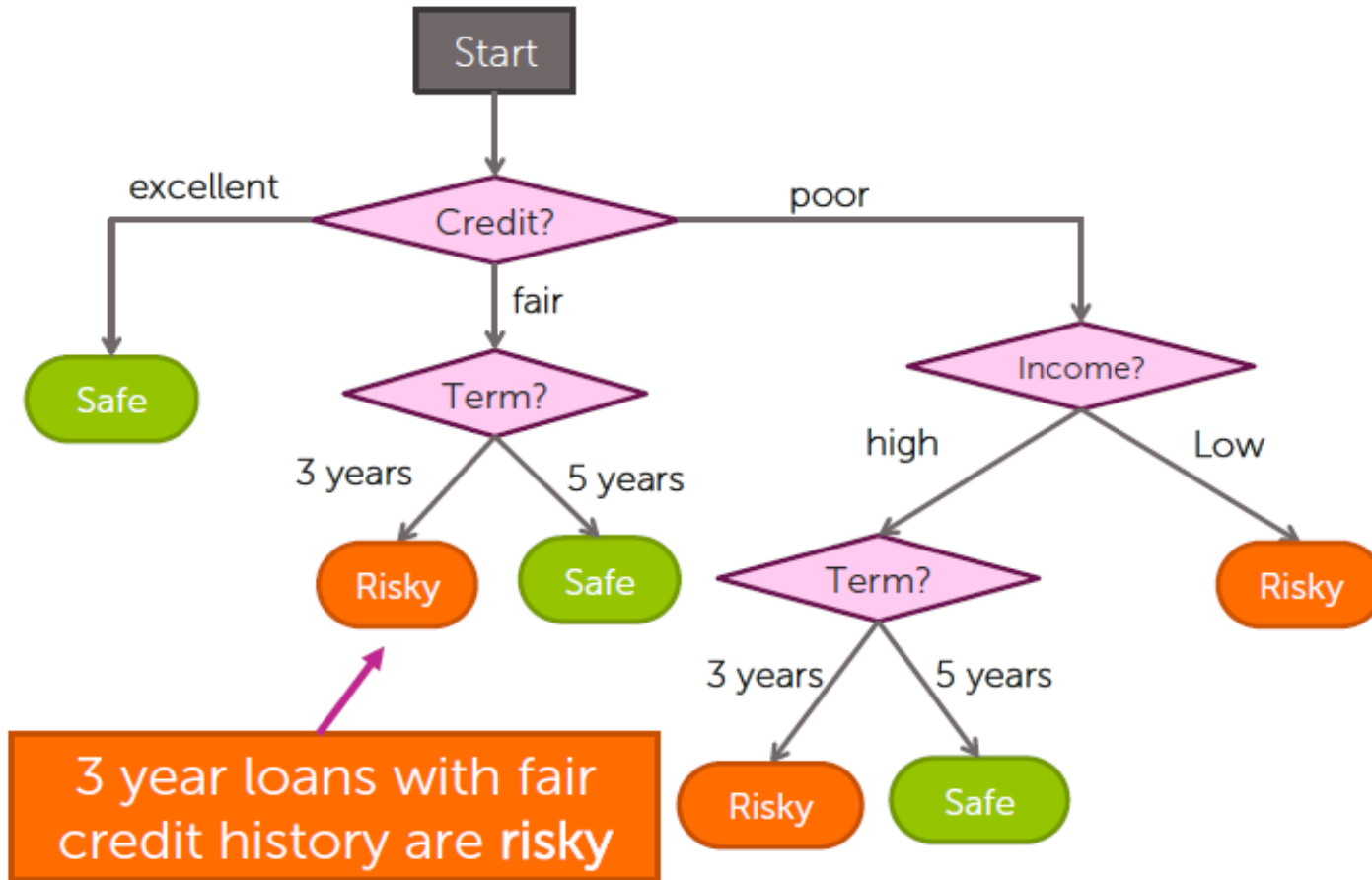
Classifier: decision trees

130



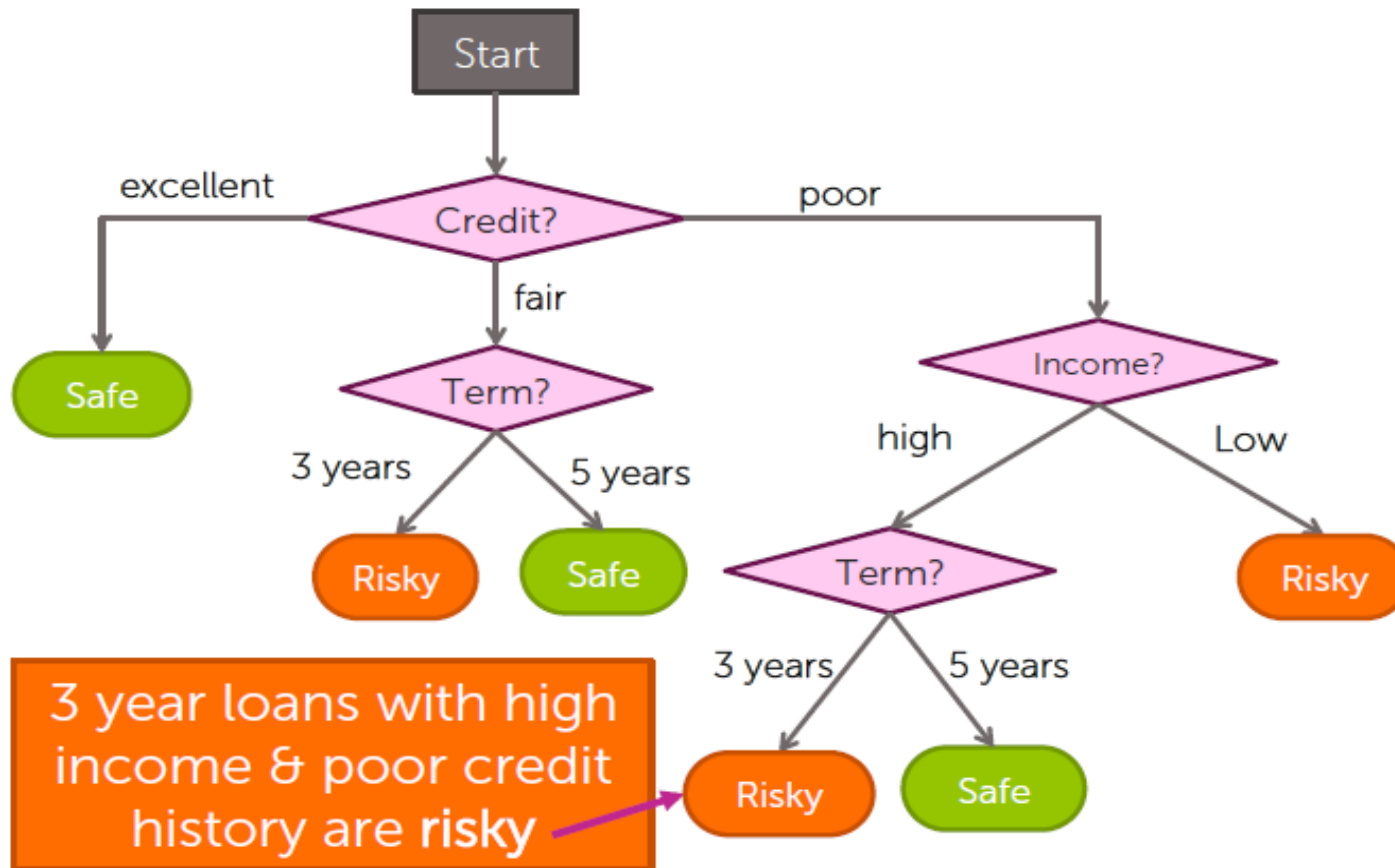
Scoring a loan application

131



Scoring a loan application

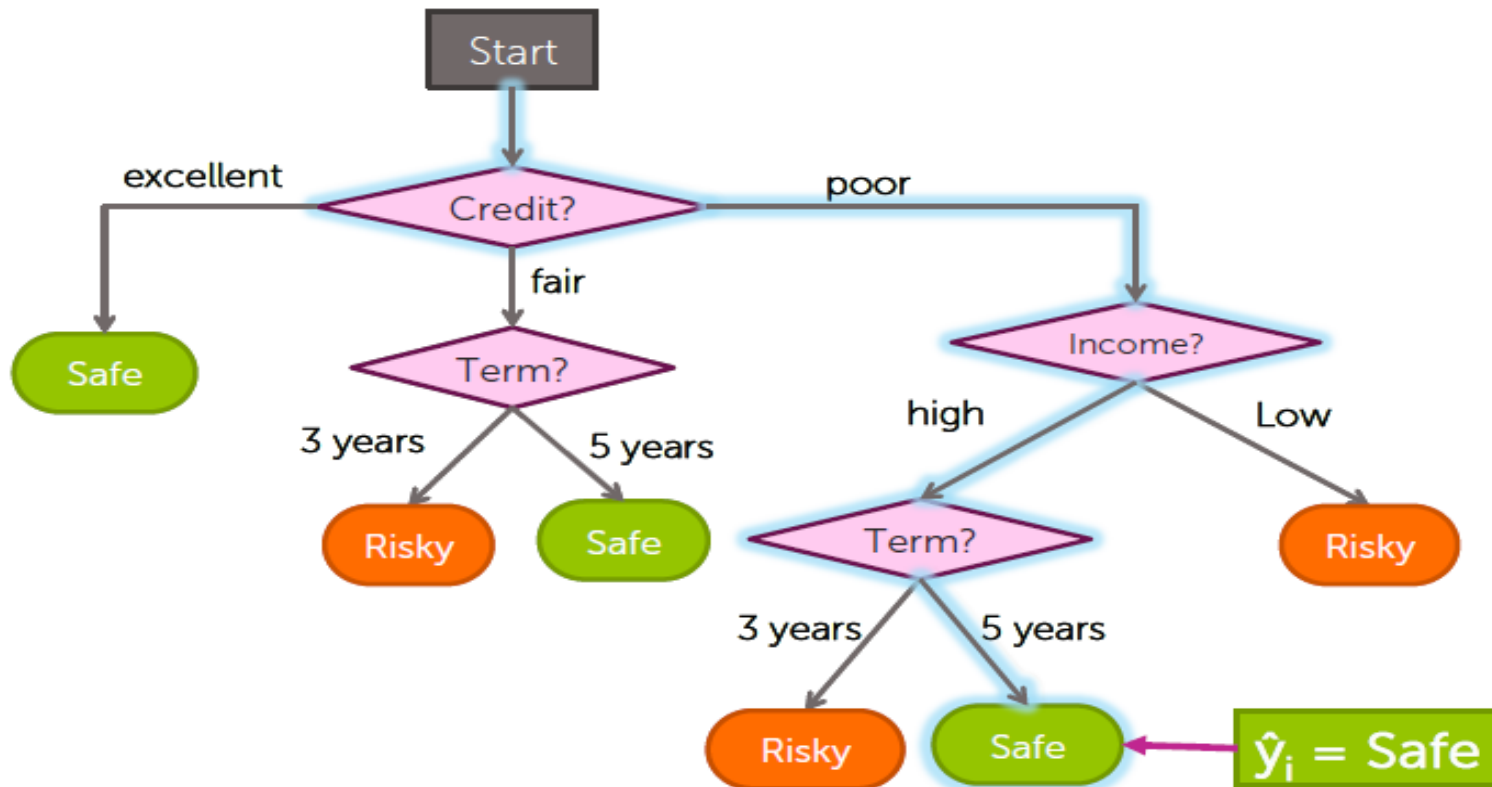
132



Scoring a loan application

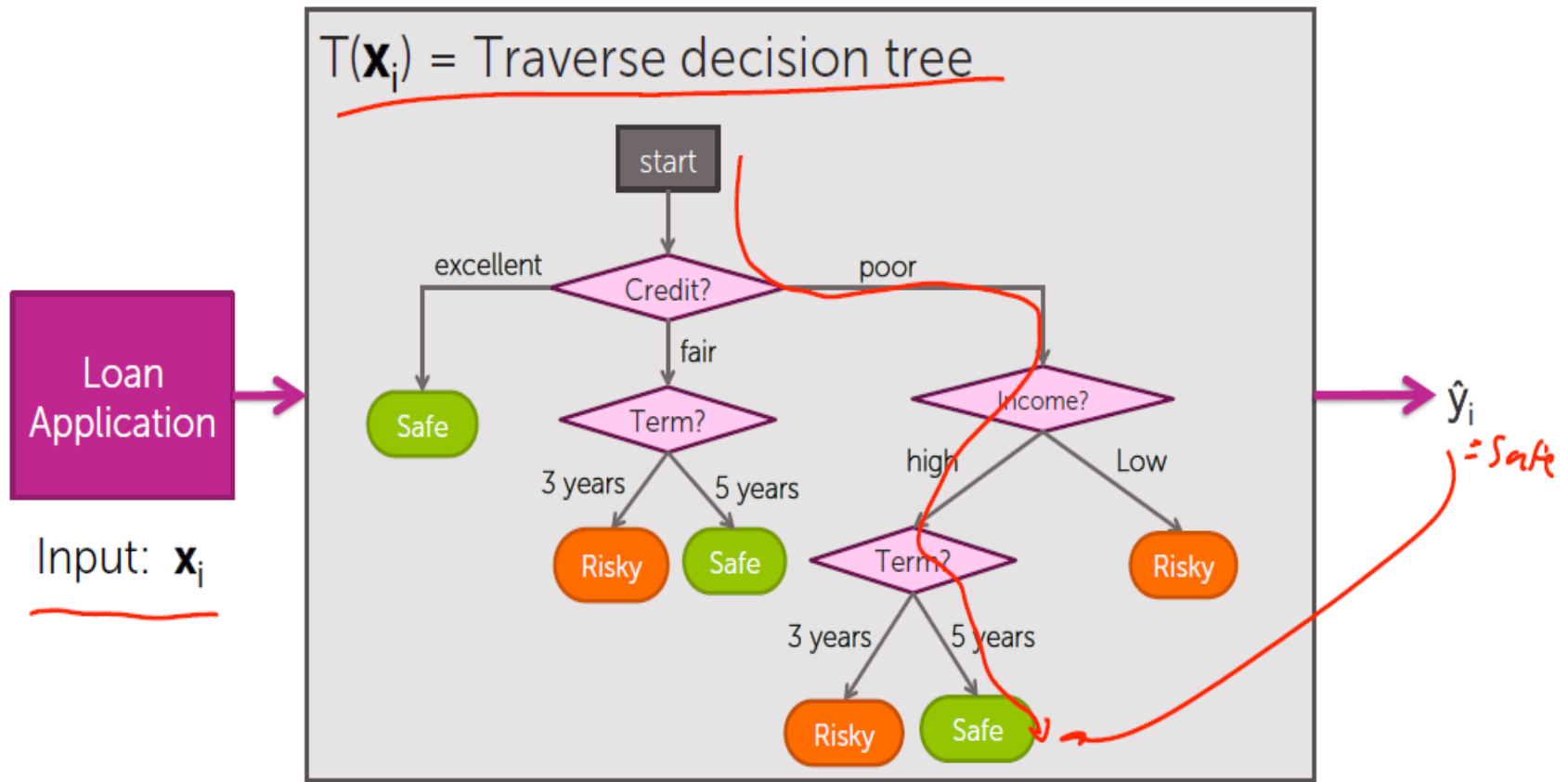
133

$\mathbf{x}_i = (\text{Credit} = \text{poor}, \text{Income} = \text{high}, \text{Term} = 5 \text{ years})$



Decision tree model

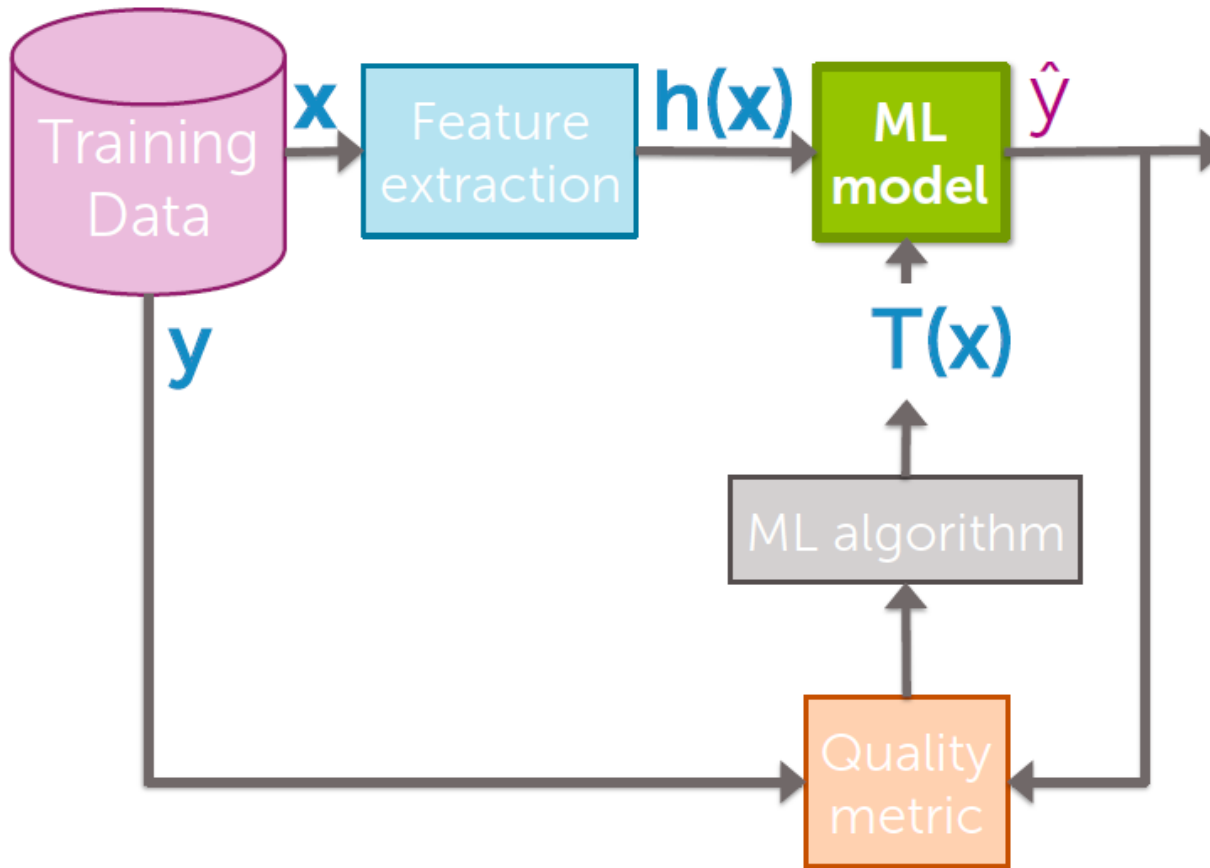
134



Flow chart:



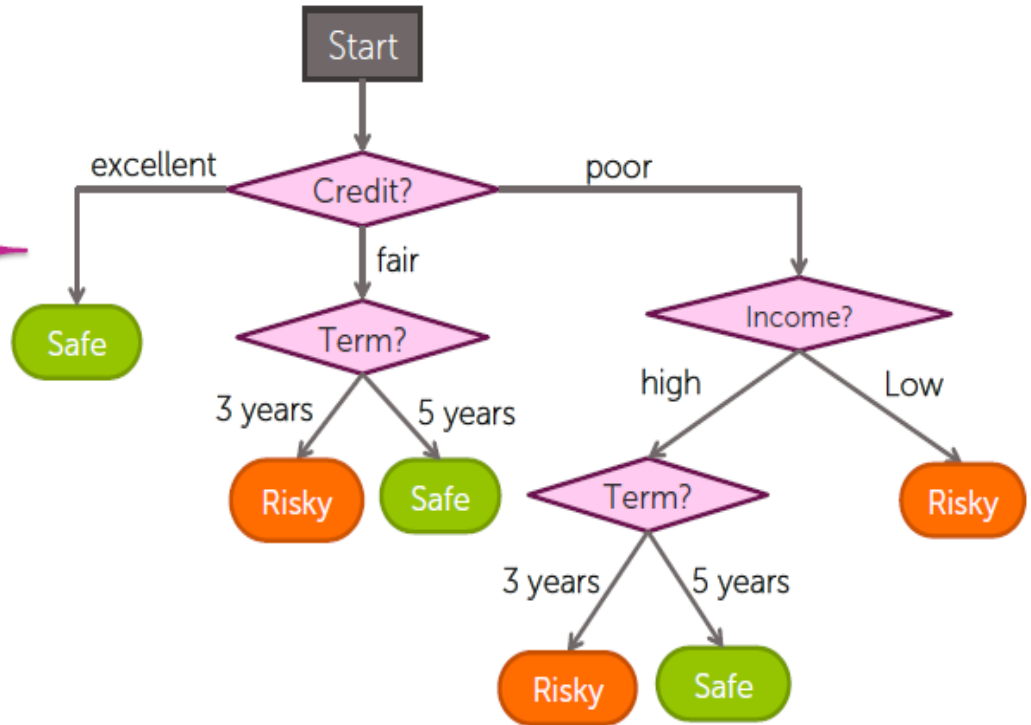
135



Learn decision tree from data

136

$h_1(x)$	$h_2(x)$	$h_3(x)$	Learn status
Credit	Term	Income	y
excellent	3 yrs	high	safe
fair	5 yrs	low	risky
fair	3 yrs	high	safe
poor	5 yrs	high	risky
excellent	3 yrs	low	risky
fair	5 yrs	low	safe
poor	3 yrs	high	risky
poor	5 yrs	low	safe
fair	3 yrs	high	safe

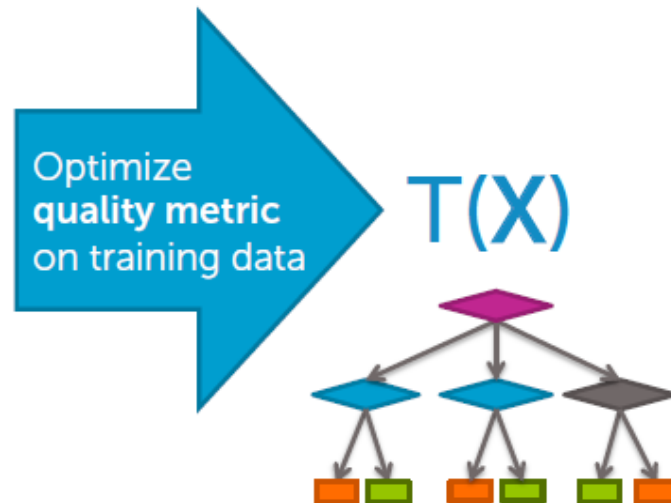


Learn decision tree from data

137

Training data: N observations (\mathbf{x}_i, y_i)

Credit	Term	Income	y
excellent	3 yrs	high	safe
fair	5 yrs	low	risky
fair	3 yrs	high	safe
poor	5 yrs	high	risky
excellent	3 yrs	low	risky
fair	5 yrs	low	safe
poor	3 yrs	high	risky
poor	5 yrs	low	safe
fair	3 yrs	high	safe



Quality metric: Classification error

138

- Error measures fraction of mistakes

$$\text{Error} = \frac{\text{\# incorrect predictions}}{\text{\# examples}}$$

- Best possible value : 0.0
- Worst possible value: 1.0

Find the tree with lowest classification error

139

Credit	Term	Income	y
excellent	3 yrs	high	safe
fair	5 yrs	low	risky
fair	3 yrs	high	safe
poor	5 yrs	high	risky
excellent	3 yrs	low	risky
fair	5 yrs	low	safe
poor	3 yrs	high	risky
poor	5 yrs	low	safe
fair	3 yrs	high	safe

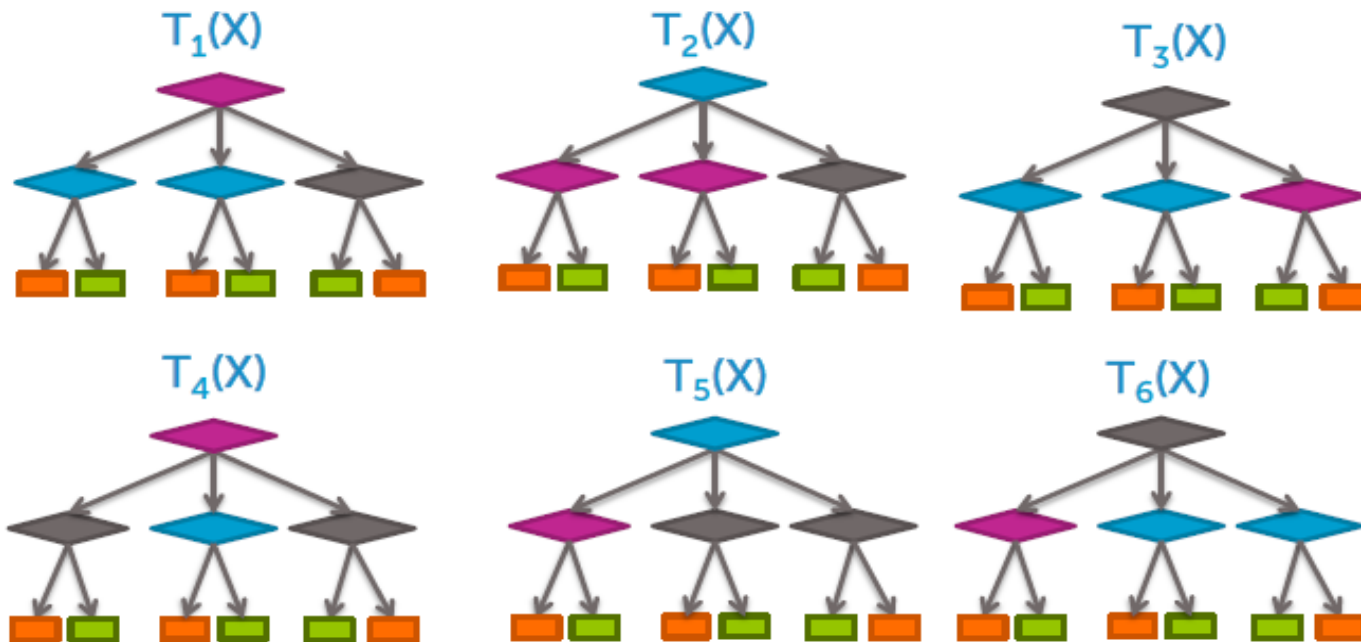
Minimize
classification error
on training data



How do we find the best tree?

140

Exponentially large number of possible trees makes decision tree learning **hard!**
(*NP-hard problem*)



Simple (greedy) algorithm finds good tree

141

Credit	Term	Income	y
excellent	3 yrs	high	safe
fair	5 yrs	low	risky
fair	3 yrs	high	safe
poor	5 yrs	high	risky
excellent	3 yrs	low	risky
fair	5 yrs	low	safe
poor	3 yrs	high	risky
poor	5 yrs	low	safe
fair	3 yrs	high	safe

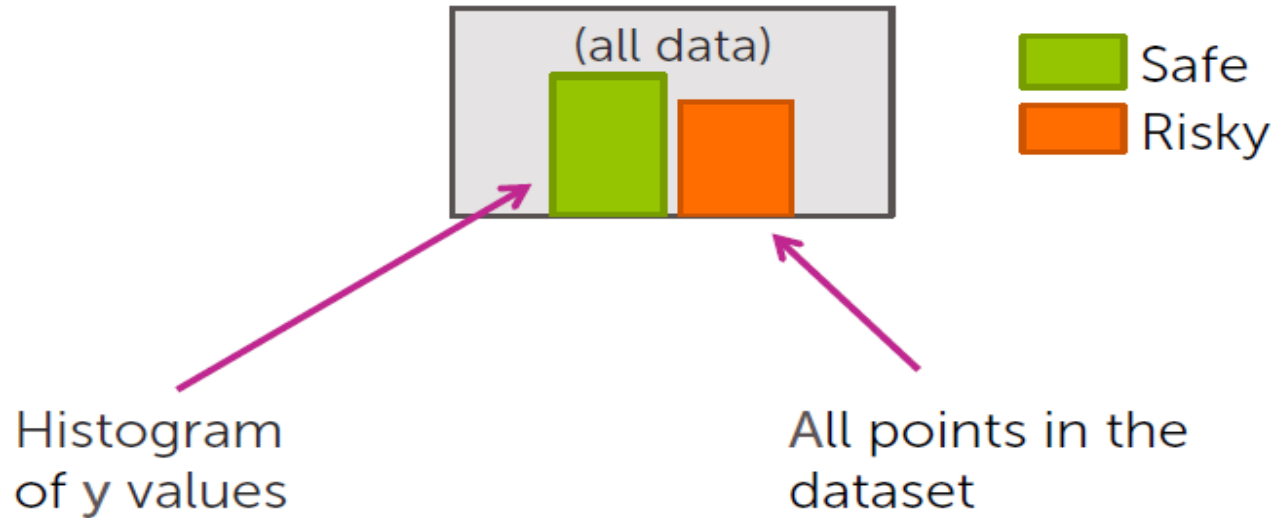
Approximately
minimize
classification error
on training data



Greedy algorithm

142

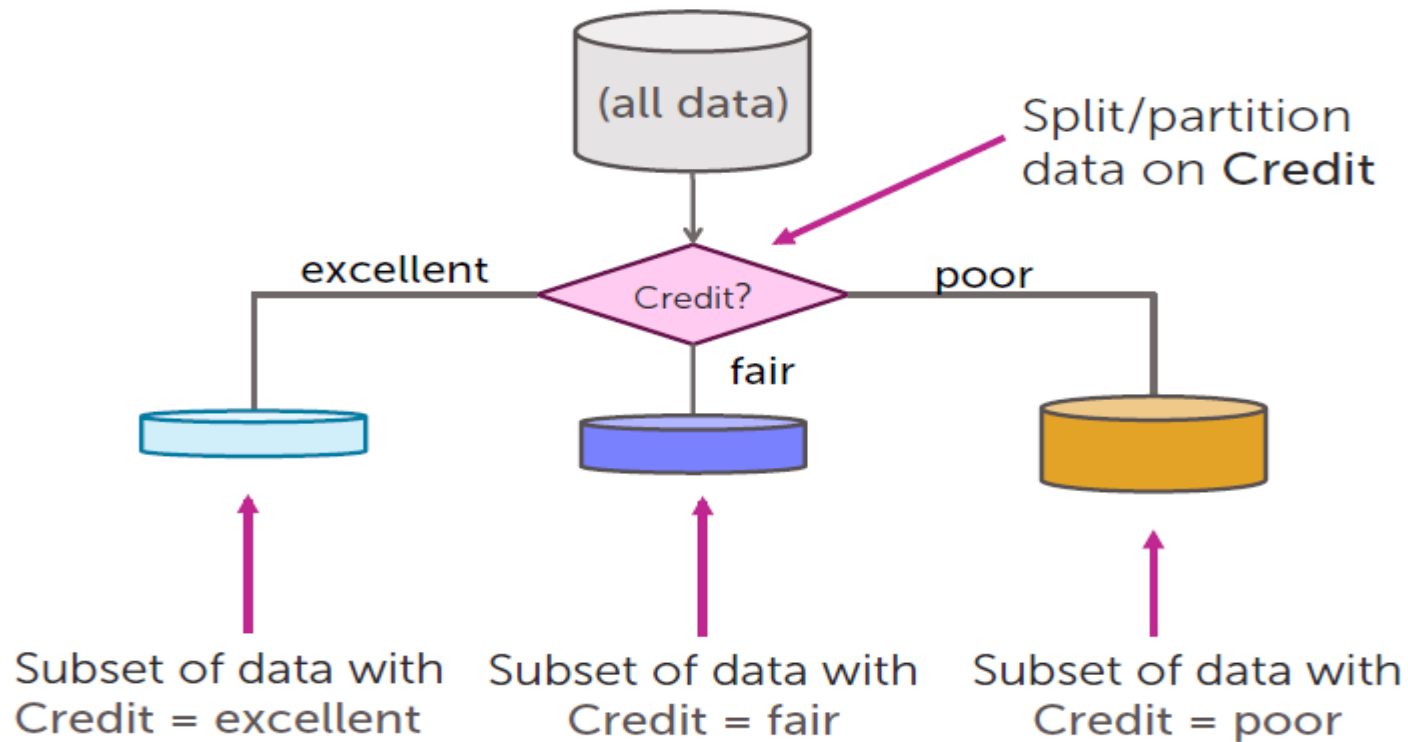
Step 1: Start with an empty tree



Greedy algorithm

143

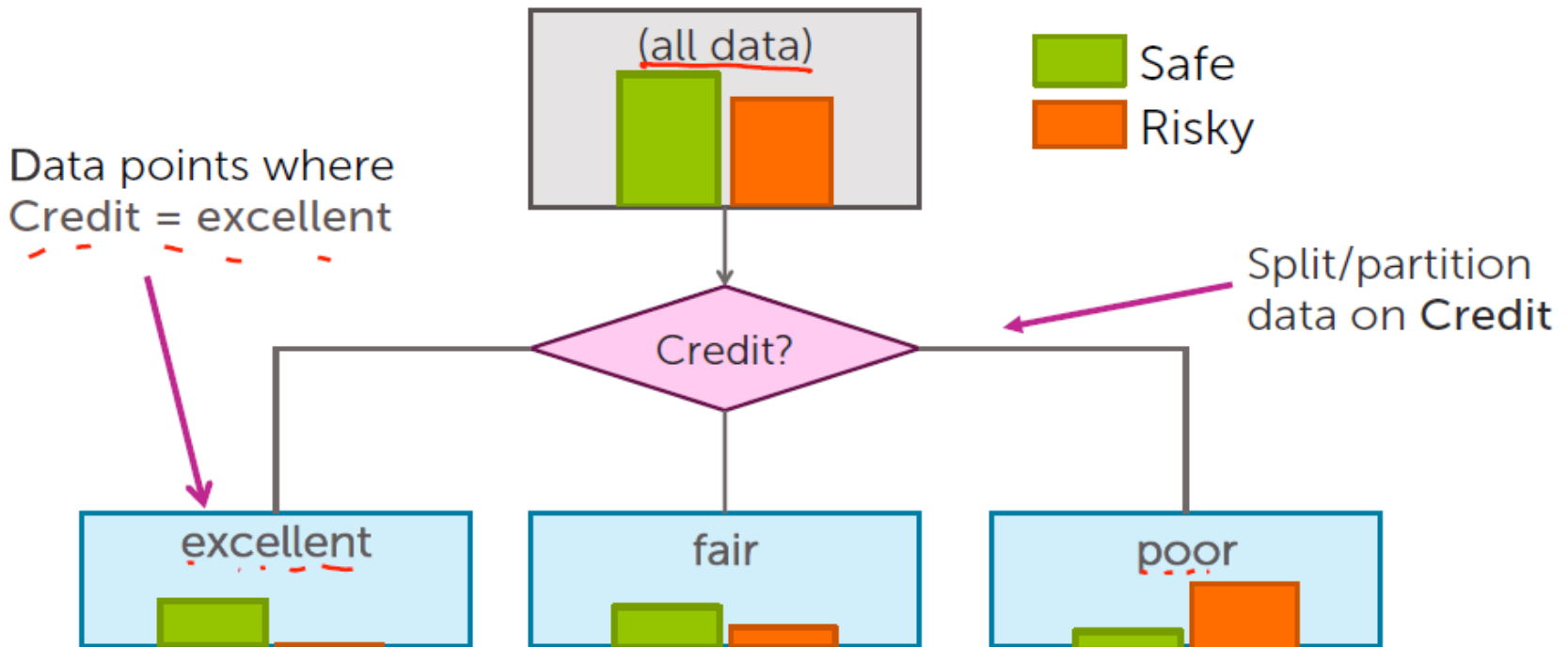
Step 2: Split on a feature



Greedy algorithm

144

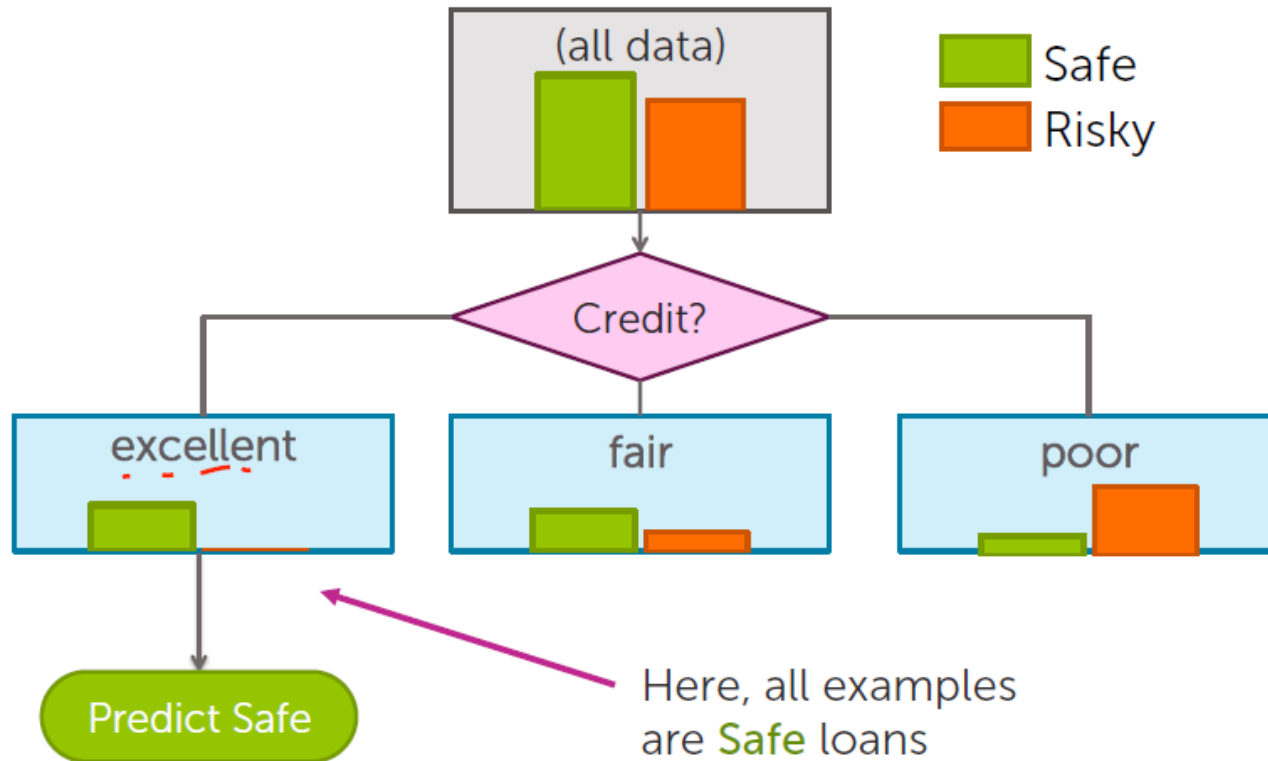
Feature split explained



Greedy algorithm

145

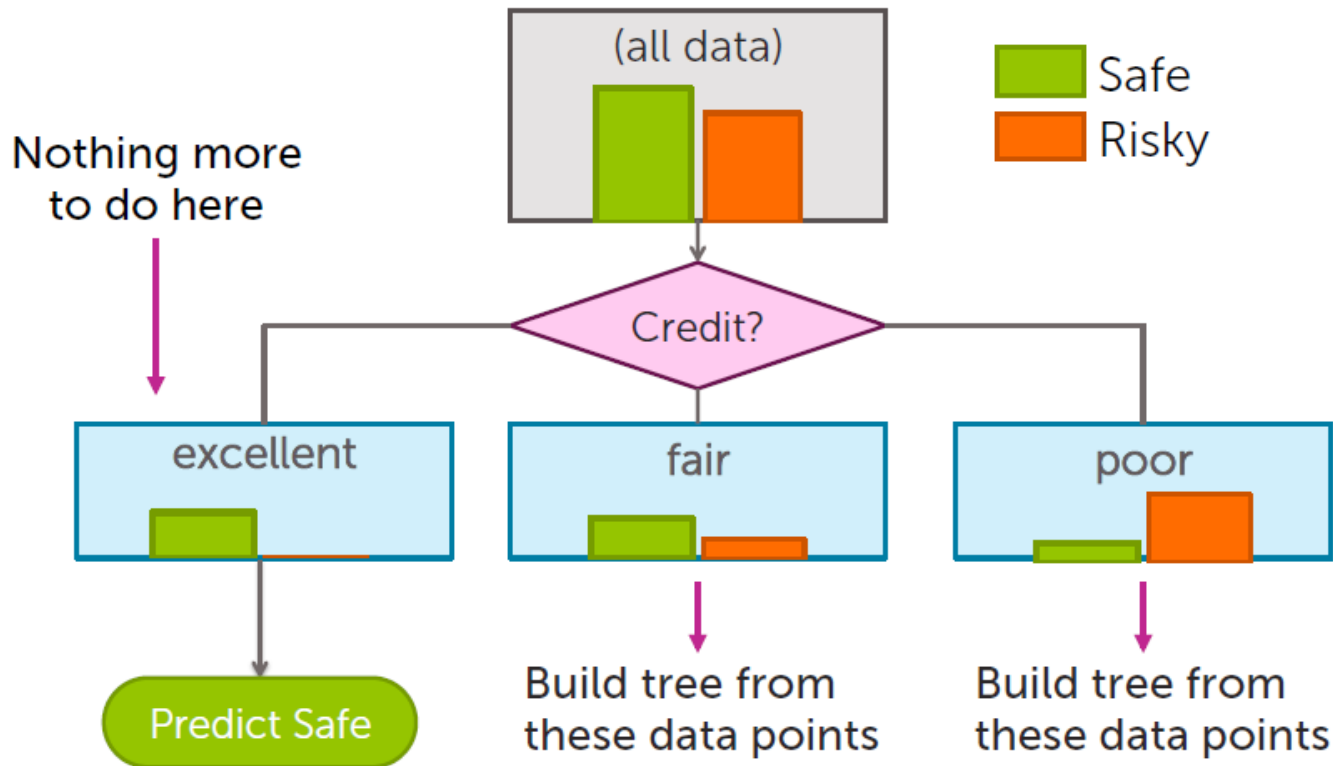
Step 3: Making predictions



Greedy algorithm

146

Step 4: Recursion



Greedy decision tree learning

147

- **Step 1:** Start with an empty tree

- **Step 2:** Select a feature to split data

- For each split of the tree:

- **Step 3:** If nothing more to, make predictions

- **Step 4:** Otherwise, go to **Step 2** & continue (recurse) on this split

Problem 1: Feature split selection

Problem 2: Stopping condition

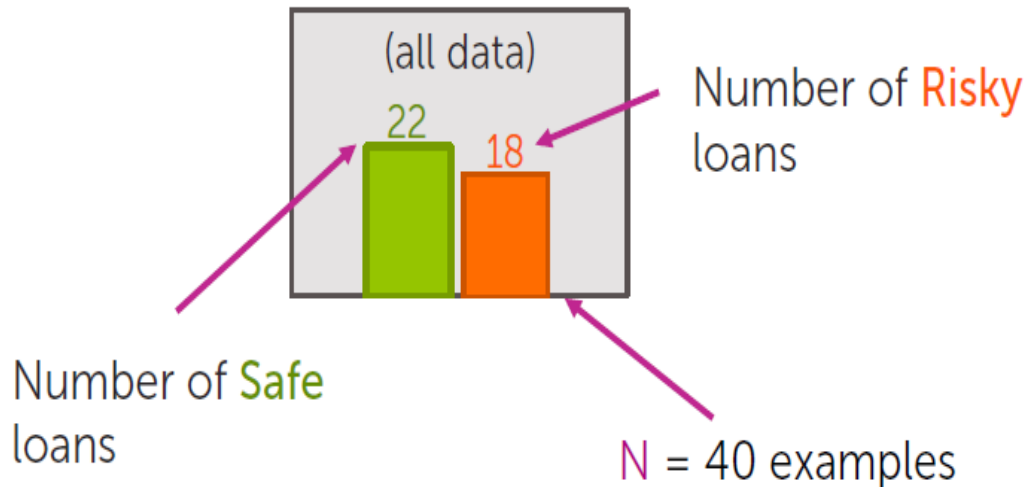
Recursion

Feature split learning

148

Start with all the data

Loan status: **Safe** **Risky**



Assume $N = 40$, 3 features

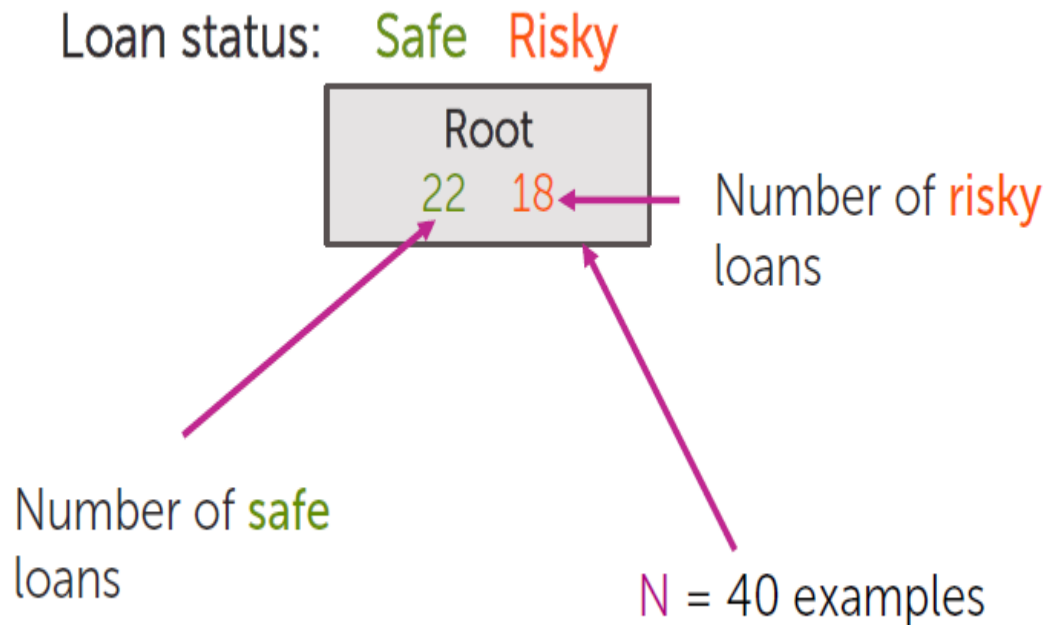
Credit	Term	Income	y
excellent	3 yrs	high	safe
fair	5 yrs	low	risky
fair	3 yrs	high	safe
poor	5 yrs	high	risky
excellent	3 yrs	low	risky
fair	5 yrs	low	safe
poor	3 yrs	high	risky
poor	5 yrs	low	safe
fair	3 yrs	high	safe

Feature split learning

149

Start with all the data

Assume $N = 40$, 3 features

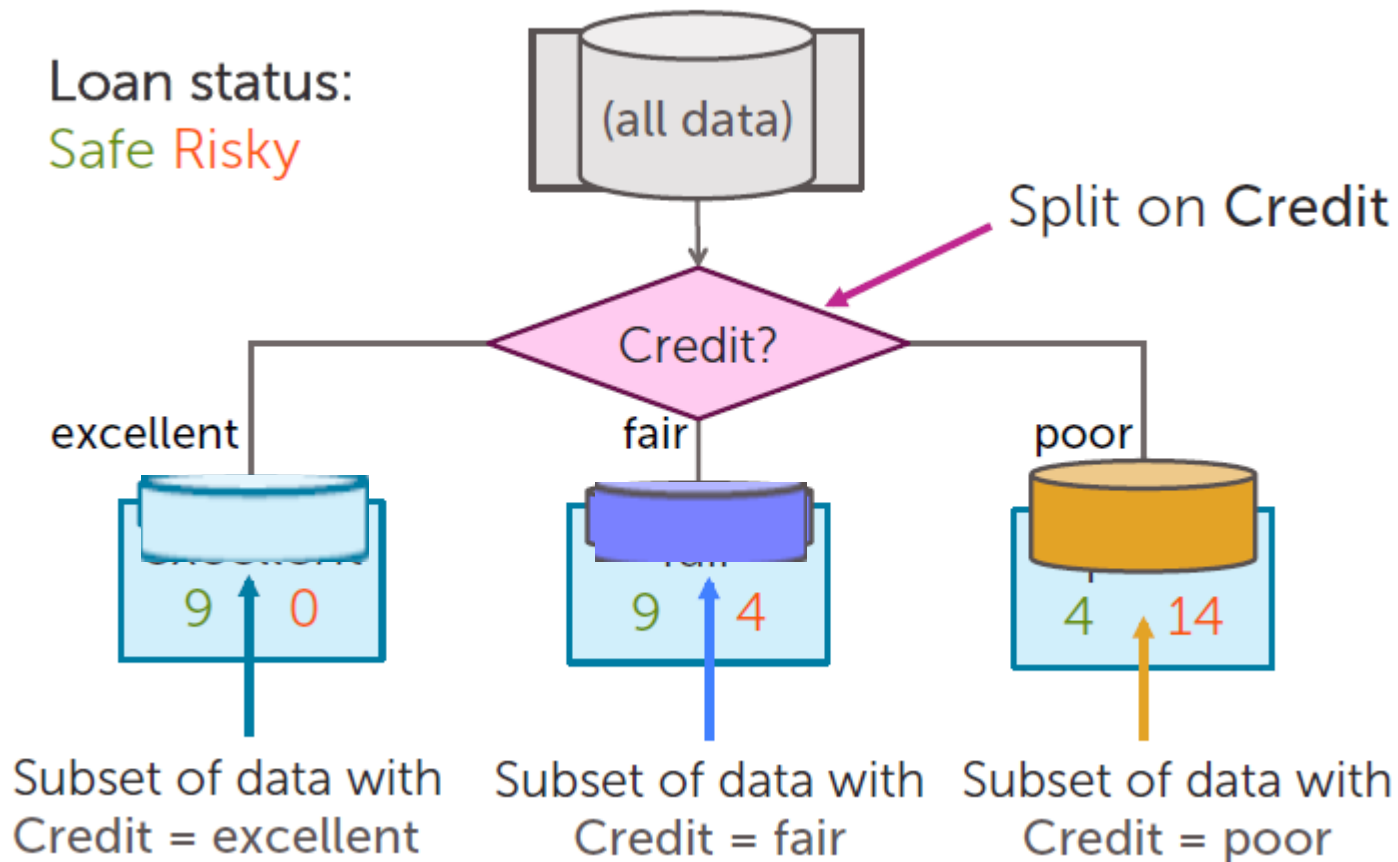


Credit	Term	Income	y
excellent	3 yrs	high	safe
fair	5 yrs	low	risky
fair	3 yrs	high	safe
poor	5 yrs	high	risky
excellent	3 yrs	low	risky
fair	5 yrs	low	safe
poor	3 yrs	high	risky
poor	5 yrs	low	safe
fair	3 yrs	high	safe

Compact notation

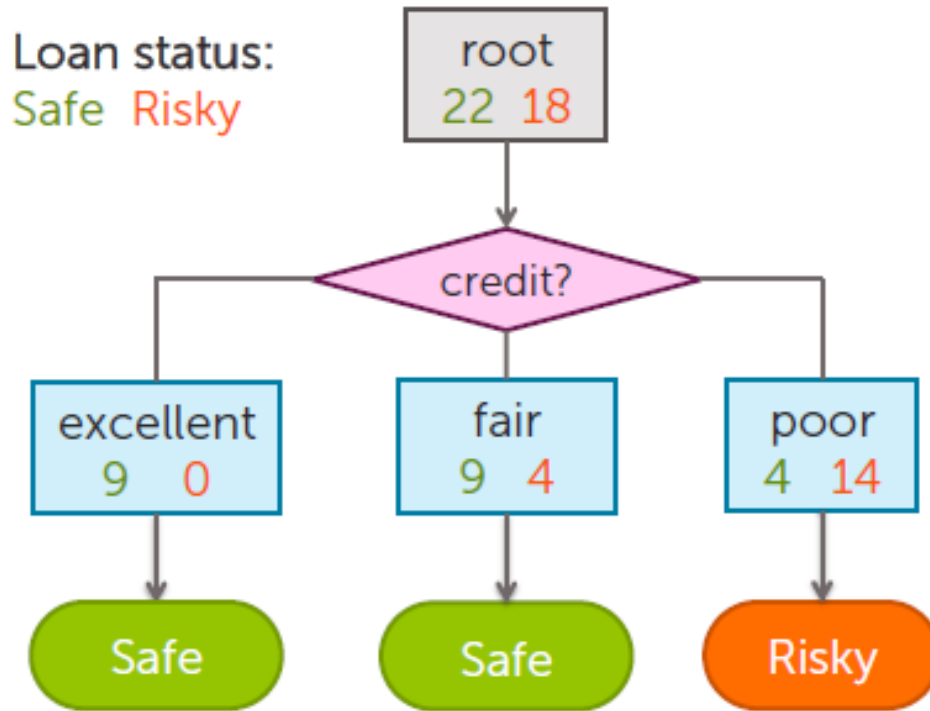
Decision stump: single level tree

150



Making predictions with a decision stump

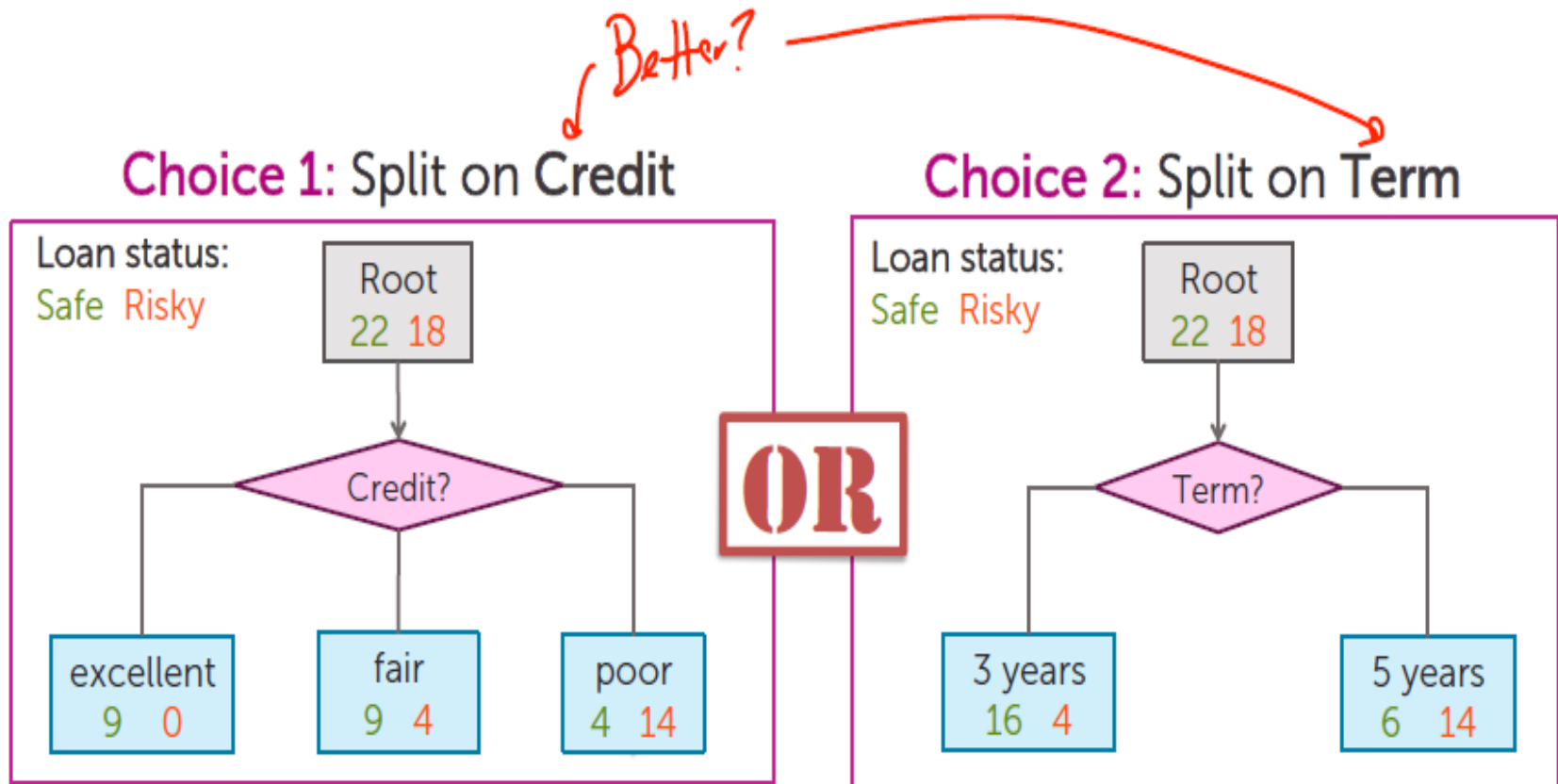
151



For each intermediate node,
set \hat{y} = majority value

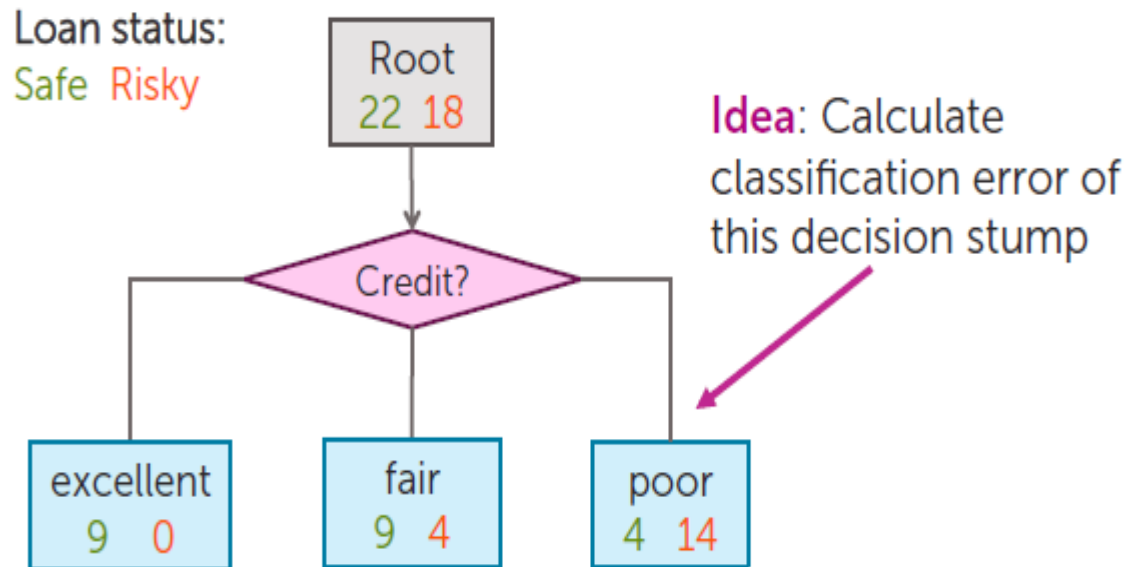
How do we select the best feature to split on?

152



How do we measure effectiveness of a split?

153

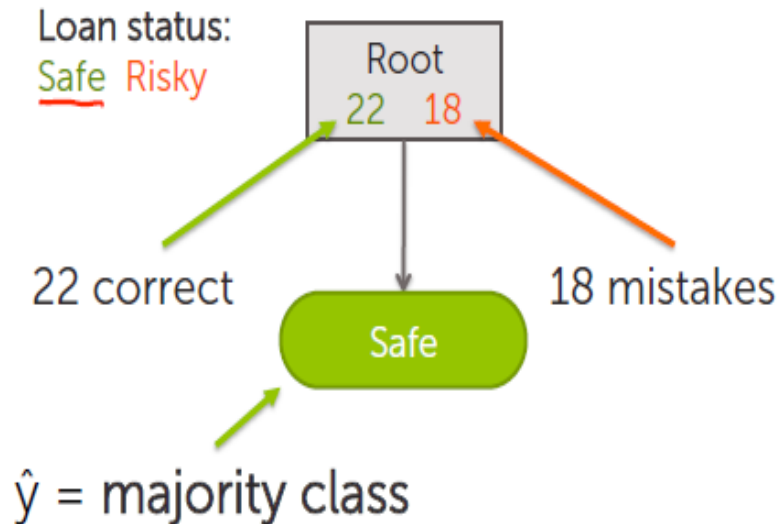


$$\text{Error} = \frac{\# \text{ mistakes}}{\# \text{ data points}}$$

Calculating classification error

154

- **Step 1:** \hat{y} = class of majority of data in node
- **Step 2:** Calculate classification error of predicting \hat{y} for this data



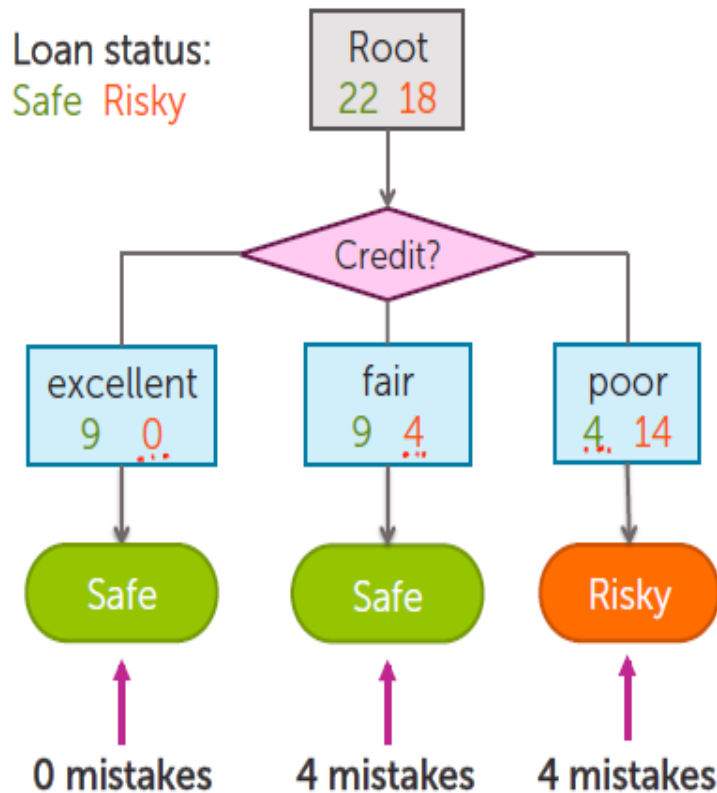
$$\text{Error} = \frac{18}{22+18}$$
$$= 0.45$$

Tree	Classification error
(root)	0.45

Classification error

155

Choice 1: Split on Credit



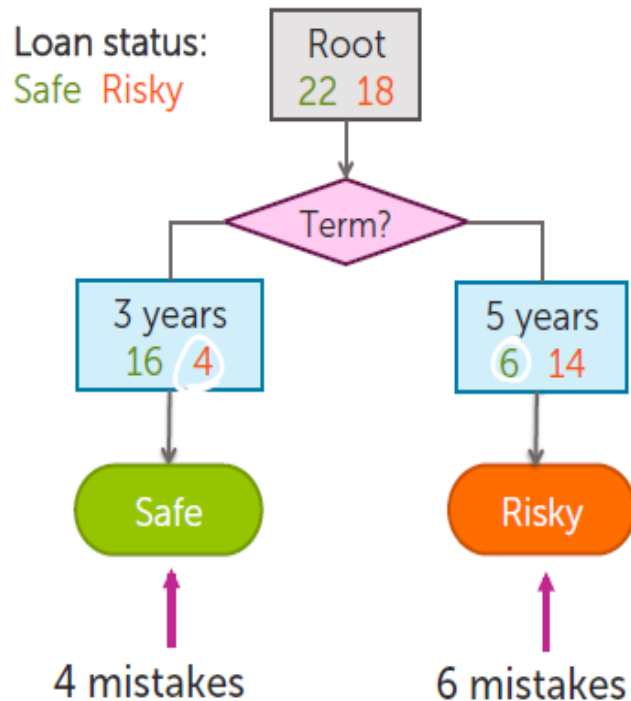
$$\text{Error} = \frac{4 + 4}{40} = 0.20$$

Tree	Classification error
(root)	0.45
Split on credit	0.2

Classification error

156

Choice 2: Split on Term



$$\text{Error} = \frac{4+6}{40} = 0.25$$

Tree	Classification error
(root)	0.45
Split on credit	0.2
Split on term	0.25

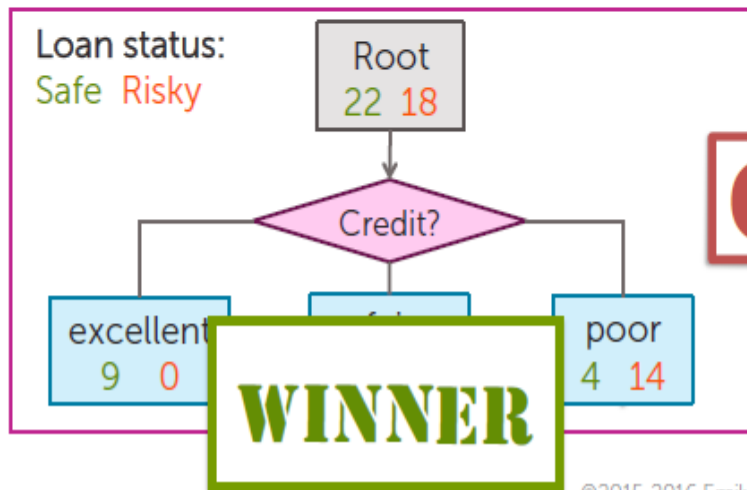
Choice 1 vs Choice 2

157

Tree	Classification error
(root)	0.45
split on <u>credit</u>	0.2
split on <u>loan term</u>	0.25

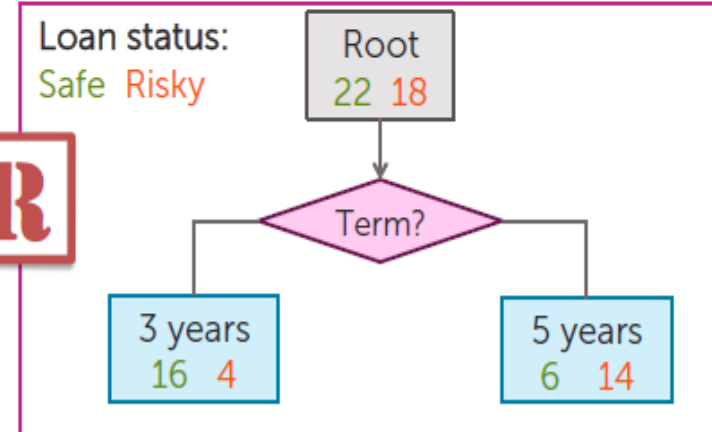
← First split!

Choice 1: Split on Credit



OR

Choice 2: Split on Term



57

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Machine Learning Specialization

9/11, 16/11, 23/11 2022

Feature split selection algorithm

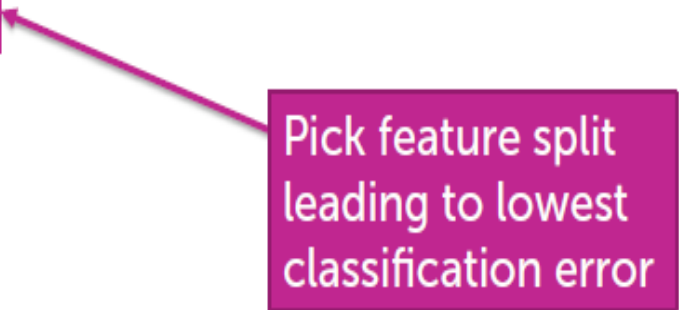
158

- Given a subset of data M (a node in a tree)
- For each feature $h_j(x)$: *credit, term, income*
 1. Split data of M according to feature $h_j(x)$
 2. Compute classification error split
- Chose feature $h^*(x)$ with lowest classification error *credit*

Greedy decision tree learning algorithm

159

- **Step 1:** Start with an empty tree
- **Step 2:** Select a feature to split data
- For each split of the tree:
 - **Step 3:** If nothing more to, make predictions
 - **Step 4:** Otherwise, go to **Step 2** & continue (recurse) on this split

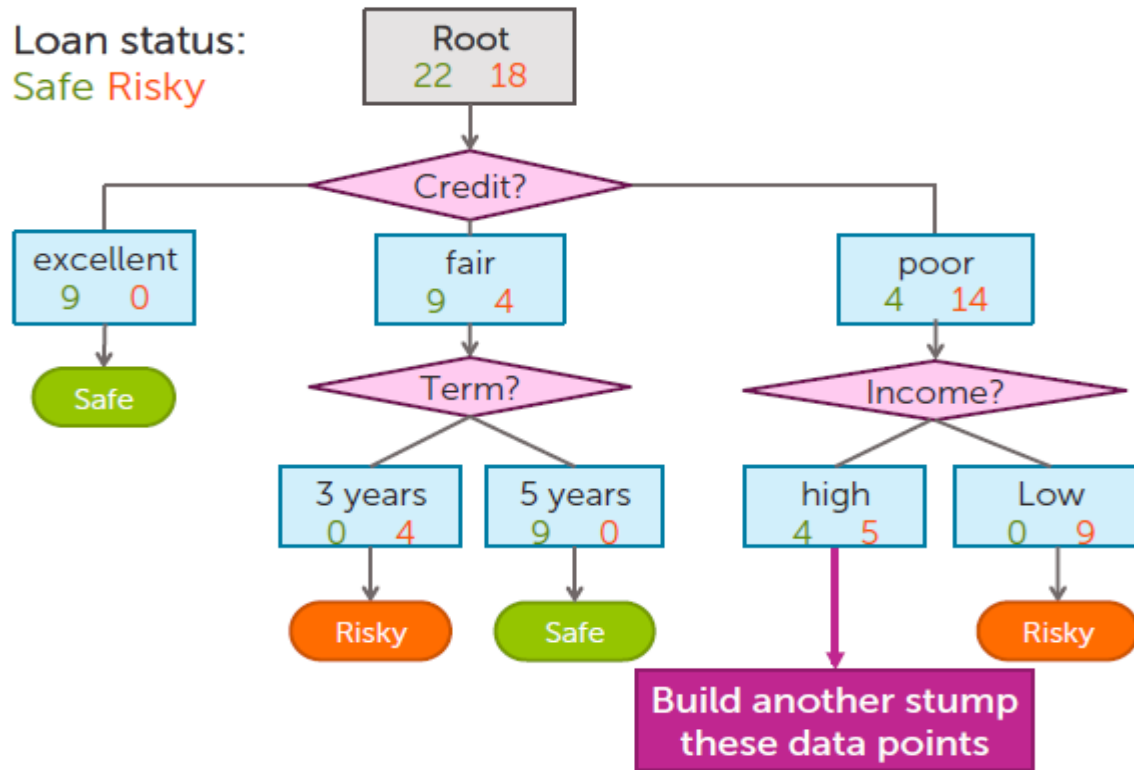


Pick feature split leading to lowest classification error

Recursive stump learning

160

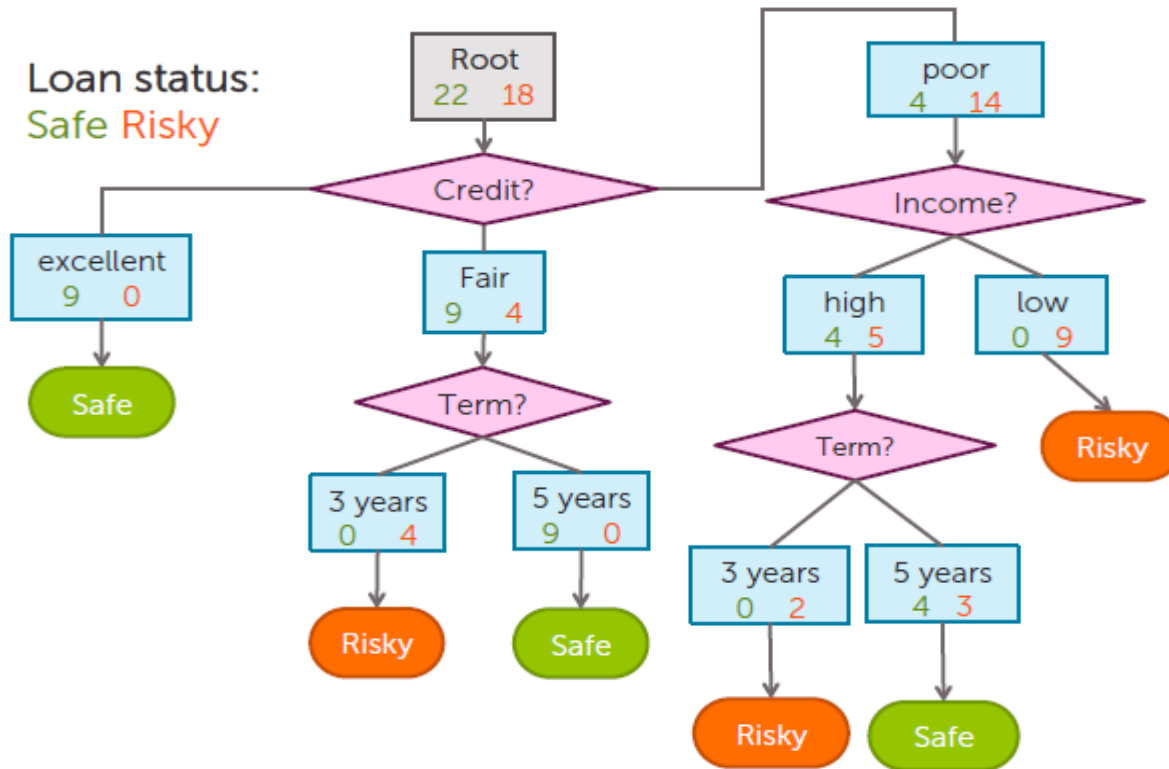
Second level



Recursive stump learning

161

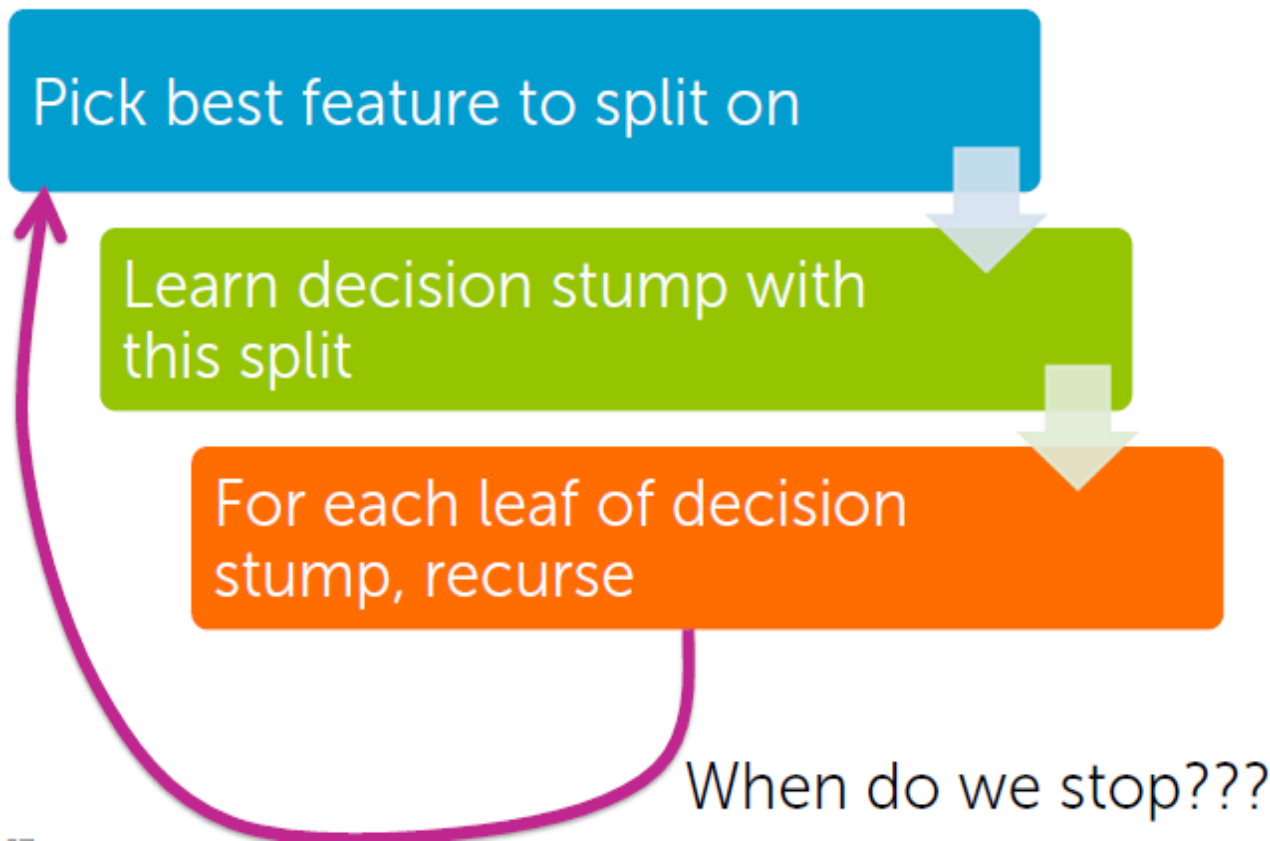
Final decision tree



Simple greedy decision tree learning

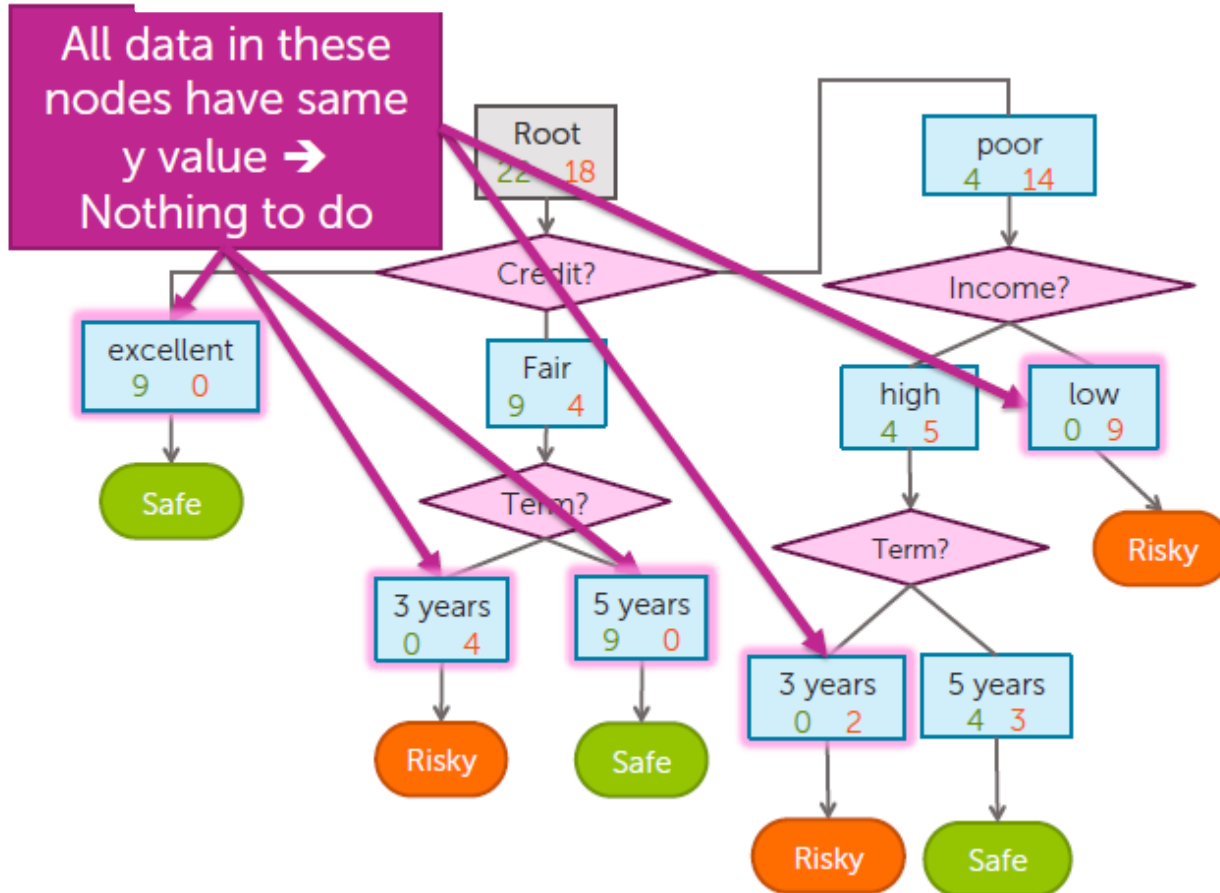
162

Recursive algorithm



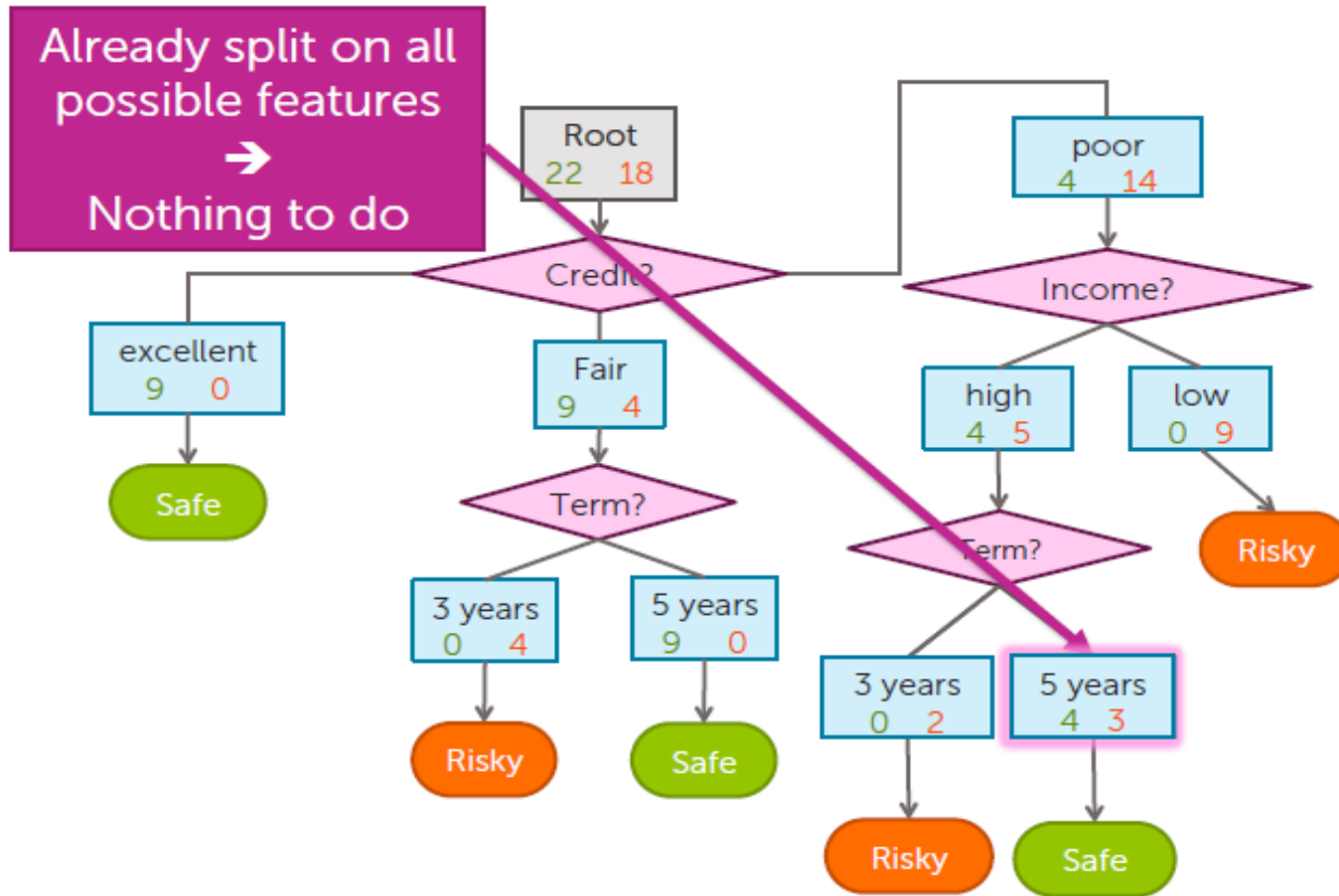
Stopping condition 1

163



Stopping condition 2

164



Greedy decision tree algorithm

165

- **Step 1:** Start with an empty tree

- **Step 2:** Select a feature to split data

- For each split of the tree:

- **Step 3:** If nothing more to, make predictions

- **Step 4:** Otherwise, go to **Step 2** & continue (recurse) on this split

Pick feature split leading to lowest classification error

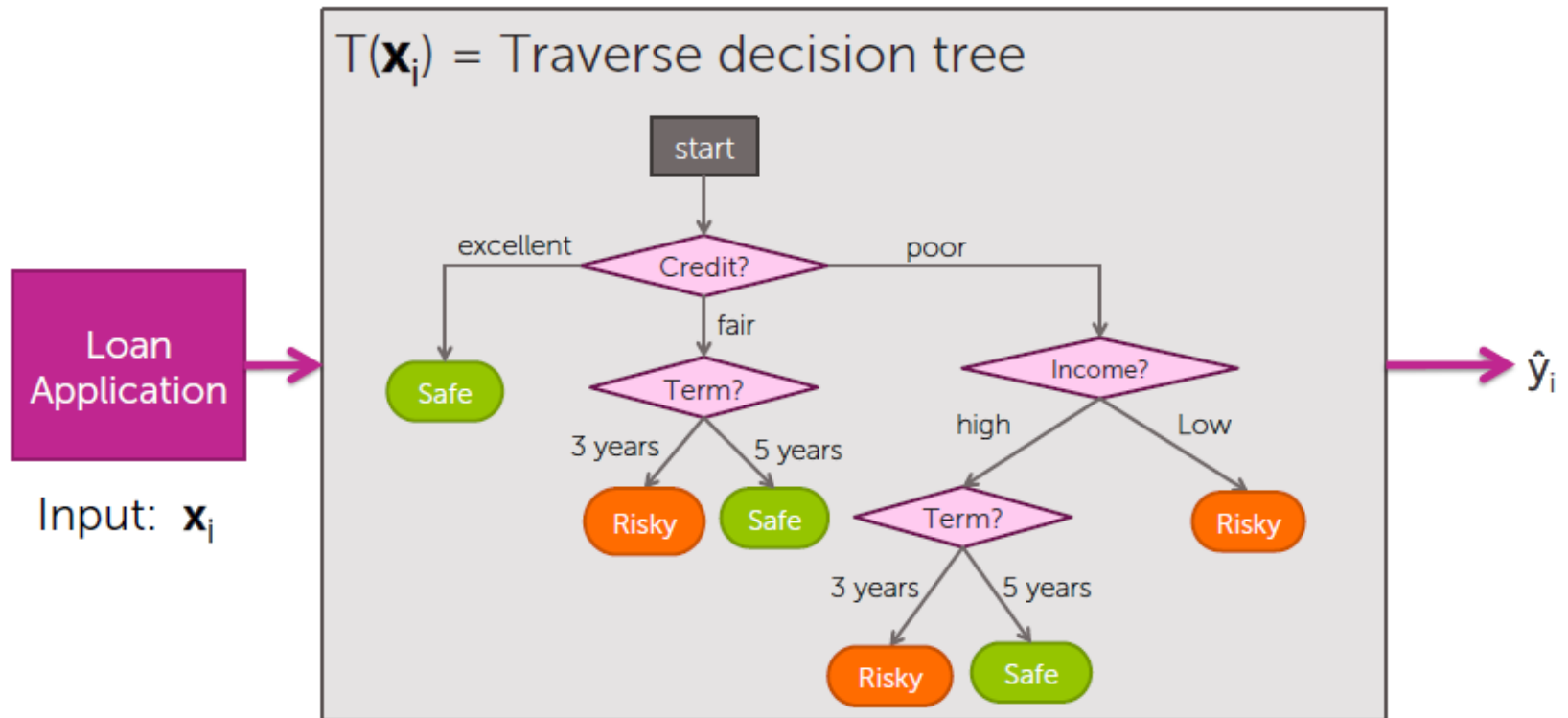
Stopping conditions 1 & 2

Recursion

Predictions with decision trees

166

Decision tree model

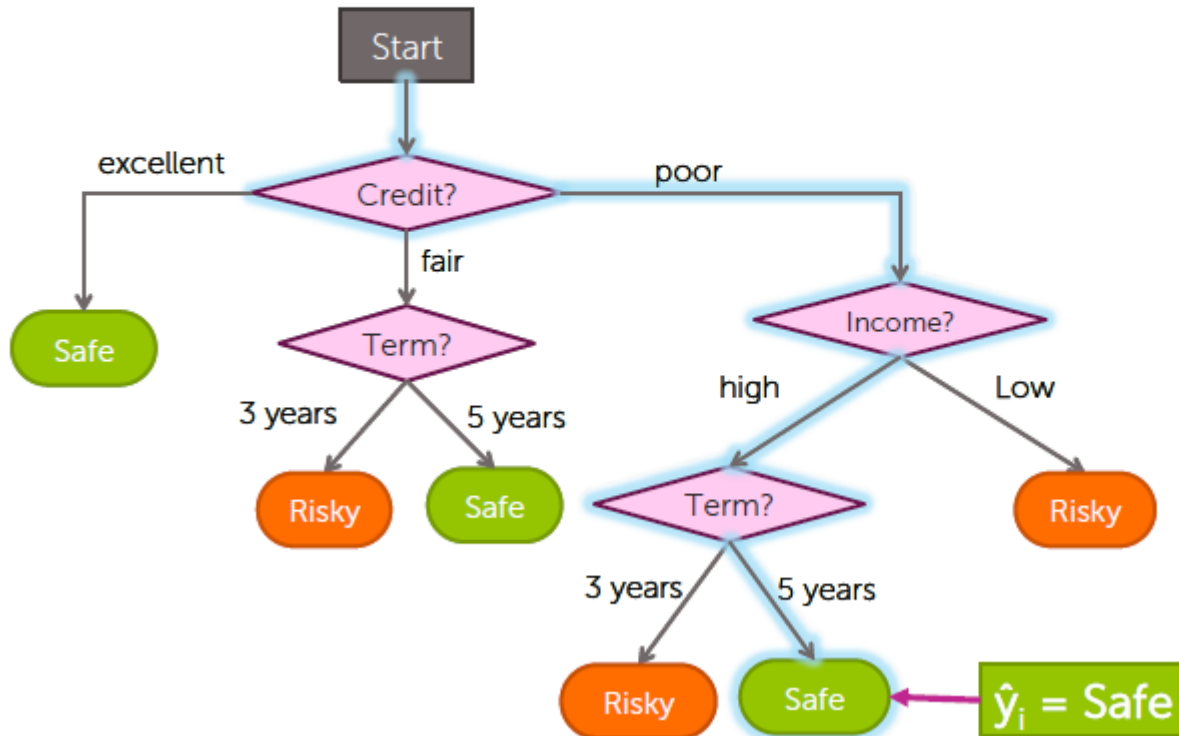


Predictions with decision trees

167

Traversing a decision tree

$x_i = (\text{Credit} = \text{poor}, \text{Income} = \text{high}, \text{Term} = 5 \text{ years})$



Predictions with decision tree

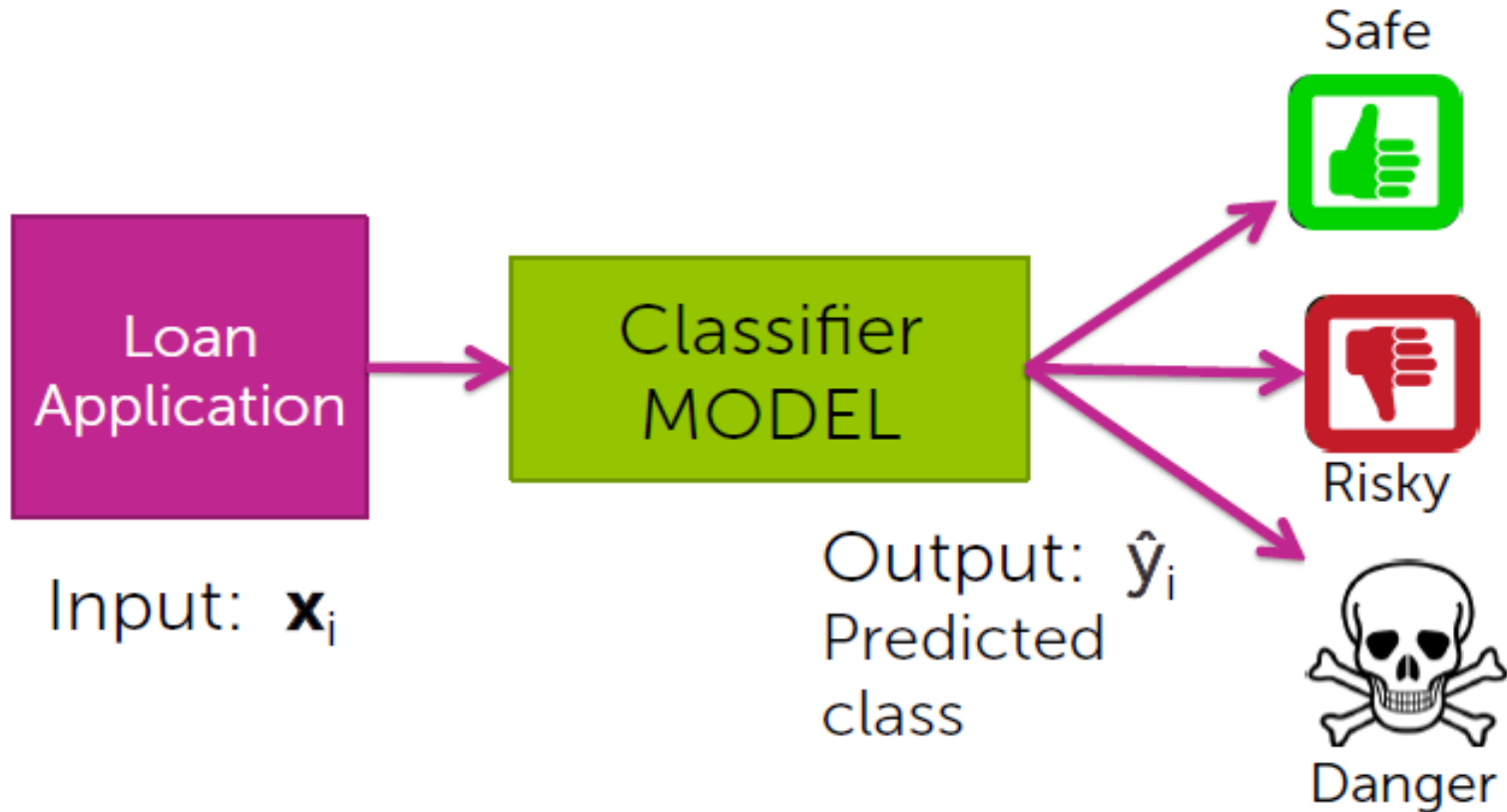
168

`predict(tree_node, input)`

- If current `tree_node` is a leaf:
 - `return` majority class of data points in leaf
- `else`:
 - `next_note` = child node of `tree_node` whose feature value agrees with input
 - `return predict(next_note, input)`

Multiclass prediction

169



Multiclass decision stump

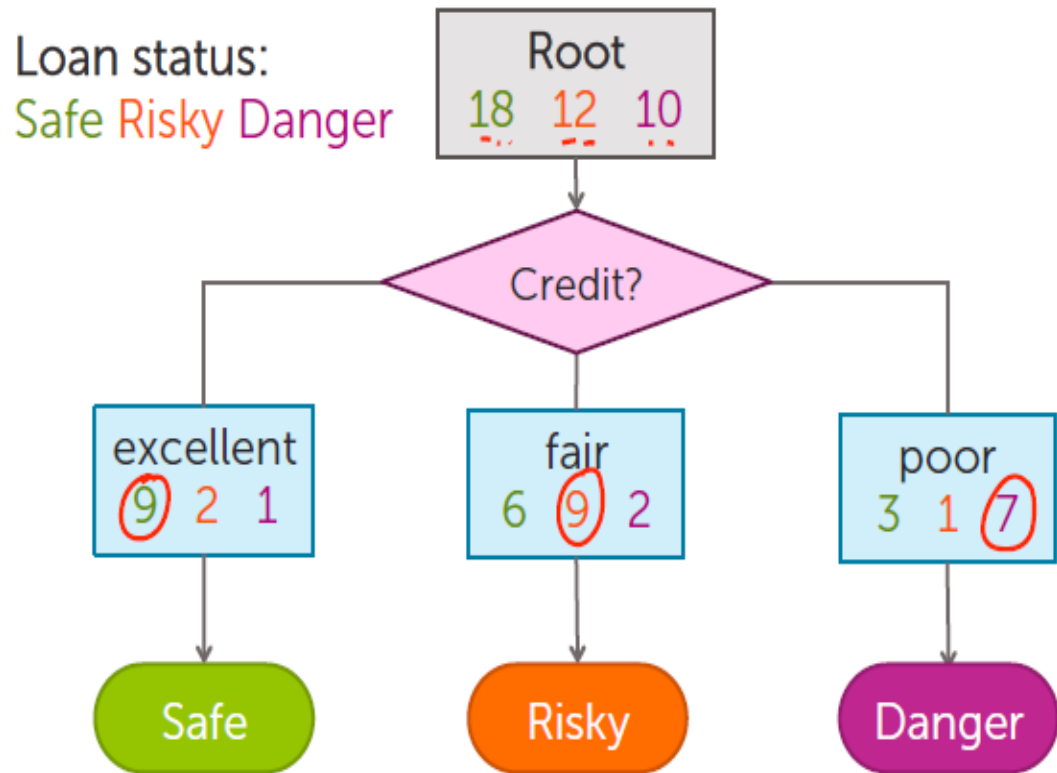
170

N = 40,
1 feature,
3 classes

Credit	y
excellent	<u>safe</u>
fair	<u>risky</u>
fair	safe
poor	<u>danger</u>
excellent	risky
fair	safe
poor	danger
poor	safe
fair	safe
...	...

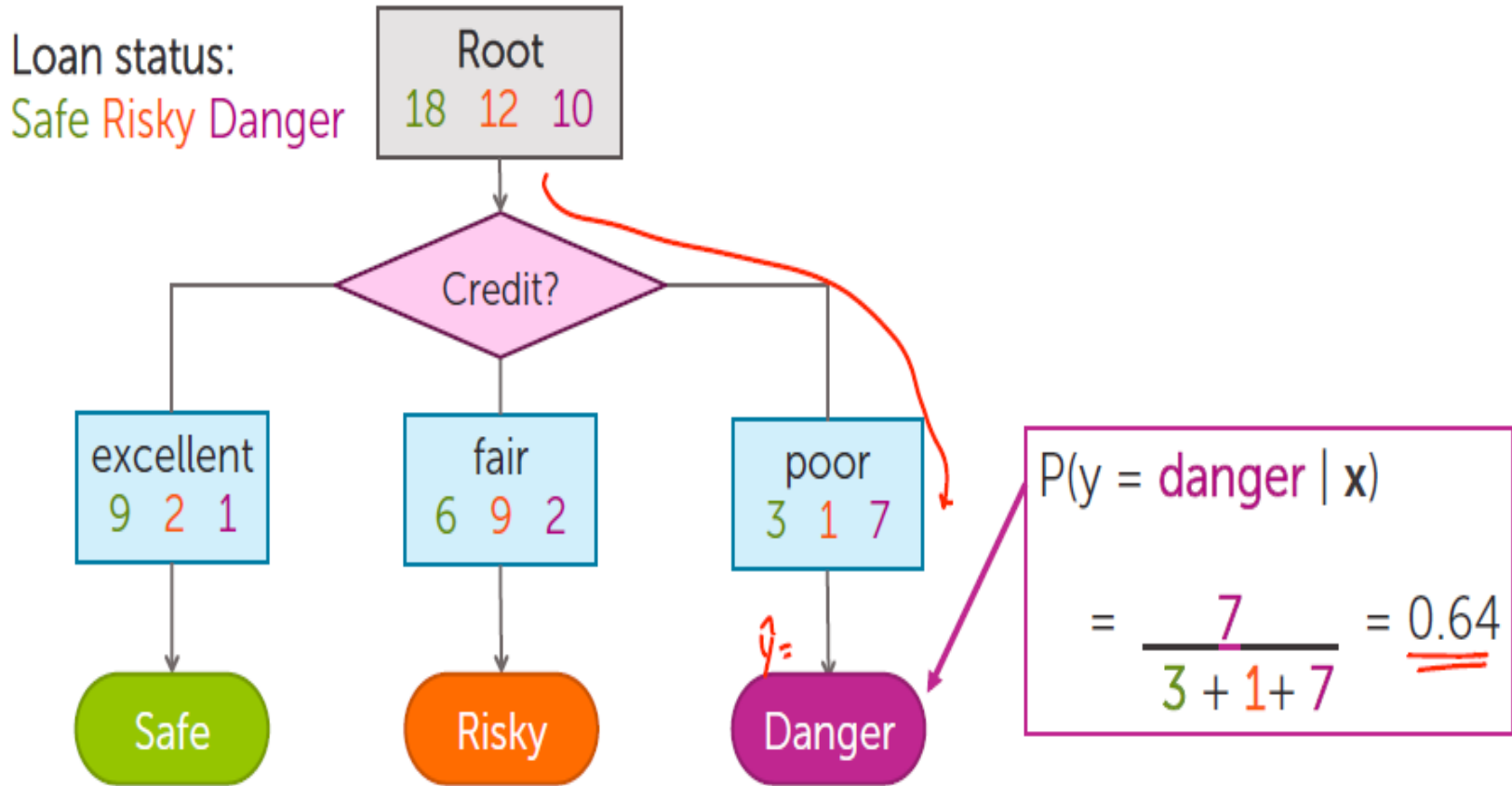


Loan status:
Safe Risky Danger



Predicting probabilities with decision trees

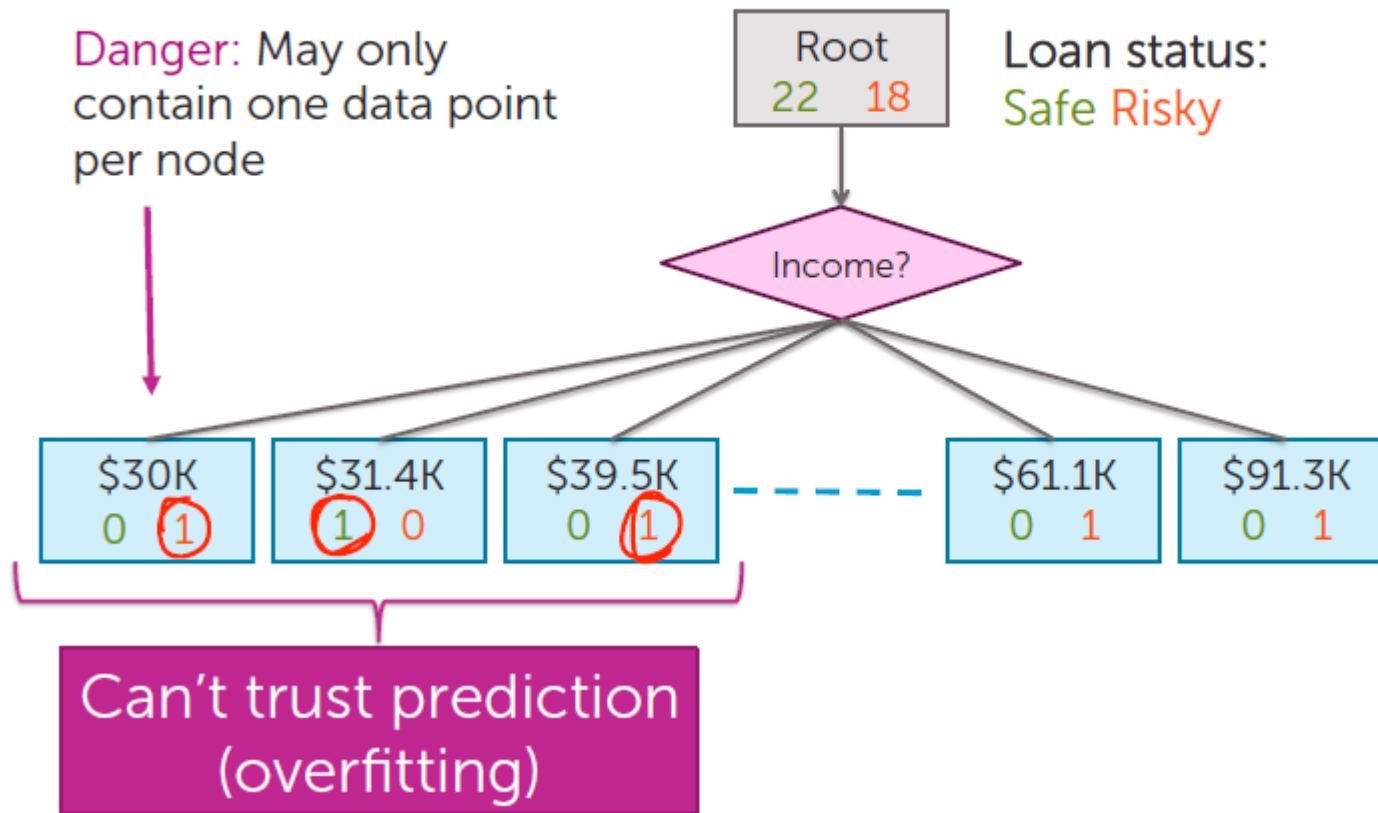
171



How to use real values inputs

172

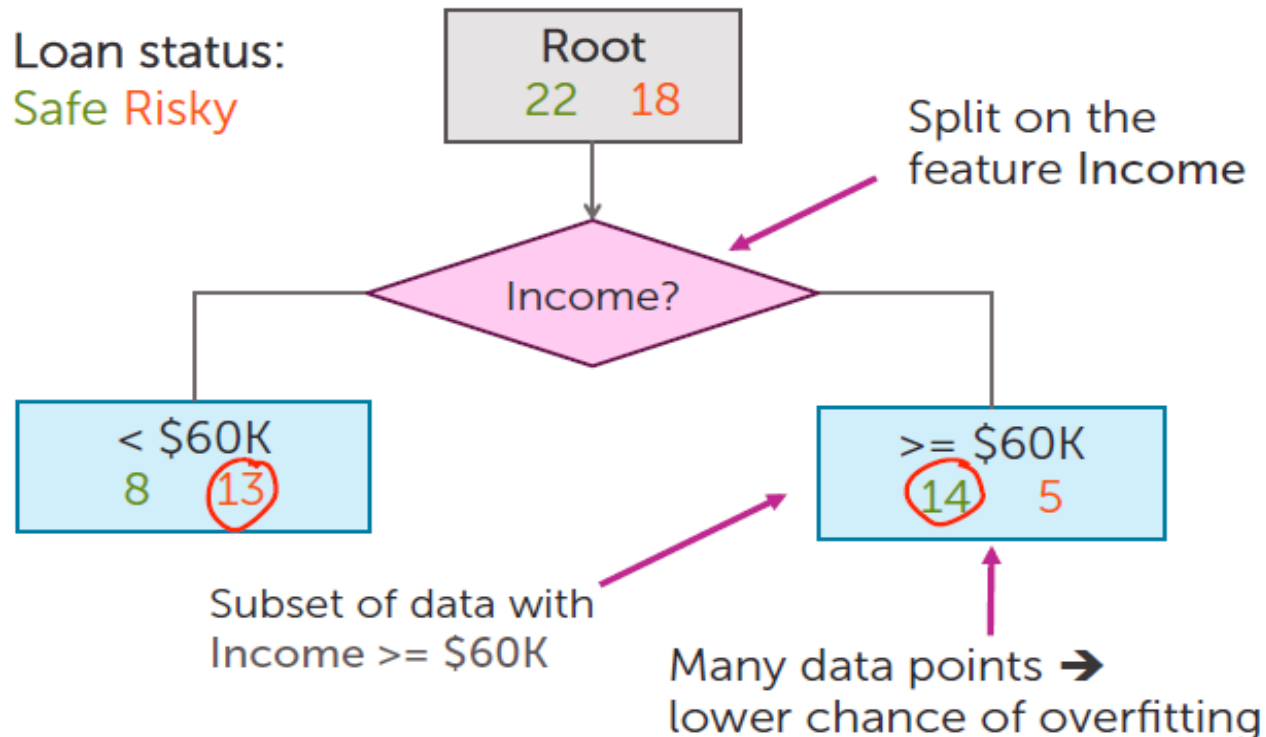
Split on each numeric value?



How to use real values inputs

173

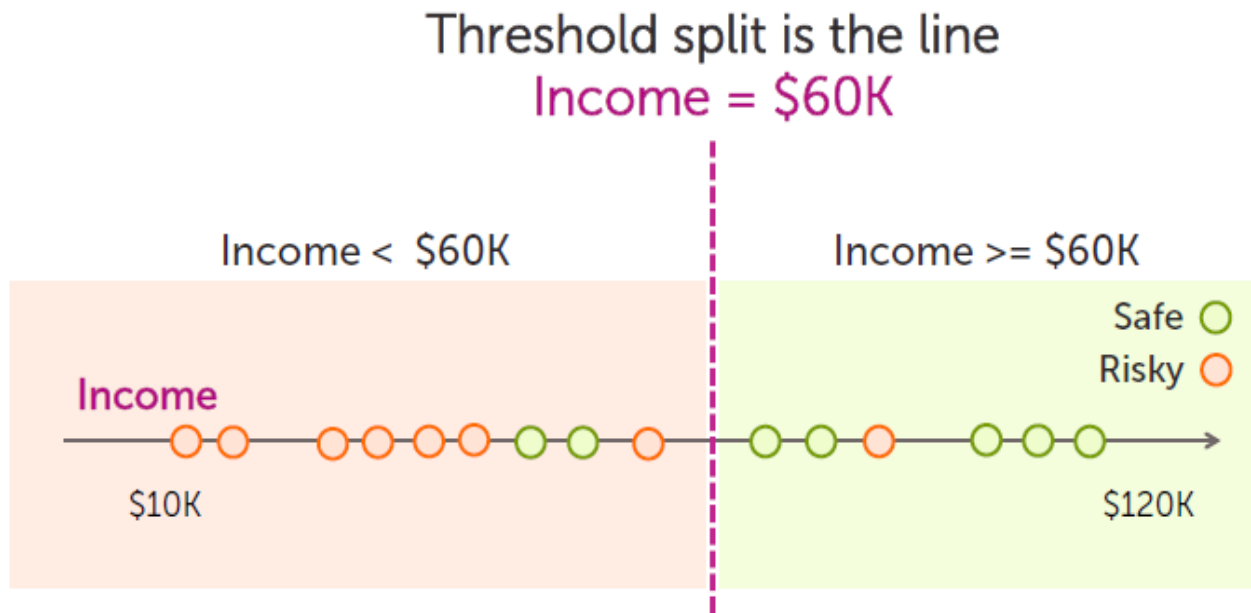
Alternative: Threshold split



Visualizing the threshold split

174

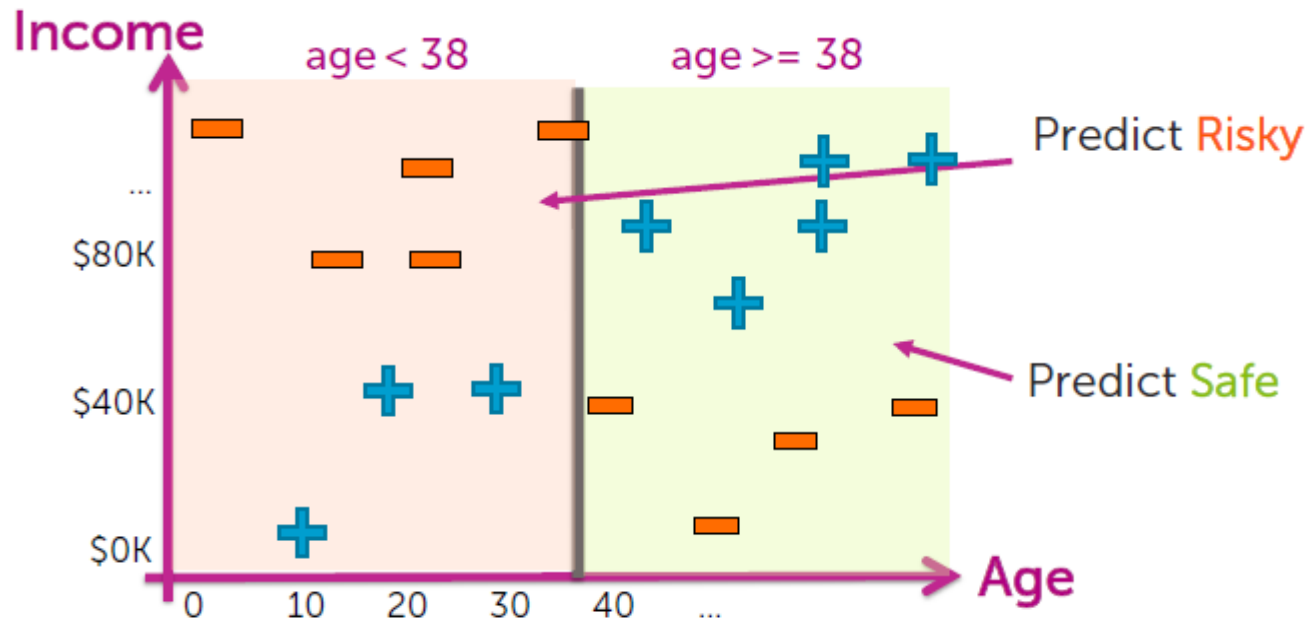
Threshold splits in 1-D



Visualizing the threshold split

175

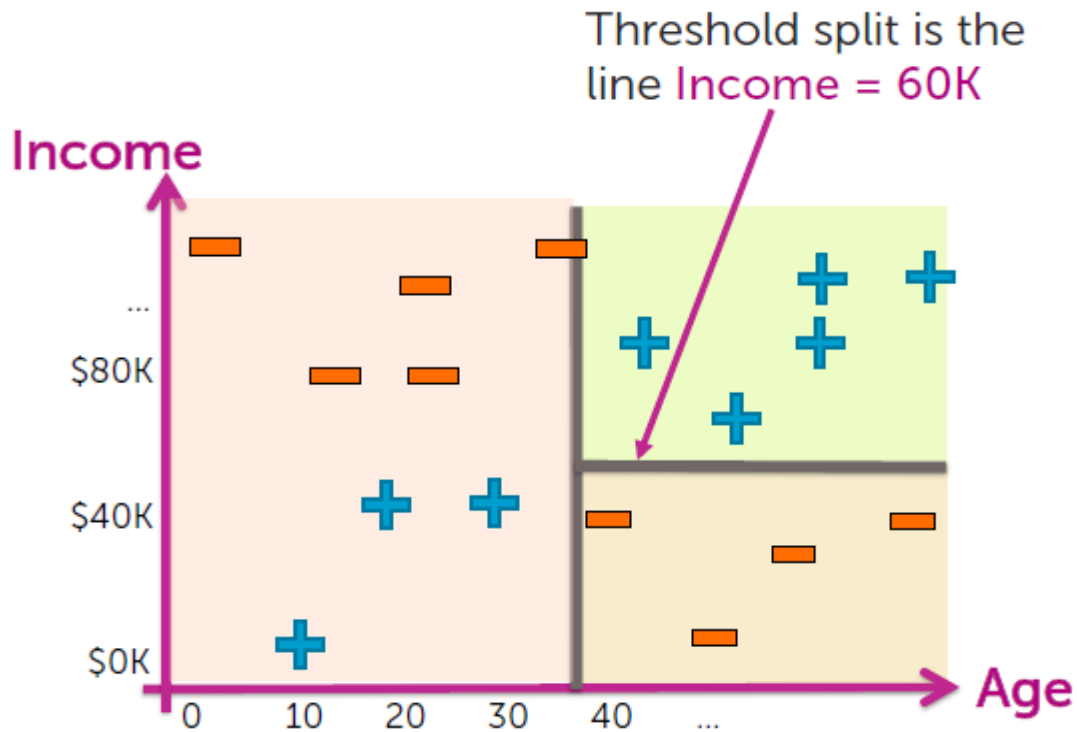
Split on Age ≥ 38



Visualizing the threshold split

176

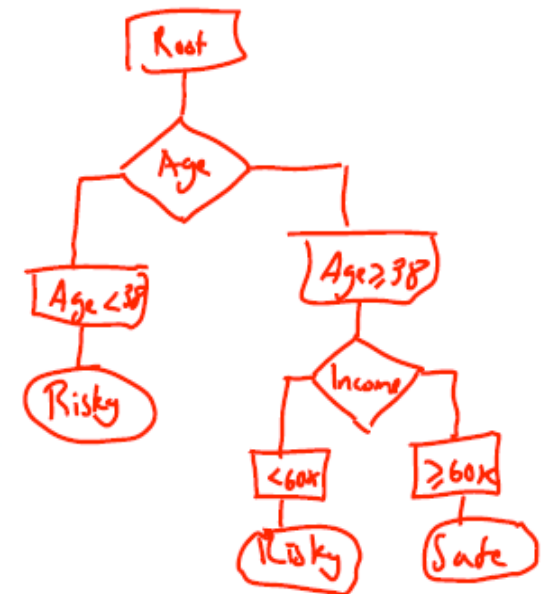
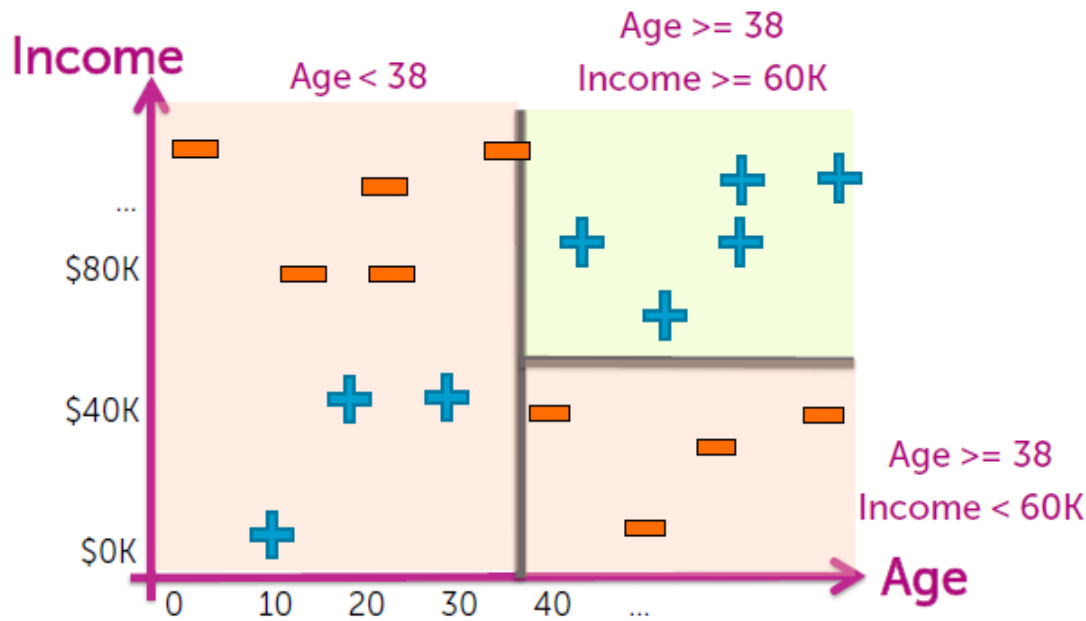
Depth 2: Split on Income \geq \$60K



Visualizing the threshold split

177

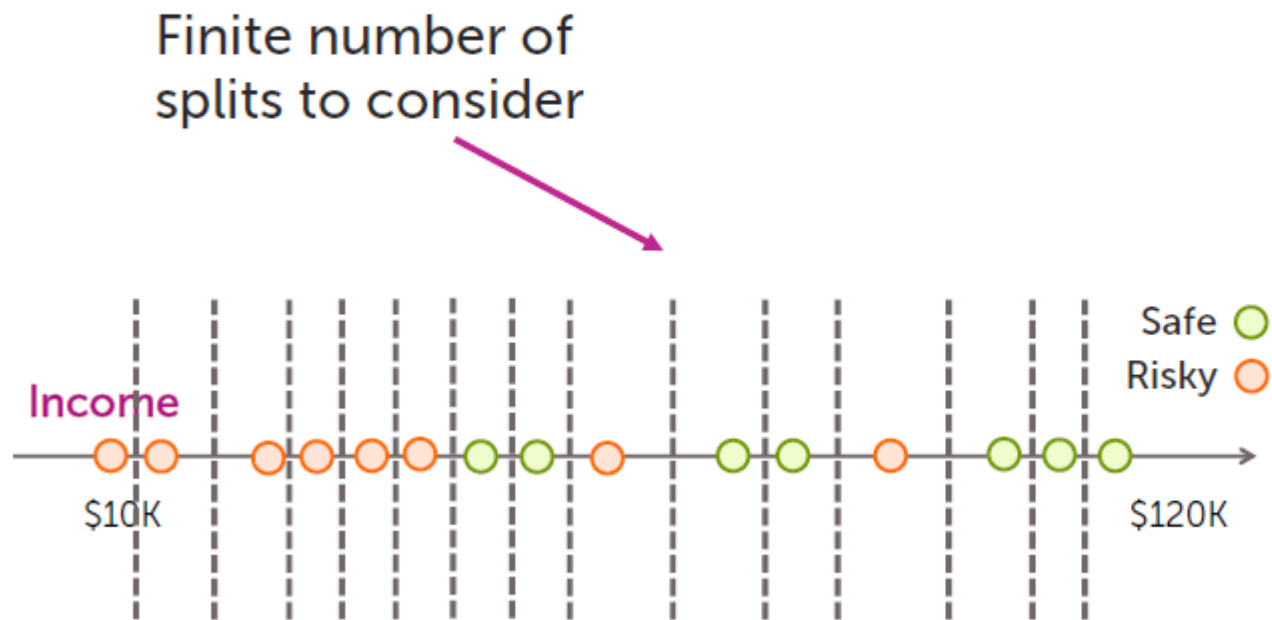
Each split partitions the 2-D space



Finding the best threshold split

178

Only need to consider mid-points



Finding the best threshold split

179

Threshold split selection algorithm

- **Step 1:** Sort the values of a feature $h_j(\mathbf{x})$:
Let $\{v_1, v_2, v_3, \dots, v_N\}$ denote sorted values
- **Step 2:**
 - For $i = 1 \dots N-1$
 - Consider split $t_i = (v_i + v_{i+1}) / 2$
 - Compute classification error for threshold split $h_j(\mathbf{x}) \geq t_i$
 - Chose the t^* with the lowest classification error

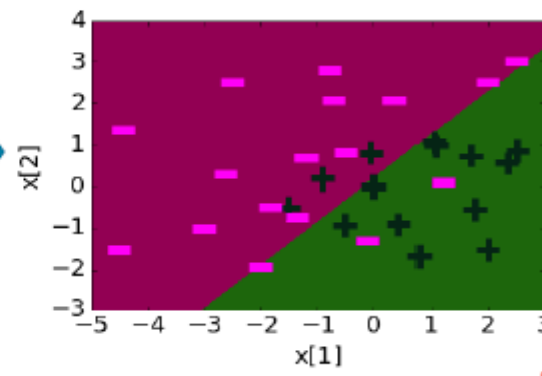
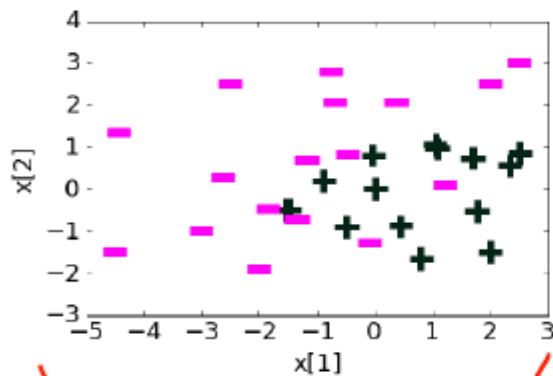
became

Decision trees vs logistic regression

180

Logistic regression

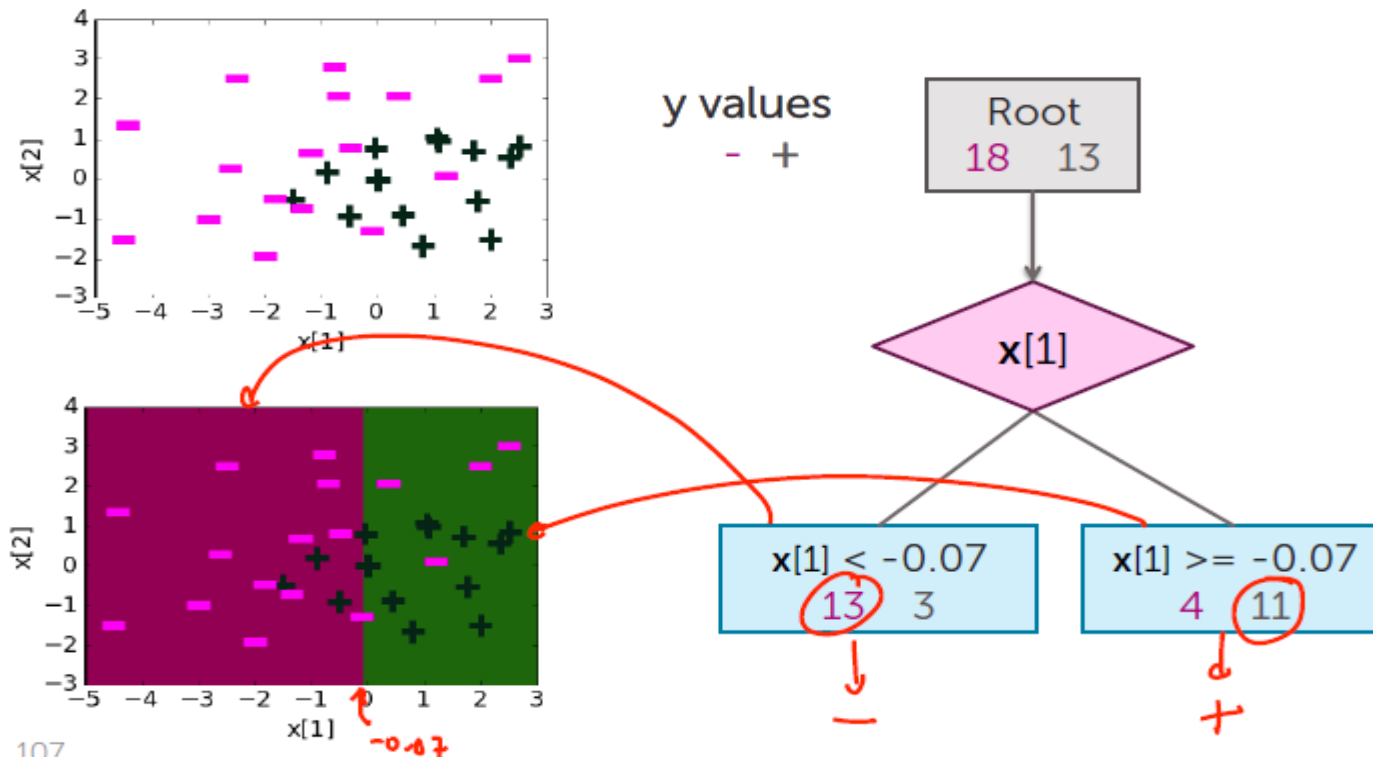
Feature	Value	Weight Learned
$h_0(x)$	1	0.22
$h_1(x)$	$x[1]$	1.12
$h_2(x)$	$x[2]$	-1.07



Decision trees vs logistic regression

181

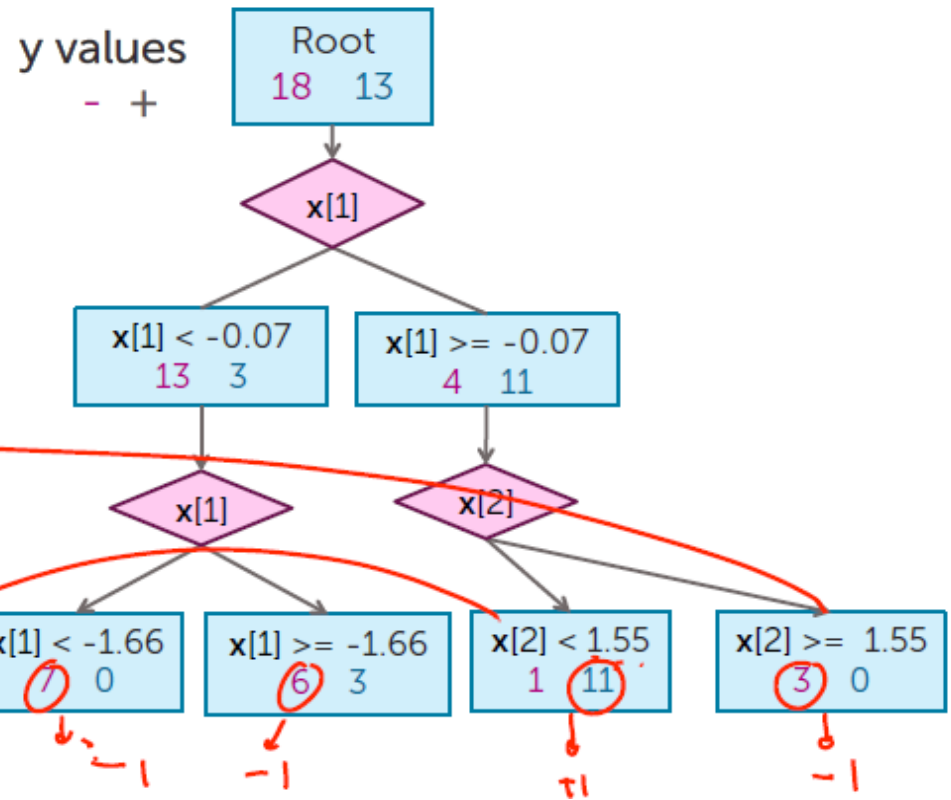
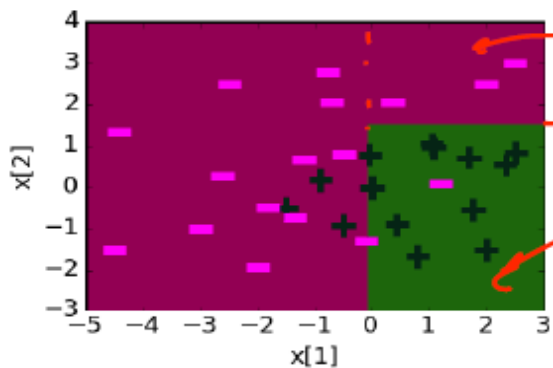
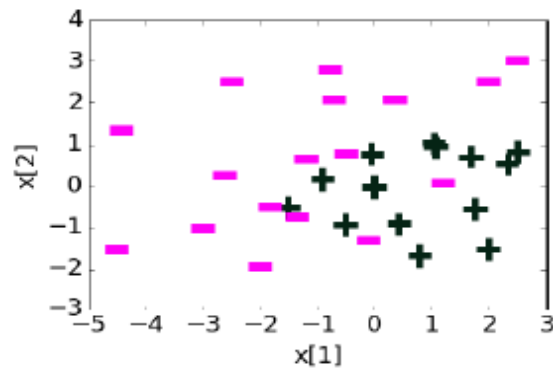
Depth 1: Split on $x[1]$



Decision trees vs logistic regression

182

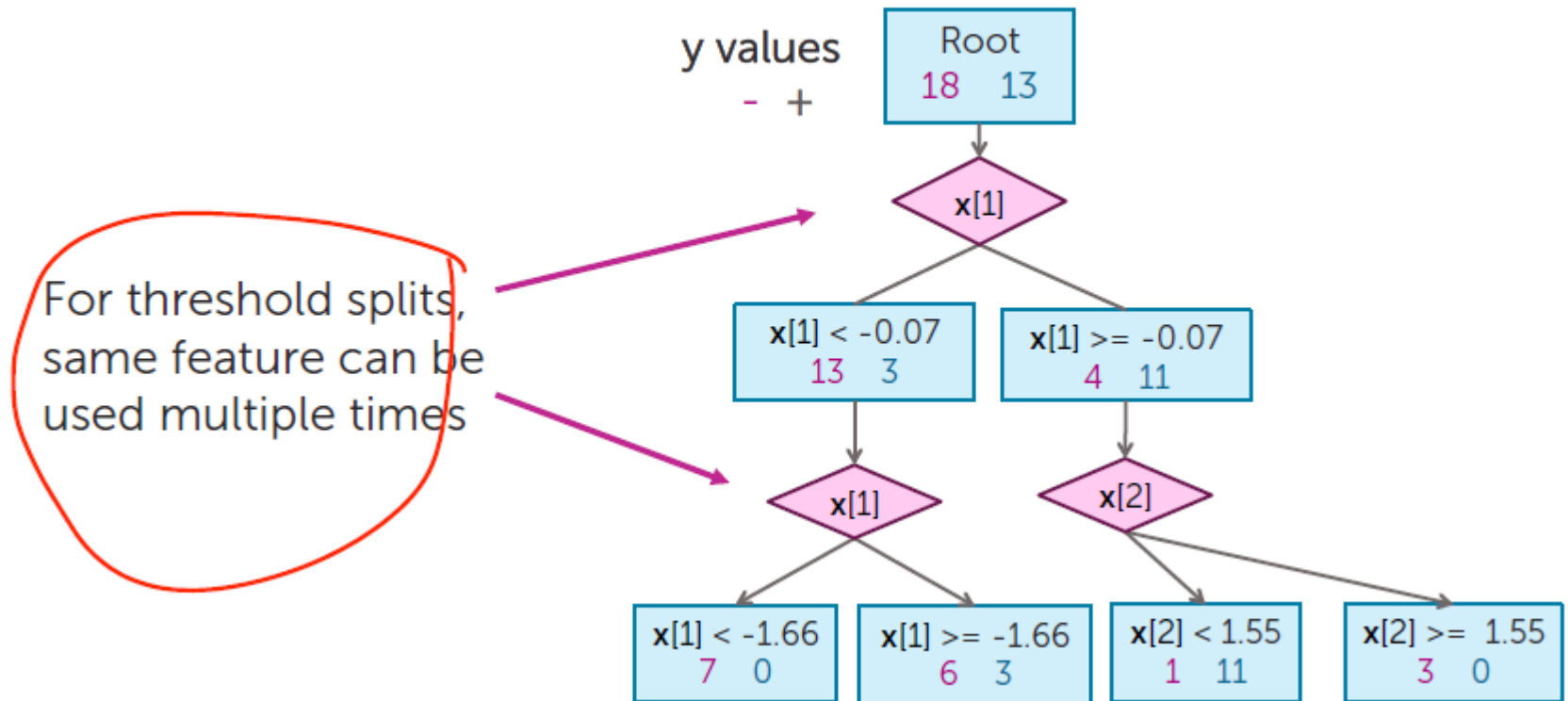
Depth 2



Decision tree vs logistic regression

183

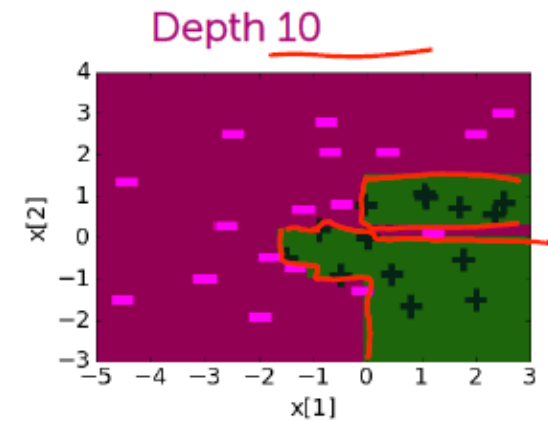
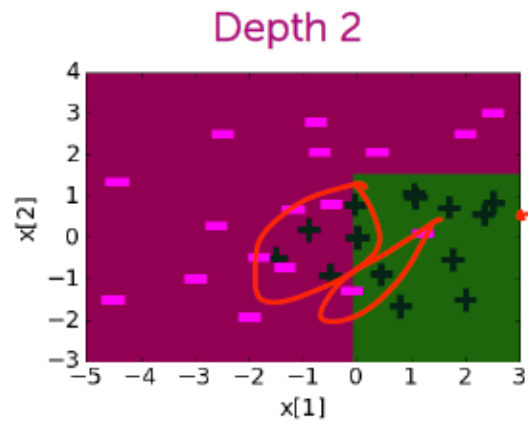
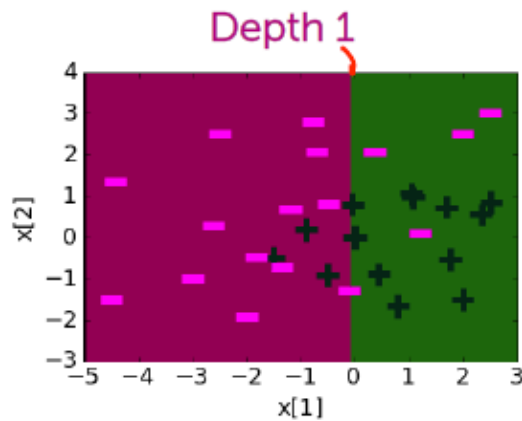
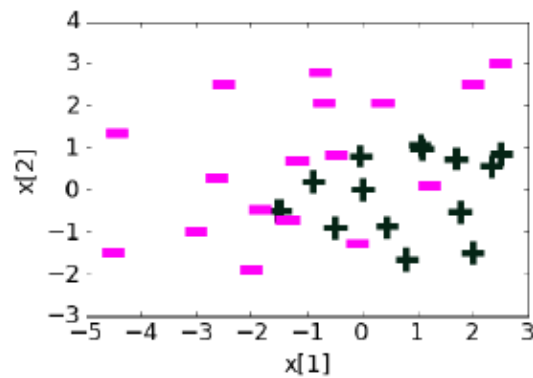
Threshold split caveat



Decision tree vs logistic regression

184

Decision boundaries

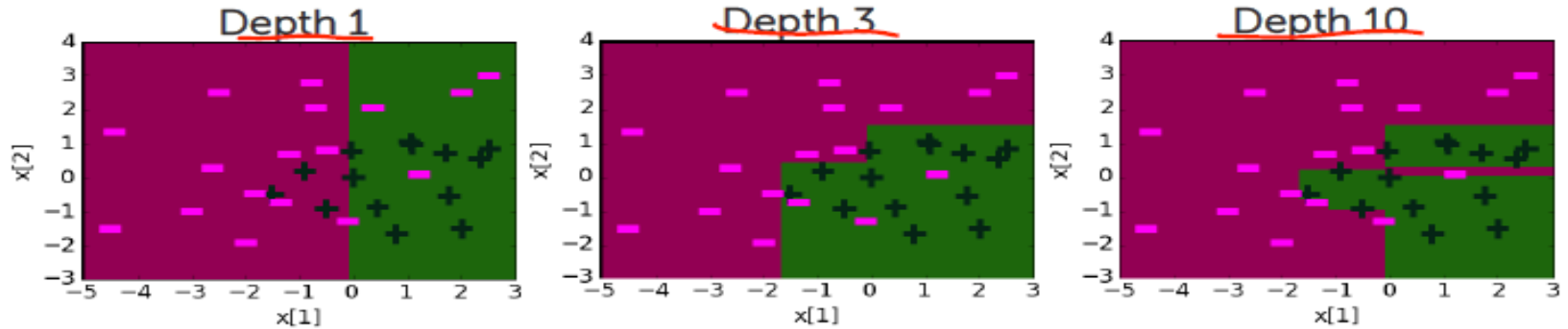


Decision tree vs logistic regression

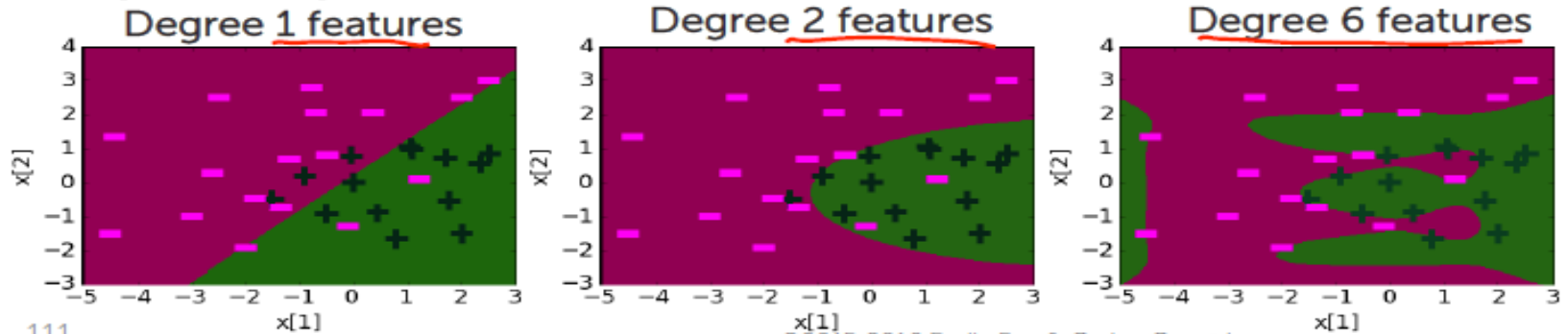
185

Comparing decision boundaries

Decision Tree



Logistic Regression



111

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9/11, 16/11, 23/11 2022

What you can do now

186

- Define a decision tree classifier
- Interpret the output of a decision trees
- Learn a decision tree classifier using greedy algorithm
- Traverse a decision tree to make predictions
 - Majority class predictions
 - Probability predictions
 - Multiclass classification

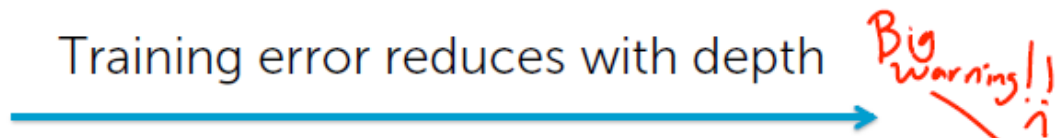
Overfitting in decision trees

Overfitting in decision tree

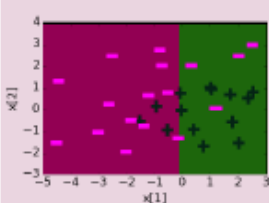
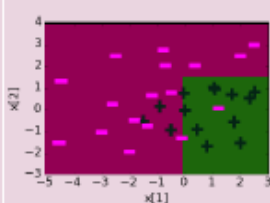
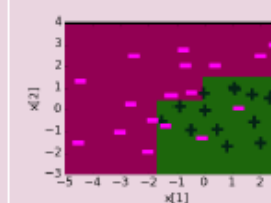
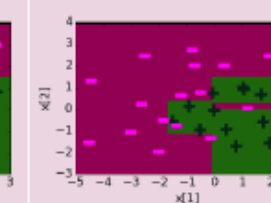
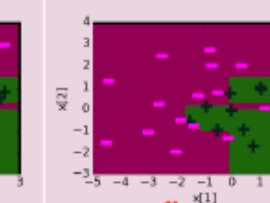
188

What happens when we increase depth?

Training error reduces with depth



Big warning!!

Tree depth	depth = 1	depth = 2	depth = 3	depth = 5	depth = 10
Training error	<u>0.22</u>	<u>0.13</u>	<u>0.10</u>	0.03	<u>0.00</u>
Decision boundary					

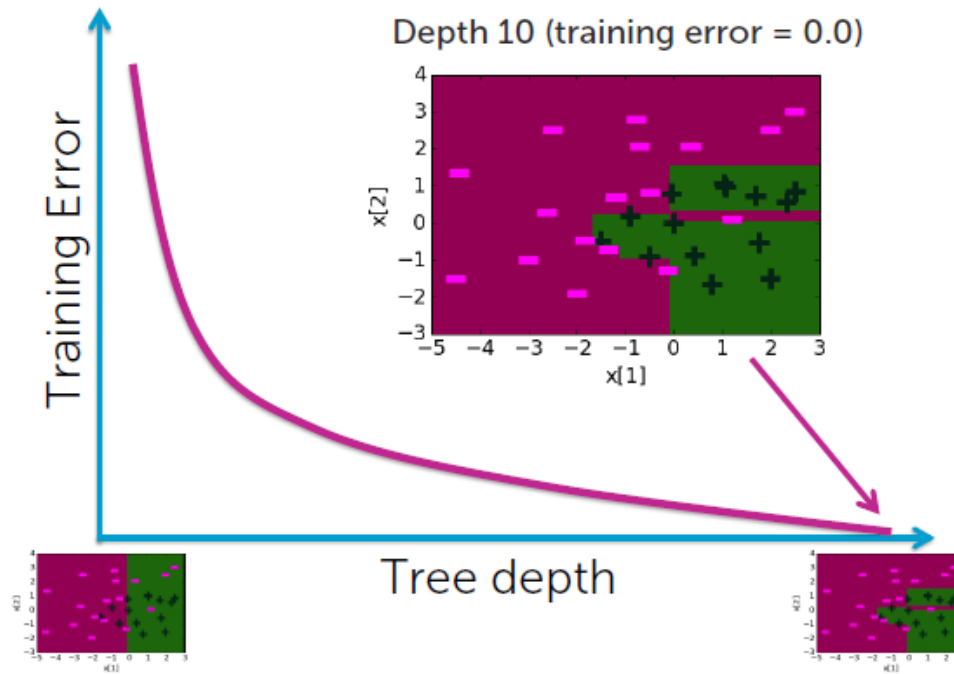
complexity of decision boundary →

1.4

Overfitting in decision tree

189

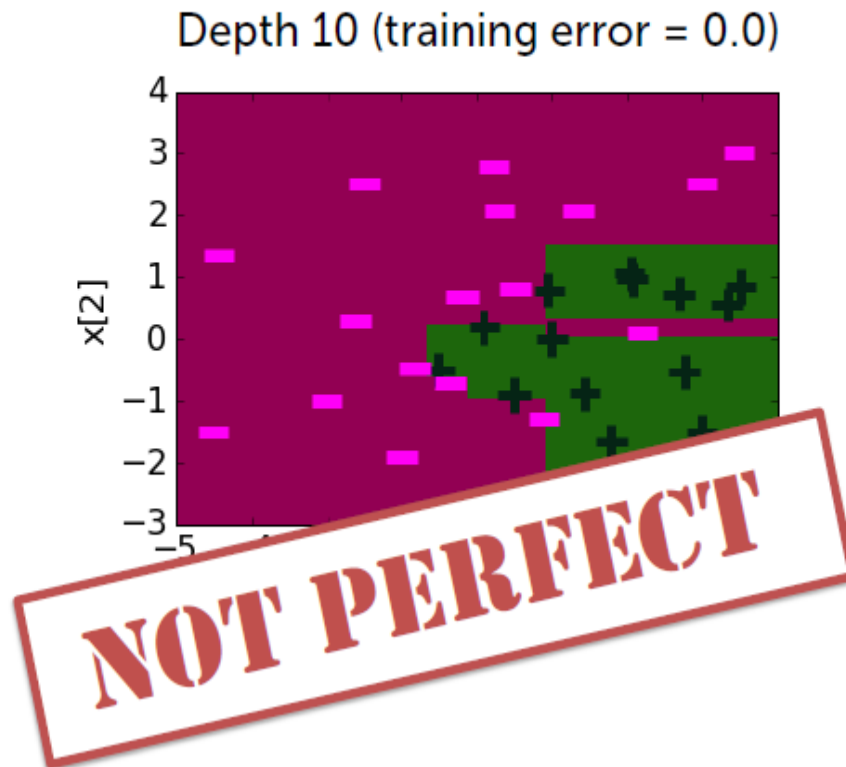
Deeper trees \rightarrow lower training error



Overfitting in decision tree

190

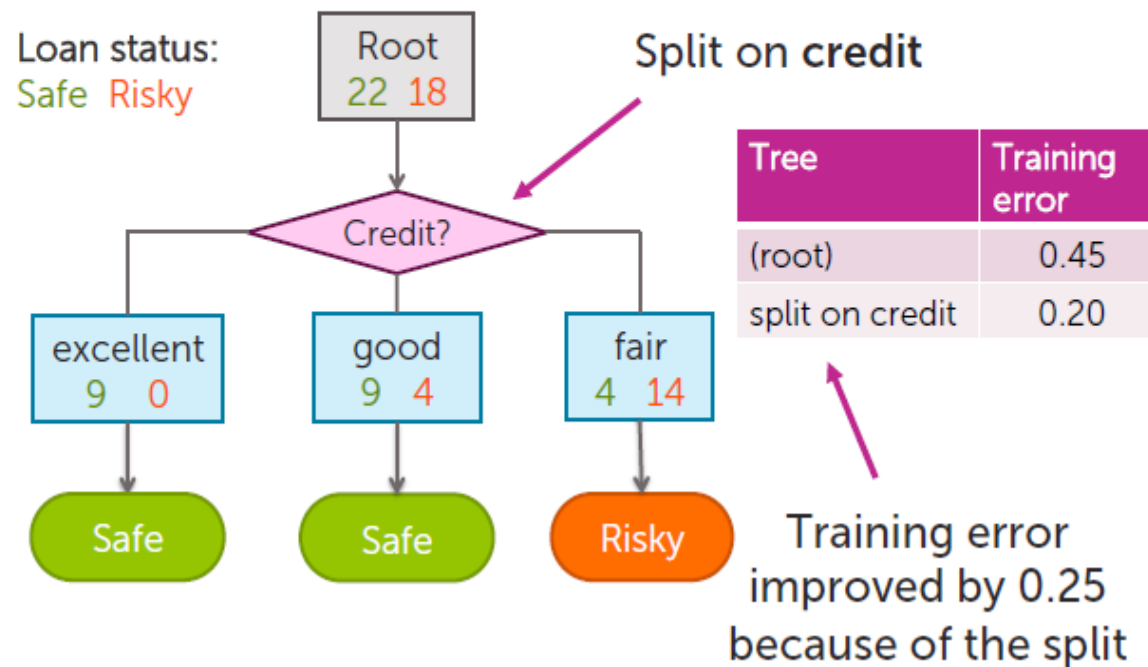
Training error = 0: Is this model perfect?



Overfitting in decision tree

191

Why training error reduces with depth?



Overfitting in decision tree

192

Feature split selection algorithm

- Given a subset of data M (a node in a tree)
- For each feature $h_j(\mathbf{x})$:
 1. Split data of M according to feature $h_j(\mathbf{x})$
 2. Compute classification error split

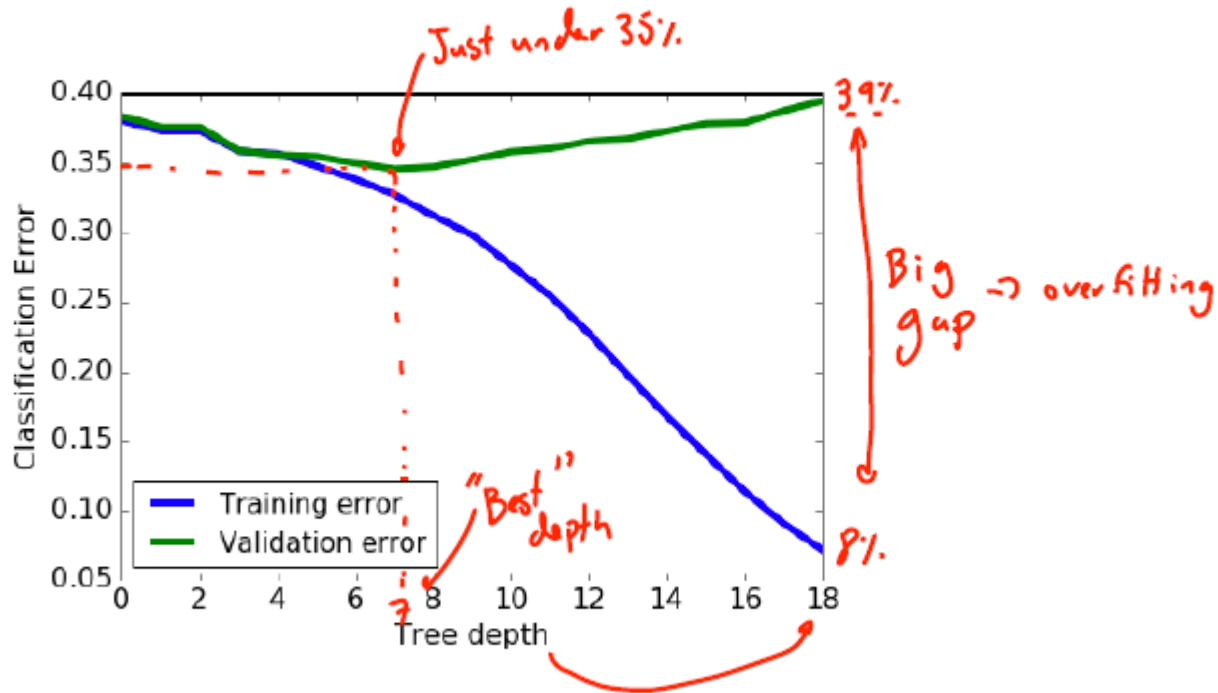
- Chose feature $h^*(\mathbf{x})$ with lowest classification error

By design, each split reduces training error

Overfitting in decision tree

193

Decision trees overfitting on loan data



Simplest tree is better

194

Principle of Occam's Razor



*"Among competing hypotheses, the one with fewest assumptions should be selected",
William of Occam, 13th Century*

Symptoms: S_1 and S_2

Diagnosis 1: 2 diseases

Two diseases D_1 and D_2 where
 D_1 explains S_1 , D_2 explains S_2

OR

SIMPLER

Diagnosis 2: 1 disease

Disease D_3 explains both
symptoms S_1 and S_2

Simplest tree is better

195

Occam's Razor for decision trees

When two trees have similar classification error on the validation set, pick the simpler one

Complexity	Train error	Validation error
Simple	0.23	0.24
Moderate	0.12	0.15
Complex	0.07	0.15
Super complex	0	0.18

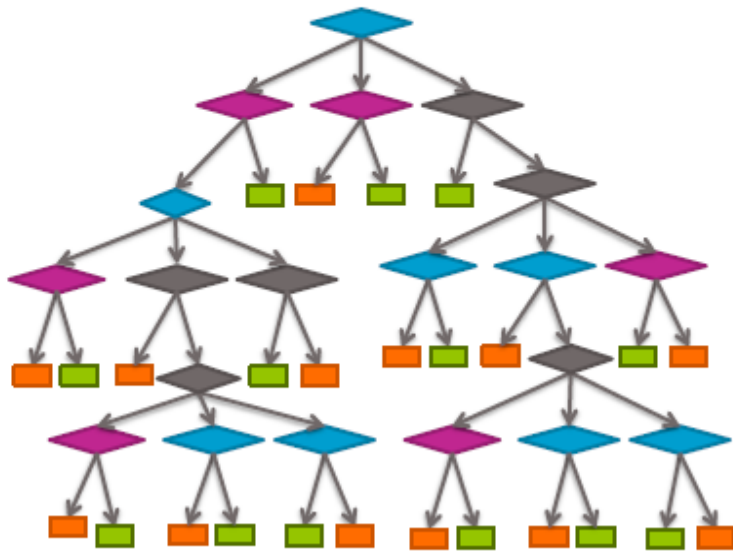
Annotations:

- Red circles around "Train error" and "Validation error" headers.
- Red arrow from "Moderate" row to "Simple" row: pick
- Red arrow from "Complex" row to "Moderate" row: bad!
- Red arrow from "Super complex" row to "Complex" row: Overfit
- Red arrow from "Moderate" row to "Validation error" header: Same validation error

Simplest tree is better

196

Which tree is simpler?



OR



Simpler

Simplest tree is better

197

How do we pick simpler trees?

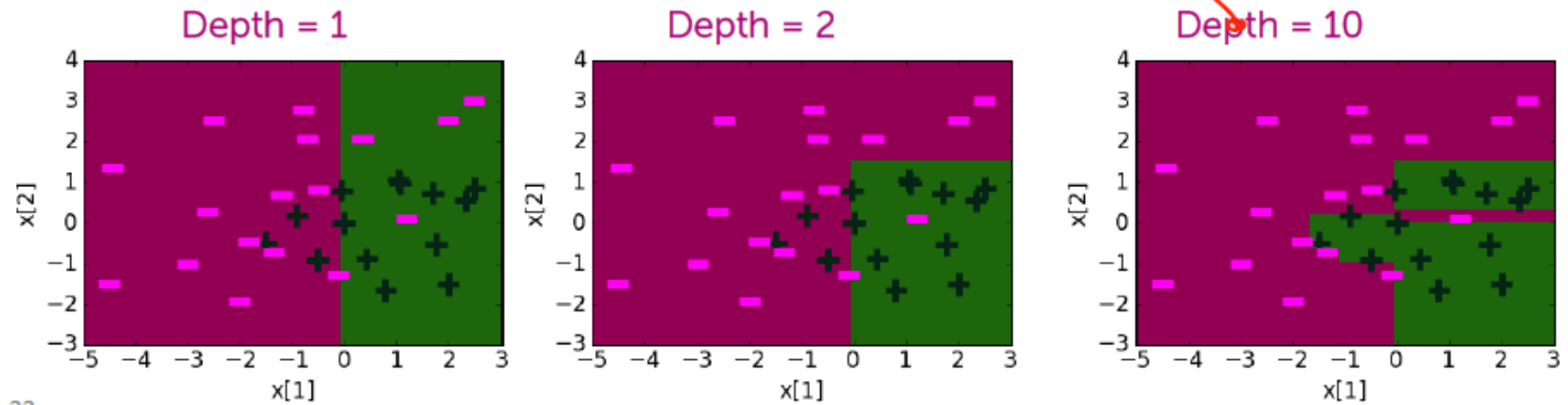
1. **Early Stopping:** Stop learning algorithm **before** tree become too complex
2. **Pruning:** Simplify tree **after** learning algorithm terminates

Early stopping for learning decision trees

198

Deeper trees →
Increasing complexity

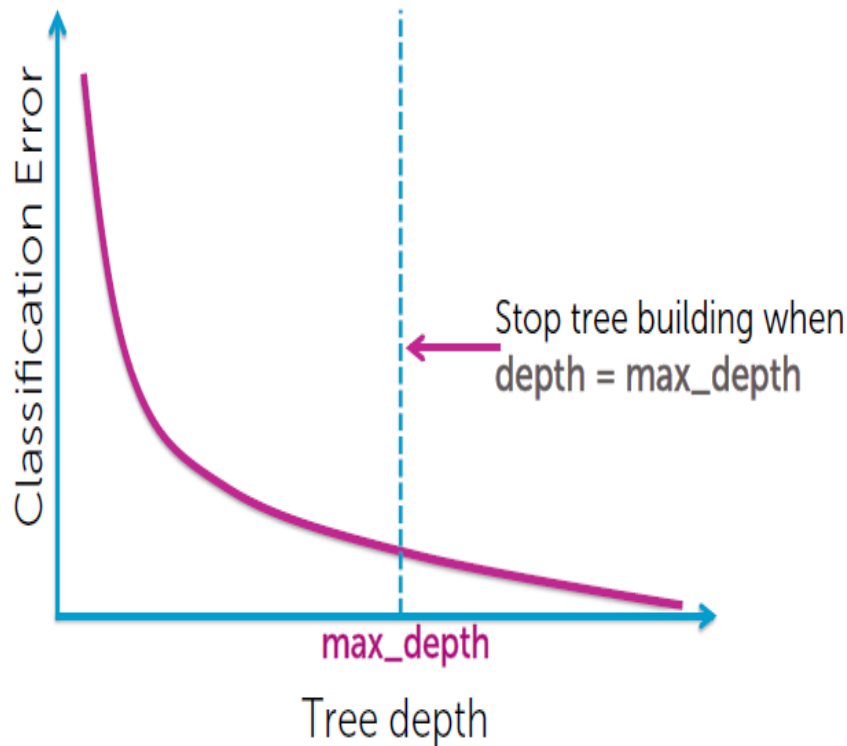
Model complexity increases with depth



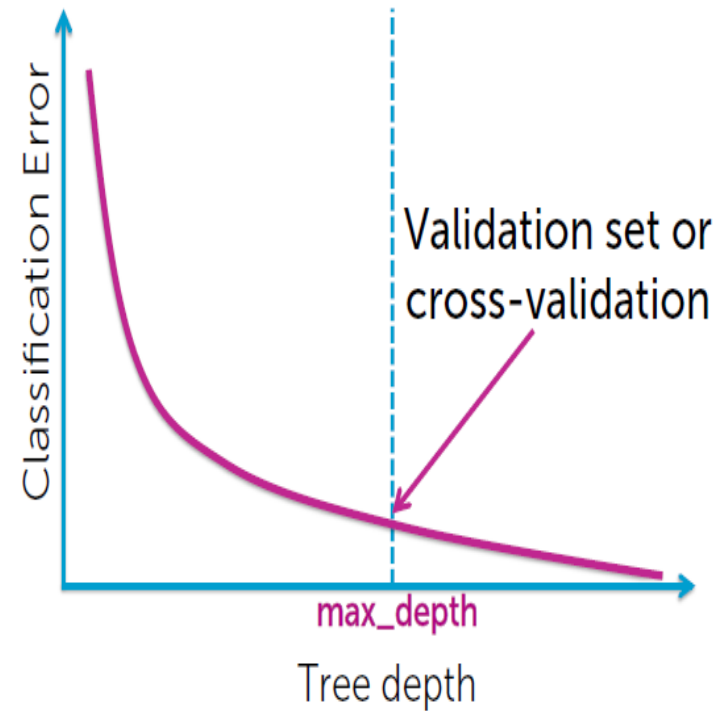
Early stopping condition 1

199

Limit depth of tree



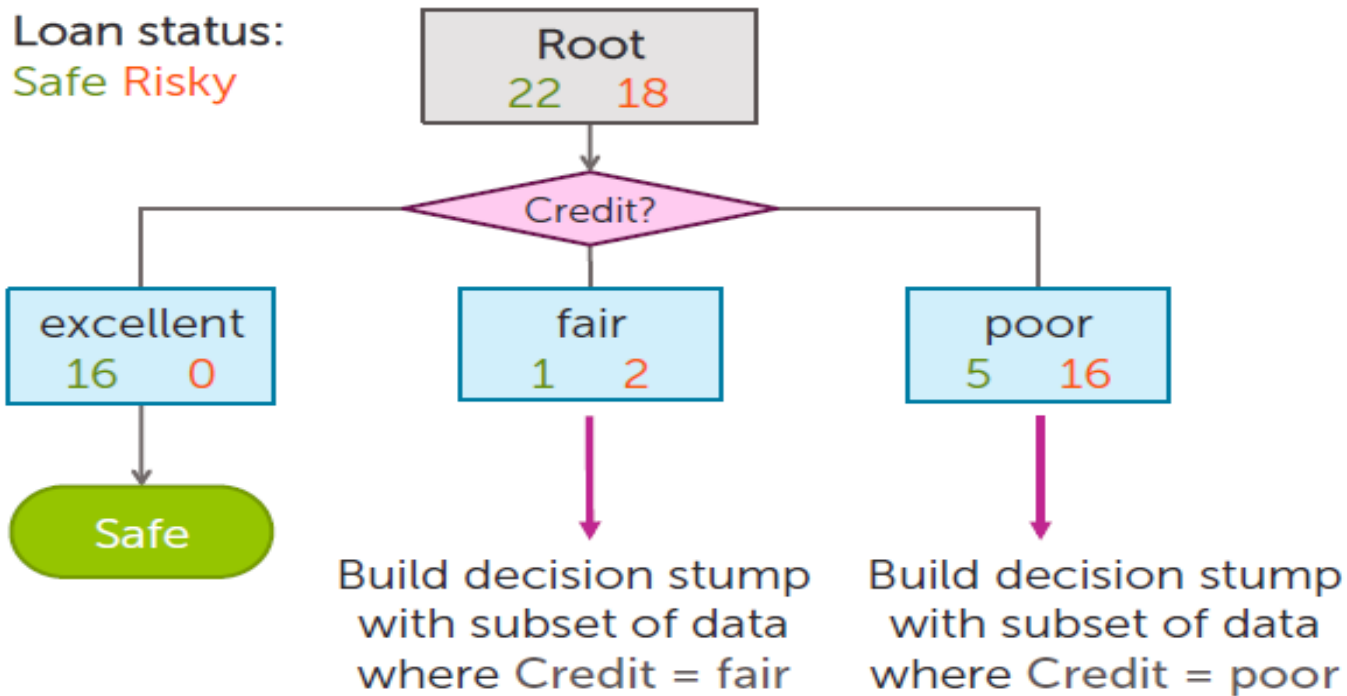
Picking value for `max_depth`???



Early stopping condition 2

200

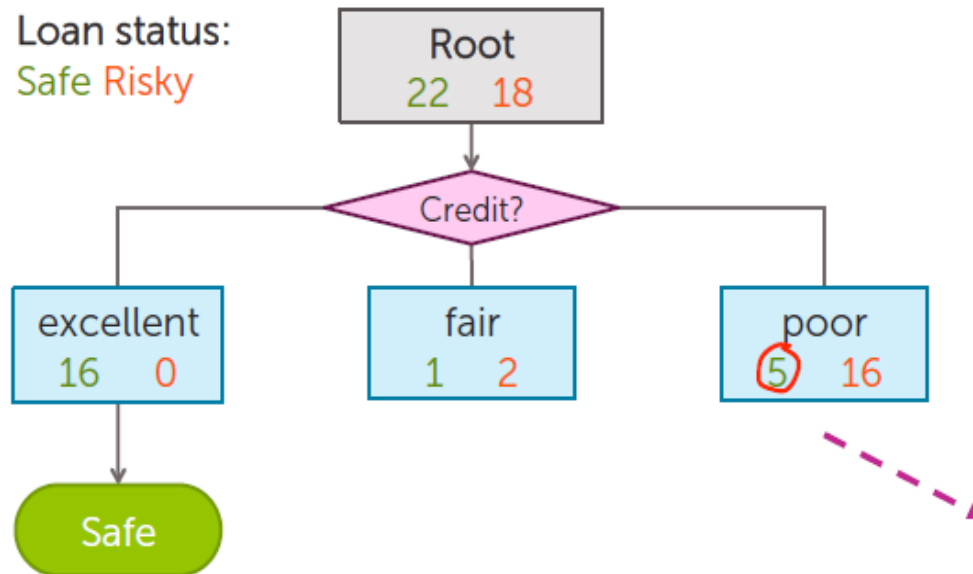
Decision tree recursion review



Early stopping condition 2

201

Split selection for credit=poor



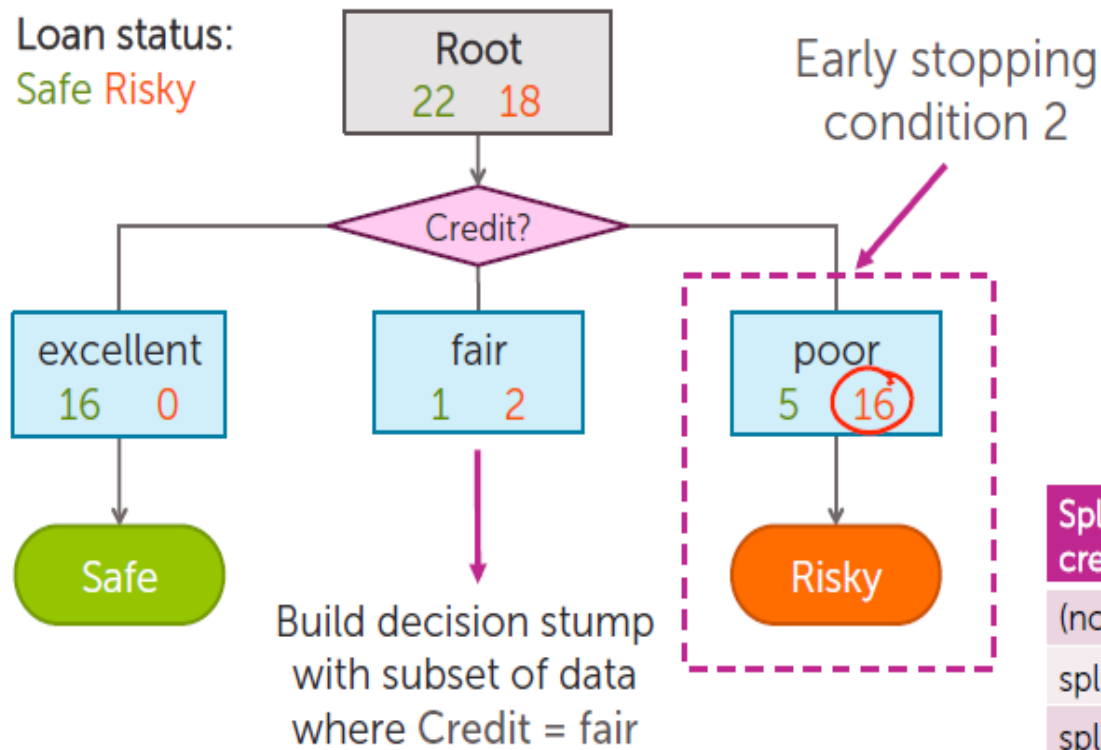
No split improves classification error
→ Stop!

Splits for credit=poor	Classification error
(no split)	<u>0.24</u>
split on term	<u>0.24</u>
split on income	<u>0.24</u>

Early stopping condition 2

202

No split improves classification error



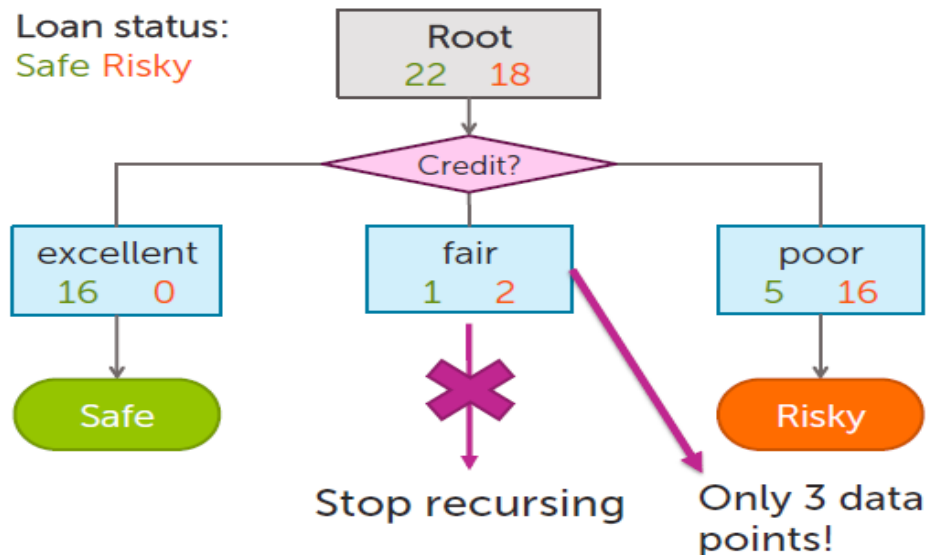
Splits for credit=poor	Classification error
(no split)	0.24
split on term	0.24
split on income	0.24

Early stopping condition 3

203

Stop if number of data points contained in a node is too small

Can we trust nodes with very few points?



Early stopping: Summary

204

1. **Limit tree depth:** Stop splitting after a certain depth
2. **Classification error:** Do not consider any split that does not cause a sufficient decrease in classification error
3. **Minimum node "size":** Do not split an intermediate node which contains too few data points

Greedy decision tree learning

205

- **Step 1:** Start with an empty tree
- **Step 2:** Select a feature to split data
- For each split of the tree:

- **Step 3:** If nothing more to, make predictions ← Majority

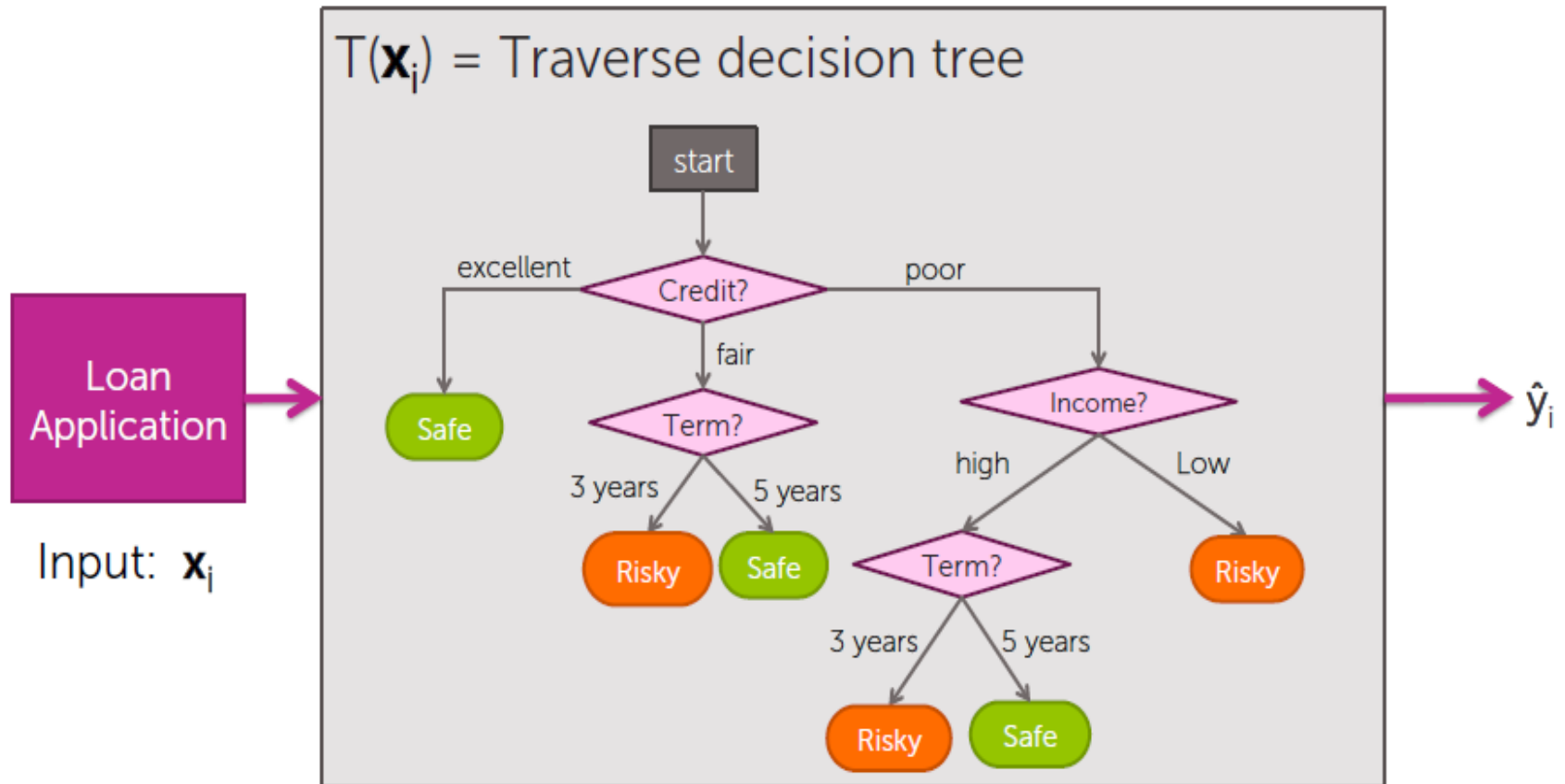
- **Step 4:** Otherwise, go to Step 2 & continue (recurse) on this split

Stopping conditions 1 & 2
or
Early stopping conditions 1, 2 & 3
Recursion

Strategies for handling missing data

Decision tree review

207



Missing data

208

Credit	Term	Income	y
excellent	3 yrs	high	safe
fair	?	low	risky
fair	3 yrs	high	safe
poor	5 yrs	high	risky
excellent	3 yrs	low	risky
fair	5 yrs	high	safe
poor	?	high	risky
poor	5 yrs	low	safe
fair	?	high	safe

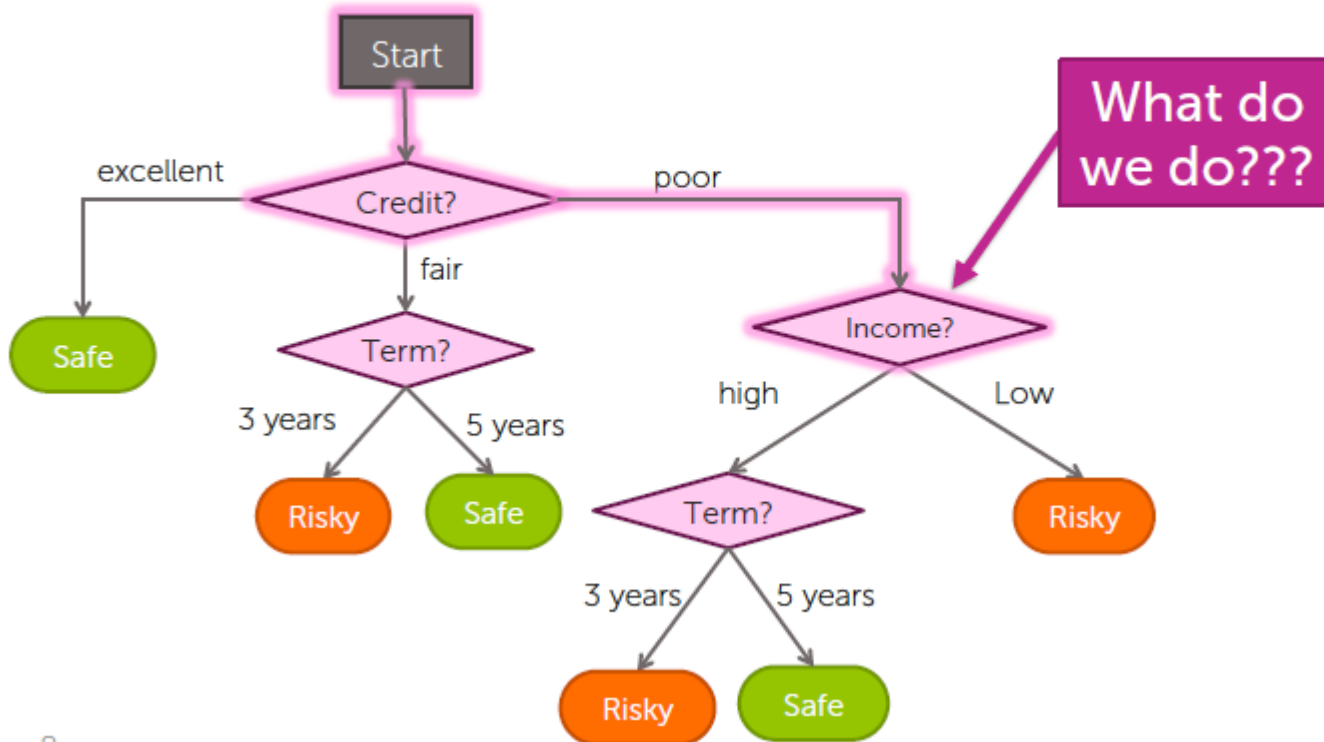
Loan application
may be
3 or 5 years

1. **Training data:** Contains "unknown" values
2. **Predictions:** Input at prediction time contains "unknown" values

Missing values during predictions

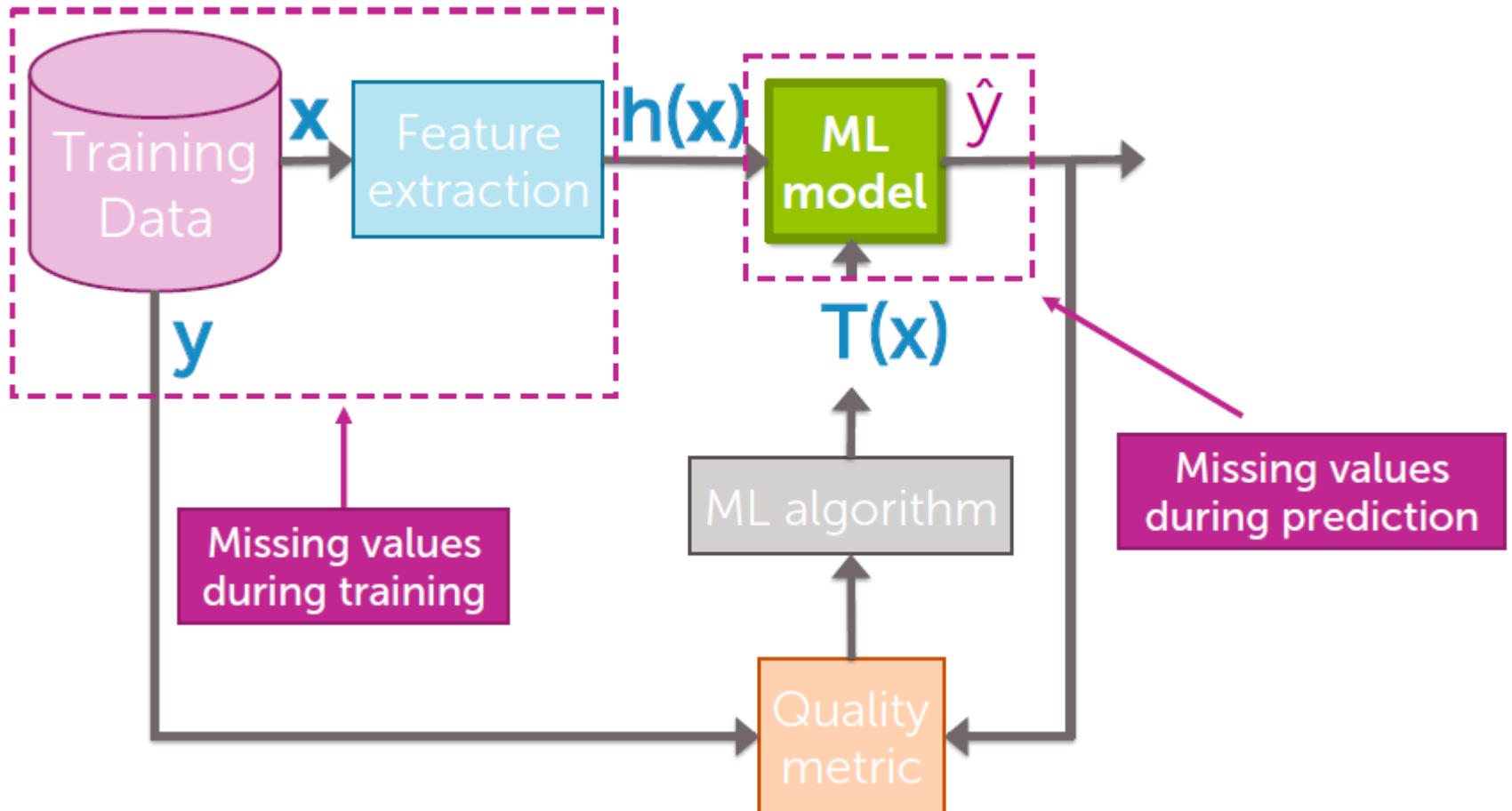
209

$x_i = (\text{Credit} = \text{poor}, \text{Income} = ?, \text{Term} = 5 \text{ years})$



Missing values

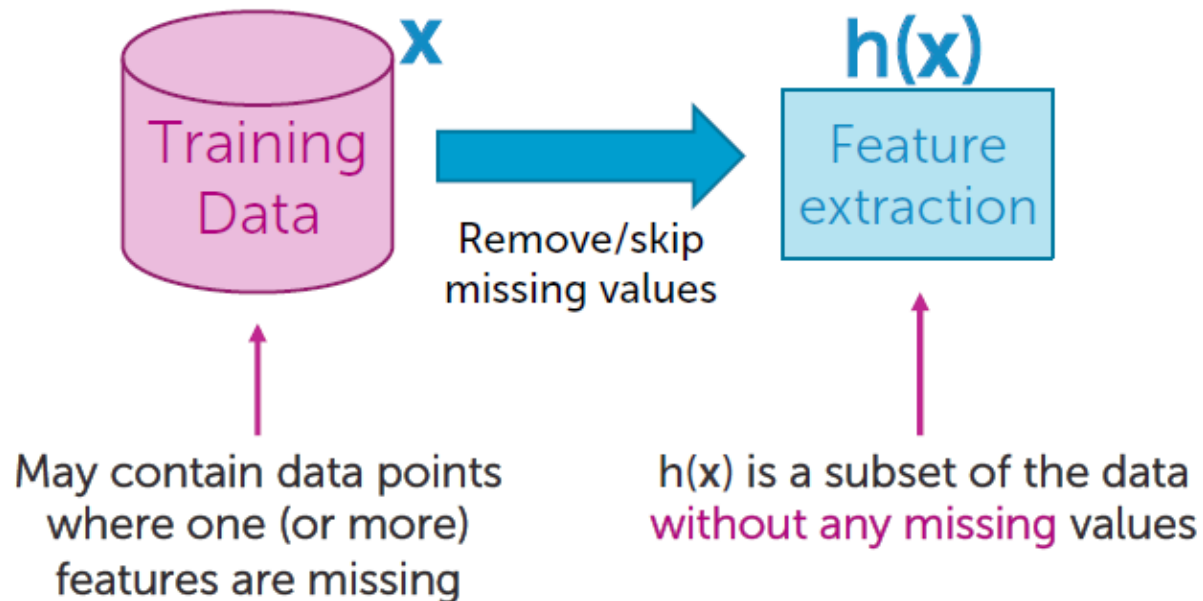
210



Handling missing data

211

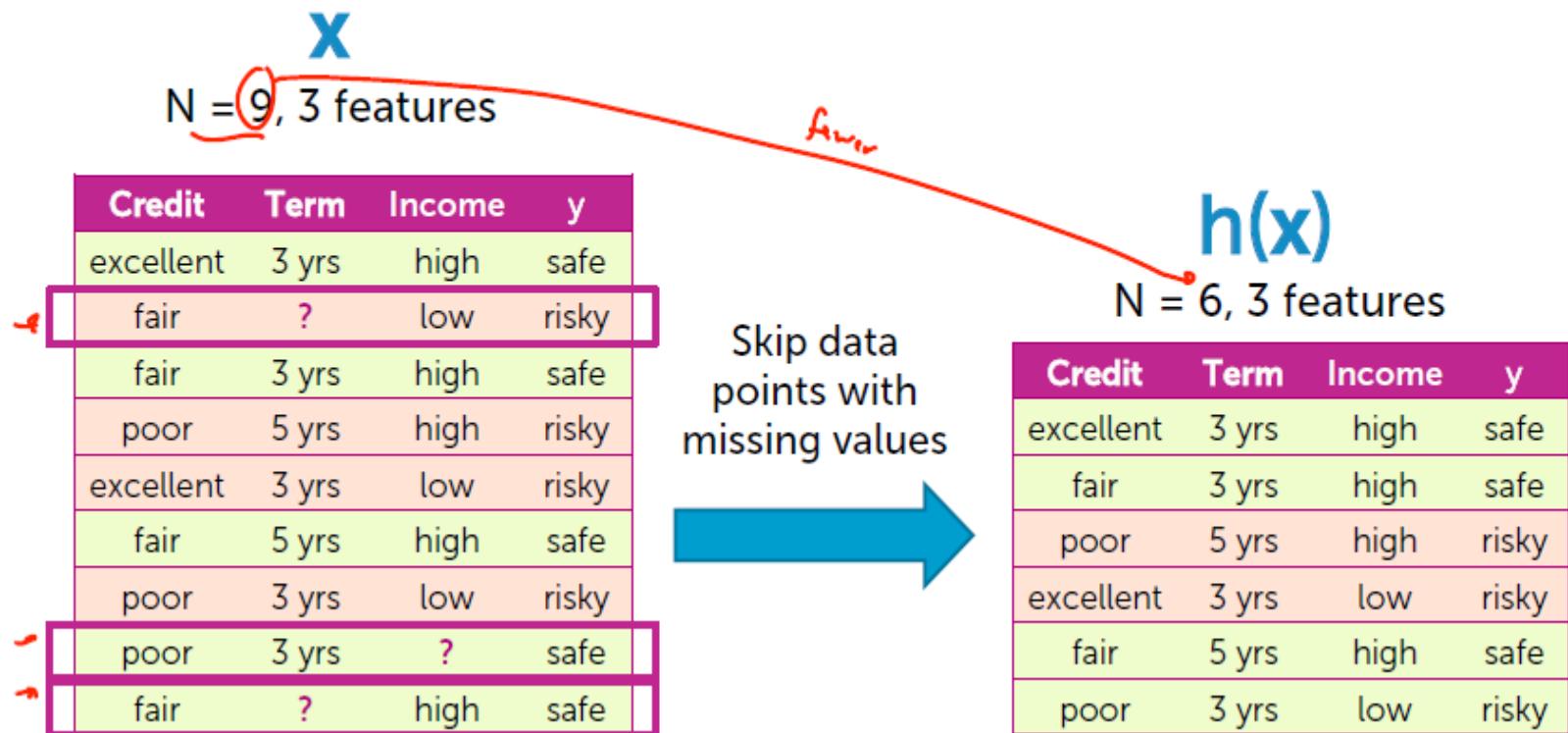
Idea 1: Purification by skipping/removing



Handling missing data

212

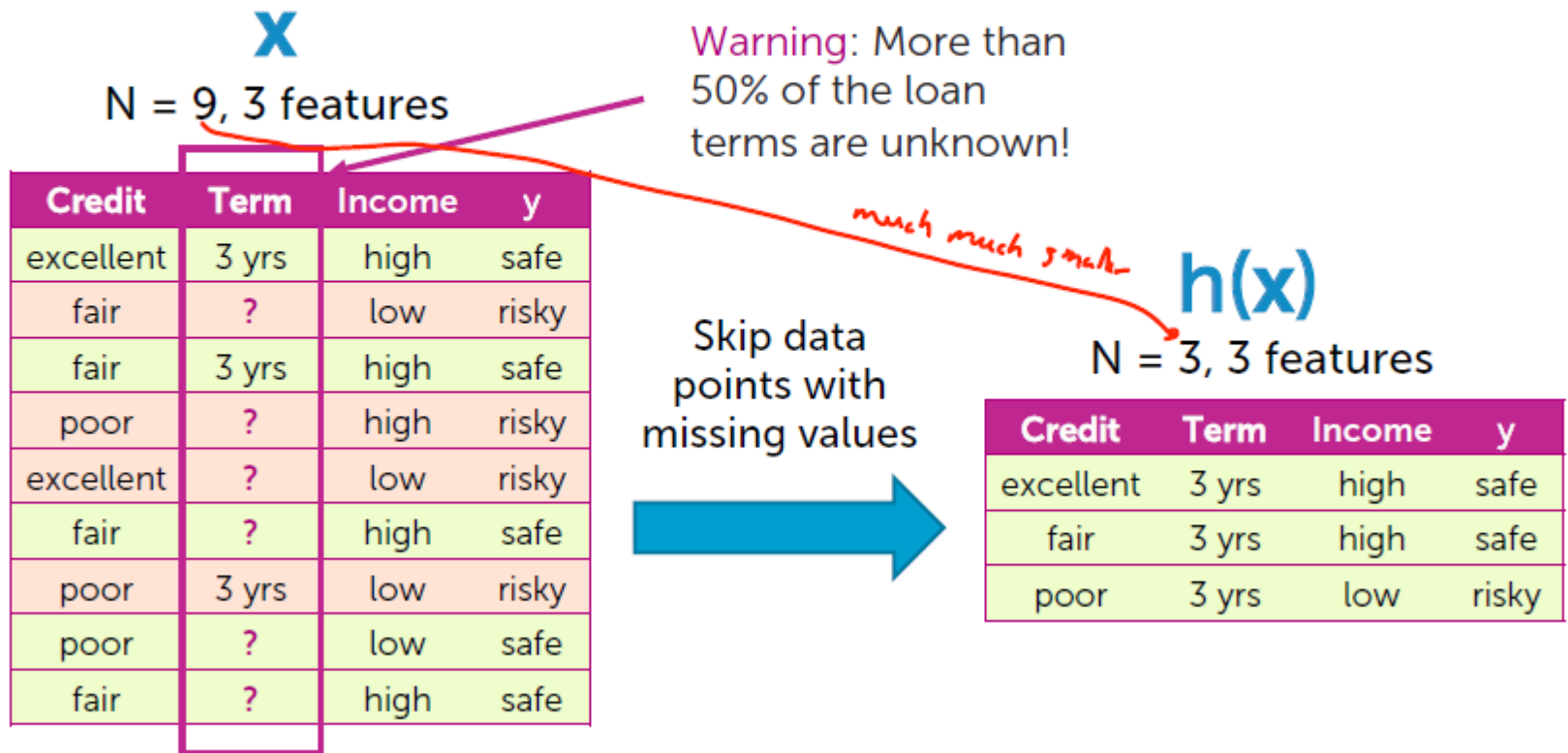
Idea 1: Skip data points with missing values



Handling missing data

213

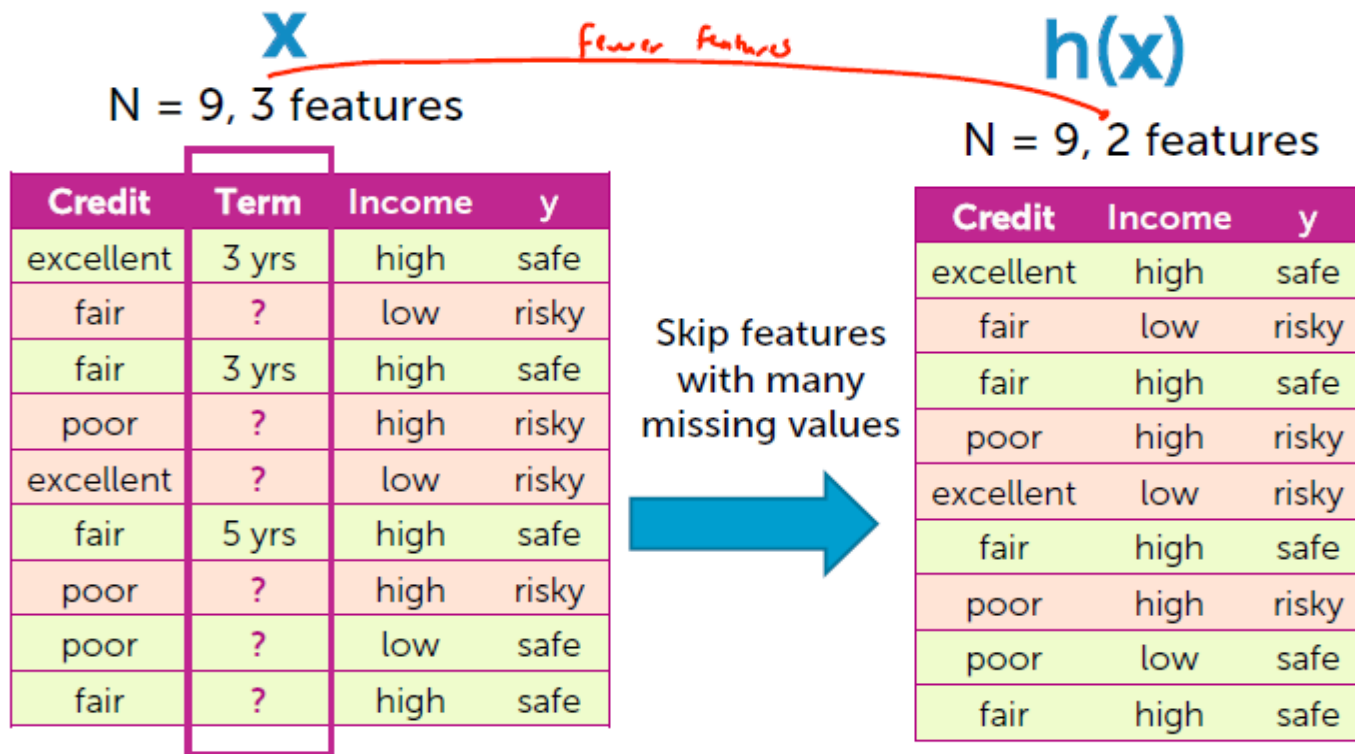
The challenge with Idea 1



Missing data

214

Idea 2: Skip features with missing values



Handling missing data

215

Missing value skipping: Ideas 1 & 2

Idea 1: Skip data points where any feature contains a missing value

- Make sure only a few data points are skipped

Idea 2: Skip an entire feature if it's missing for many data points

- Make sure only a few features are skipped

Handling missing data

216

Missing value skipping: Pros and Cons

Pros

- Easy to understand and implement
- Can be applied to any model (decision trees, logistic regression, linear regression,...)

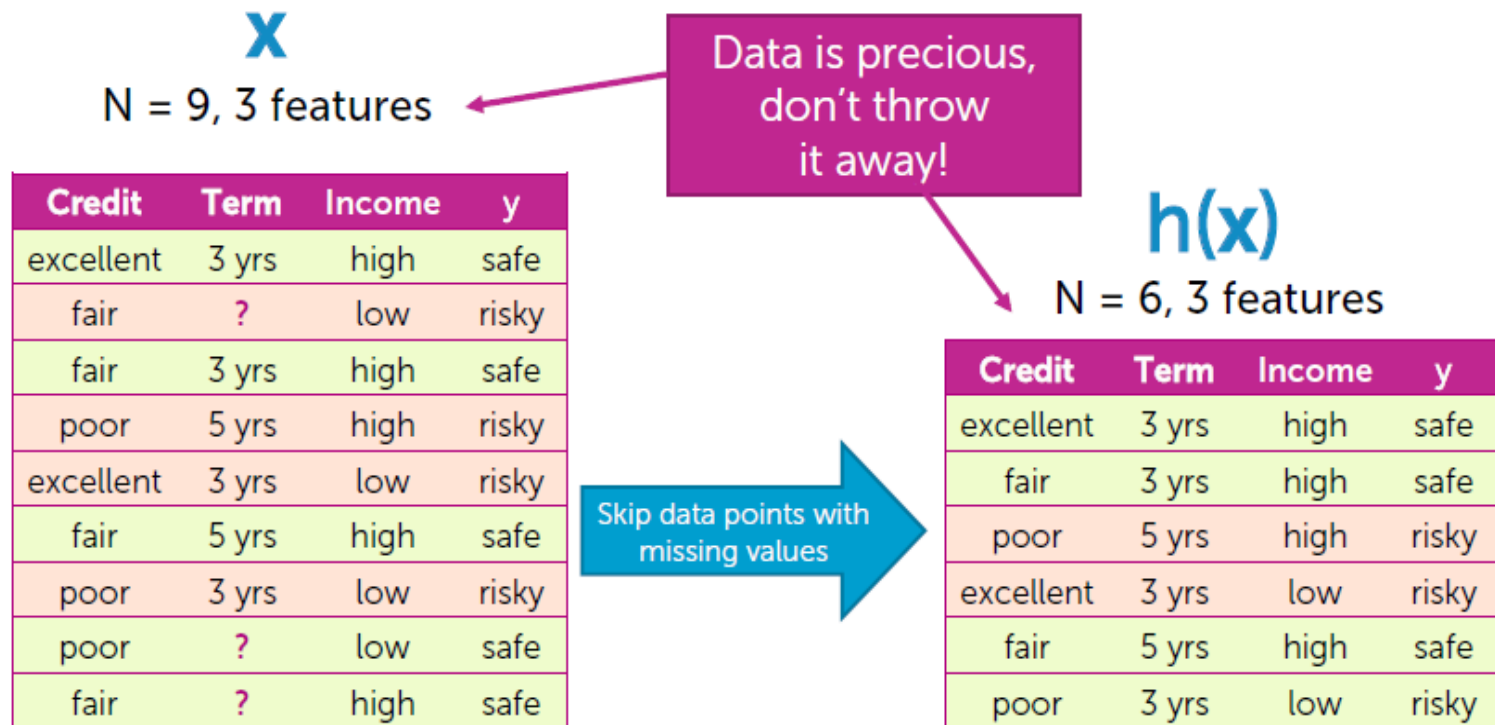
Cons

- Removing data points and features may remove important information from data
- Unclear when it's better to remove data points versus features
- Doesn't help if data is missing at prediction time

Data is precious

217

Main drawback of skipping strategy



Data is precious

218

Can we keep all the data?

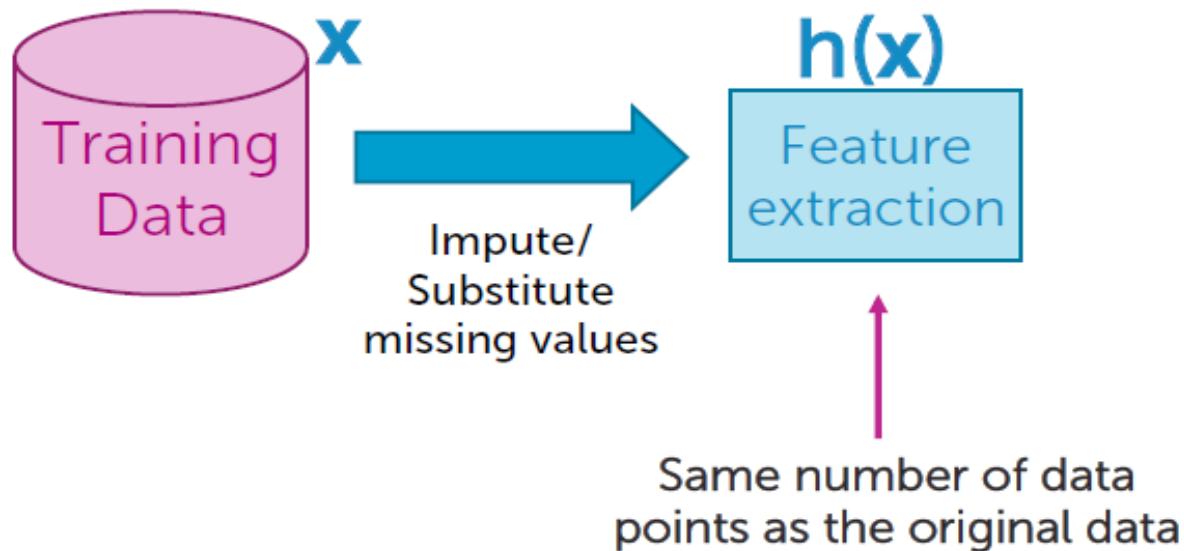
credit	term	income	y
excellent	3 yrs	high	safe
fair	?	low	risky
fair	3 yrs	high	safe
poor	5 yrs	high	risky
excellent	3 yrs	low	risky
fair	5 yrs	high	safe
poor	3 yrs	high	risky
poor	?	low	safe
fair	?	high	safe

Use other data points in x to "guess" the "?"

Handling missing data

219

Idea 2: Purification by imputing



Handling missing data

220

Idea 2: Imputation/Substitution

N = 9, 3 features

Credit	Term	Income	y
excellent	3 yrs	high	safe
fair	?	low	risky
fair	3 yrs	high	safe
poor	5 yrs	high	risky
excellent	3 yrs	low	risky
fair	5 yrs	high	safe
poor	3 yrs	high	risky
poor	?	low	safe
fair	?	high	safe

Fill in each missing value with a calculated guess



N = 9, 3 features

Credit	Term	Income	y
excellent	3 yrs	high	safe
fair	3 yrs	low	risky
fair	3 yrs	high	safe
poor	5 yrs	high	risky
excellent	3 yrs	low	risky
fair	5 yrs	high	safe
poor	3 yrs	high	risky
poor	3 yrs	low	safe
fair	3 yrs	high	safe

28

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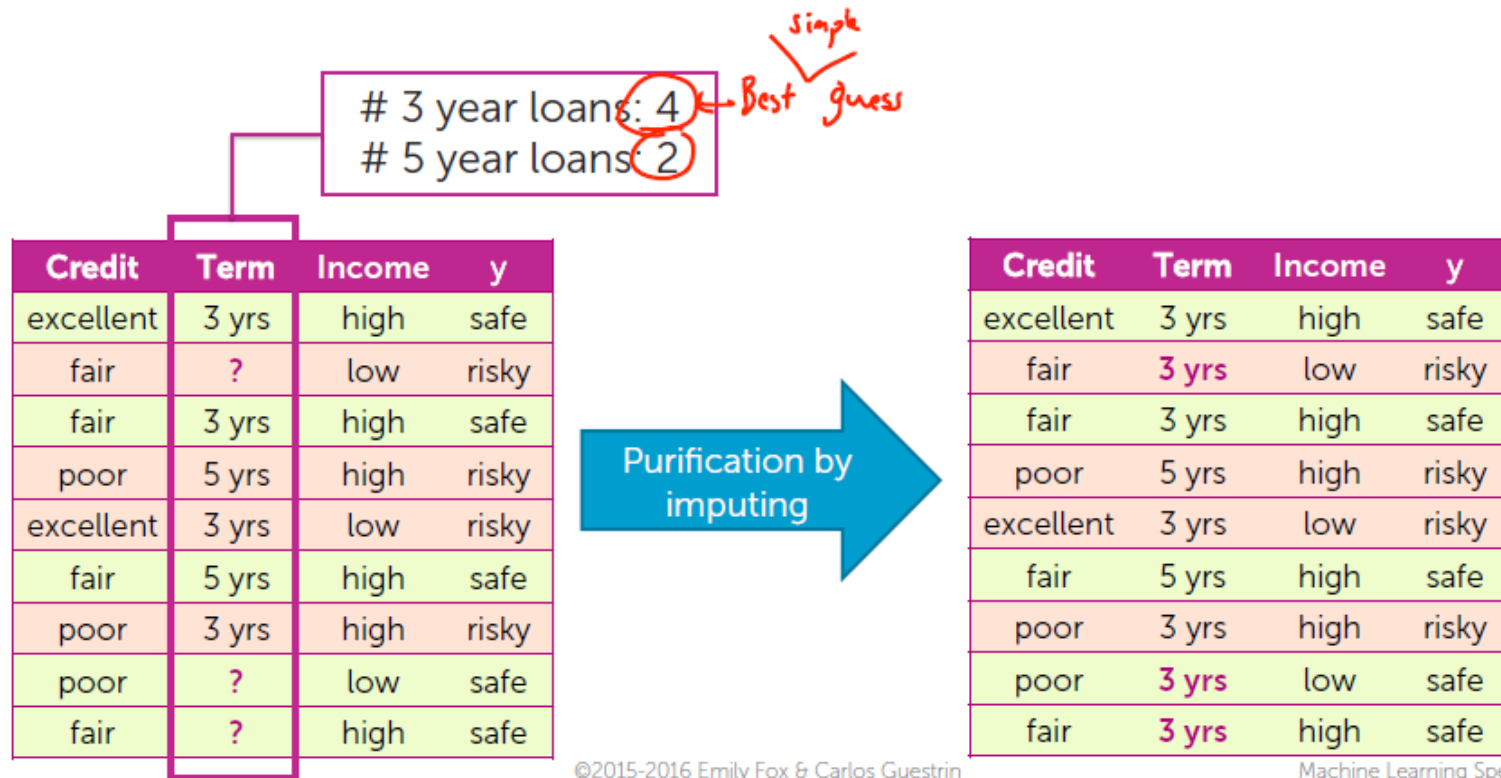
Machine Learning Practice

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Example

221

Example: Replace ? with most common value



29

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Machine Learning Specialization

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Handling missing data

222

Common (simple) rules for purification by imputation

Credit	Term	Income	y
excellent	3 yrs	high	safe
fair	?	low	risky
fair	3 yrs	high	safe
poor	5 yrs	high	risky
excellent	3 yrs	low	risky
fair	5 yrs	high	safe
poor	3 yrs	high	risky
poor	?	low	safe
fair	?	high	safe

Impute each feature with missing values:

1. Categorical features use mode: Most popular value (mode) of non-missing x_i
2. Numerical features use average or median: Average or median value of non-missing x_i

Many advanced methods exist, e.g., expectation-maximization (EM) algorithm

Handling missing data

223

Missing value imputation: Pros and Cons

Pros

- Easy to understand and implement
- Can be applied to any model
(decision trees, logistic regression, linear regression,...)
- Can be used at prediction time: use same imputation rules

Cons

- May result in systematic errors

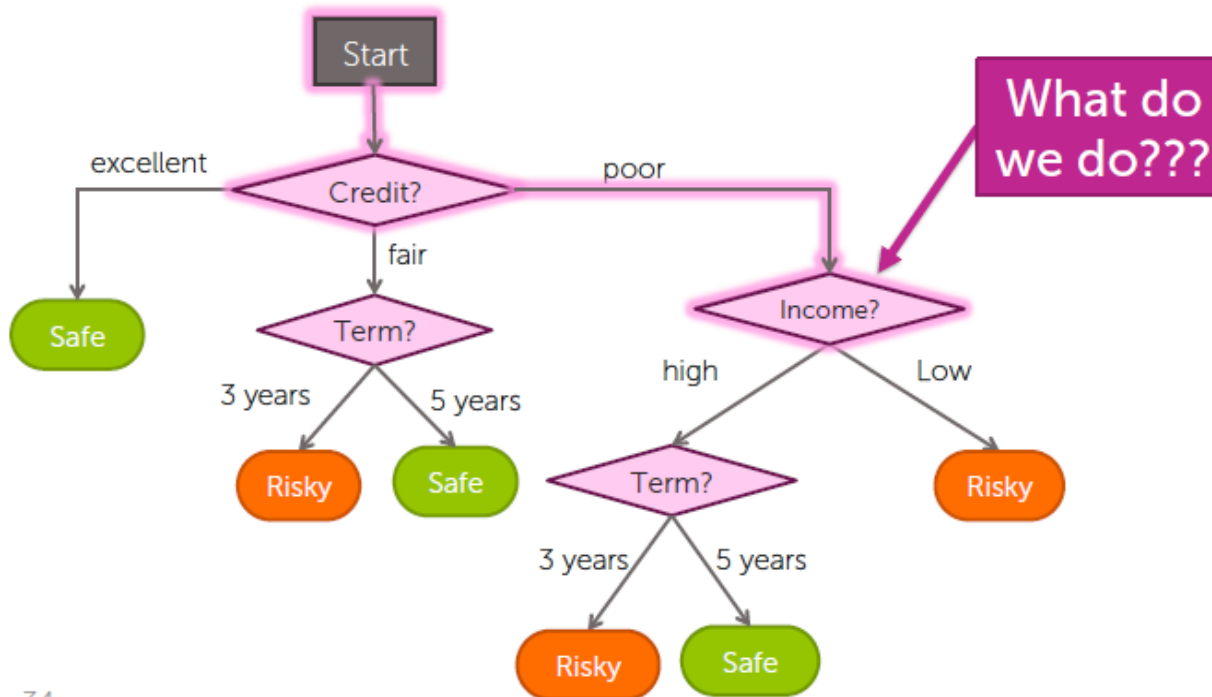
Example: Feature "age" missing in all banks in Washington by state law

Strategy 3: adapt algorithm

224

Missing values during prediction: *revisited*

$x_i = (\text{Credit} = \text{poor}, \text{Income} = ?, \text{Term} = 5 \text{ years})$



34

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Machine Learning Spring

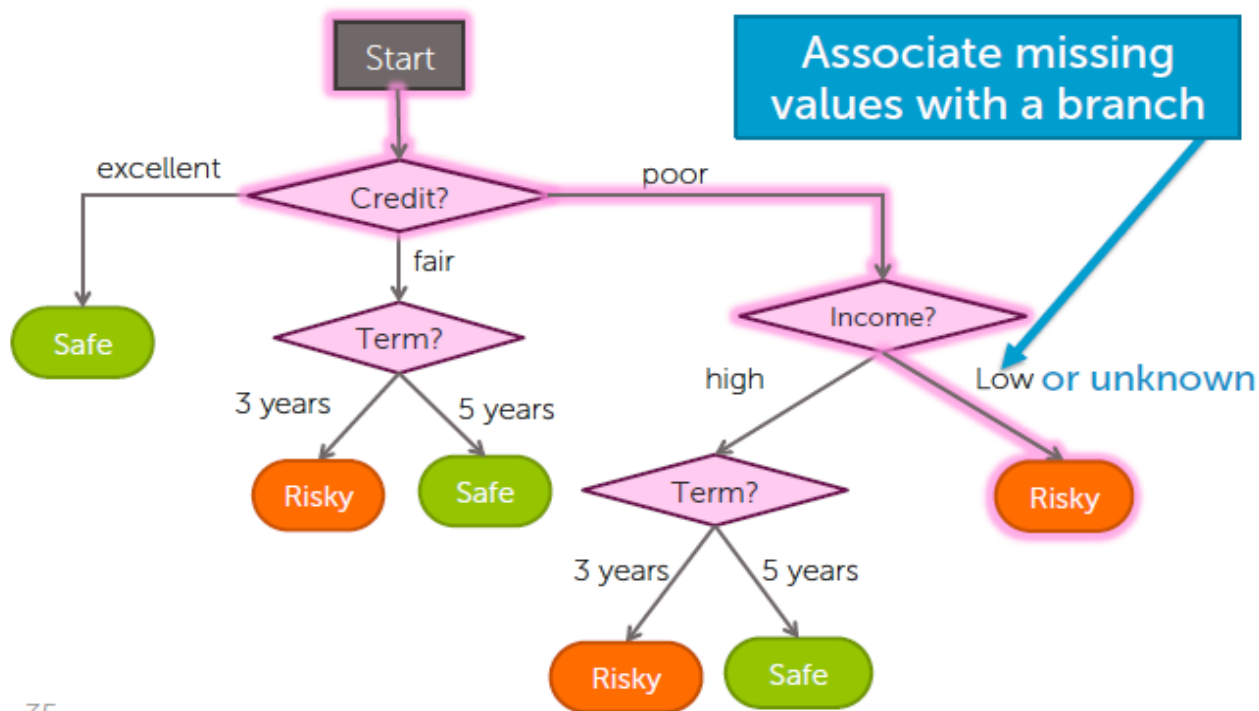
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Strategy 3: adapt algorithm

225

Add missing values to the tree definition

$x_i = (\text{Credit} = \text{poor}, \text{Income} = ?, \text{Term} = 5 \text{ years})$



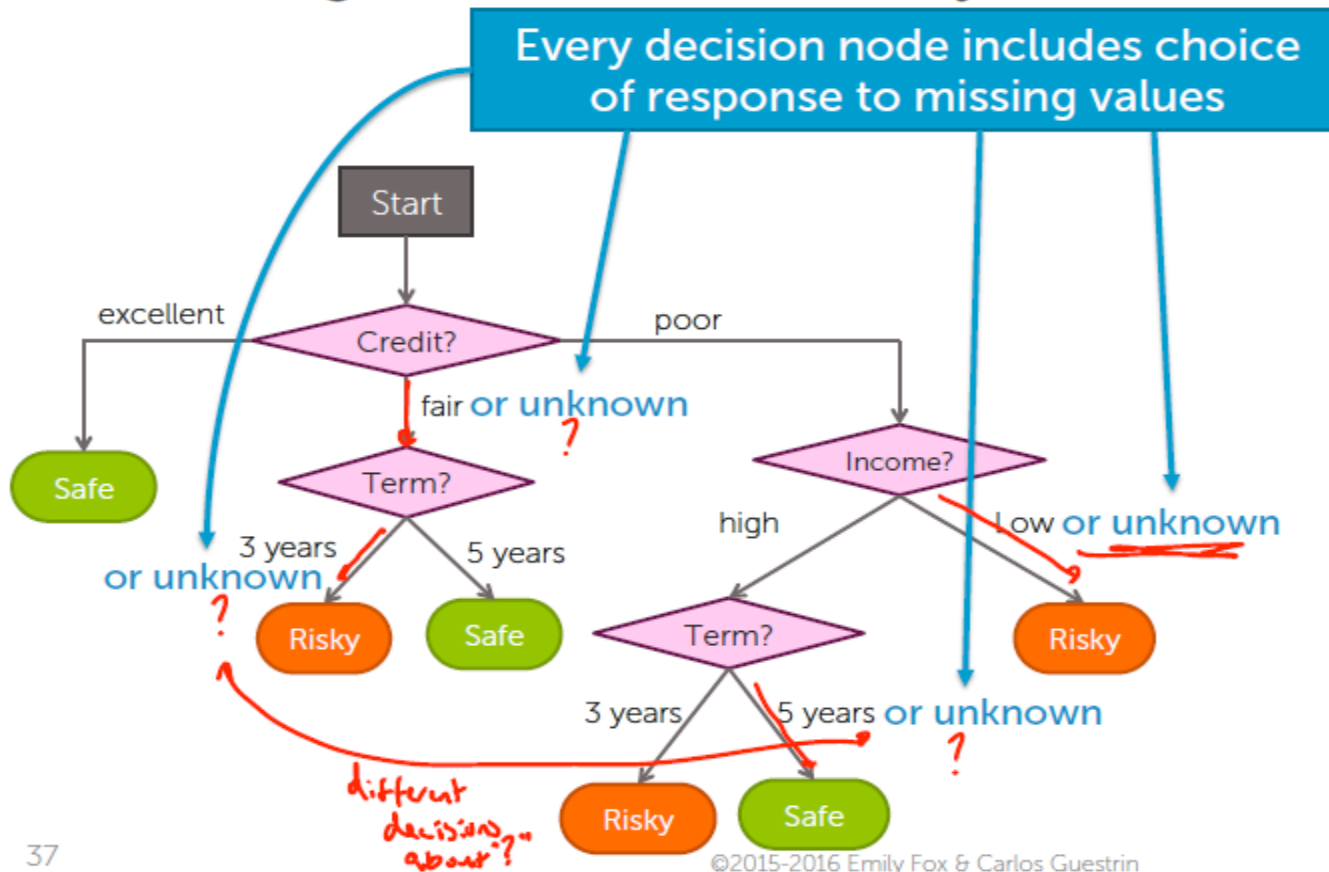
35

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Strategy 3: adapt algorithm

226

Add missing value choice to every decision node



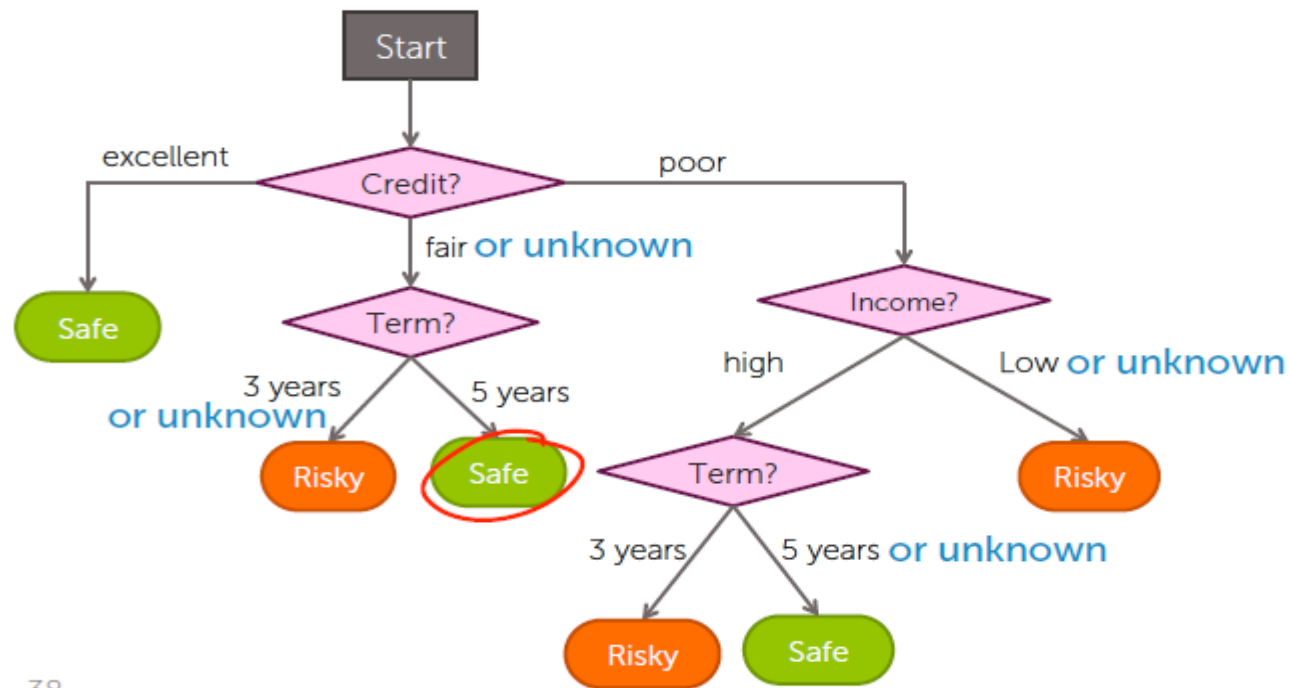
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Strategy 3: adapt algorithm

227

Prediction with missing values becomes simple

$x_i = (\text{Credit} = ?, \text{Income} = \text{high}, \text{Term} = 5 \text{ years})$



38

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Machine

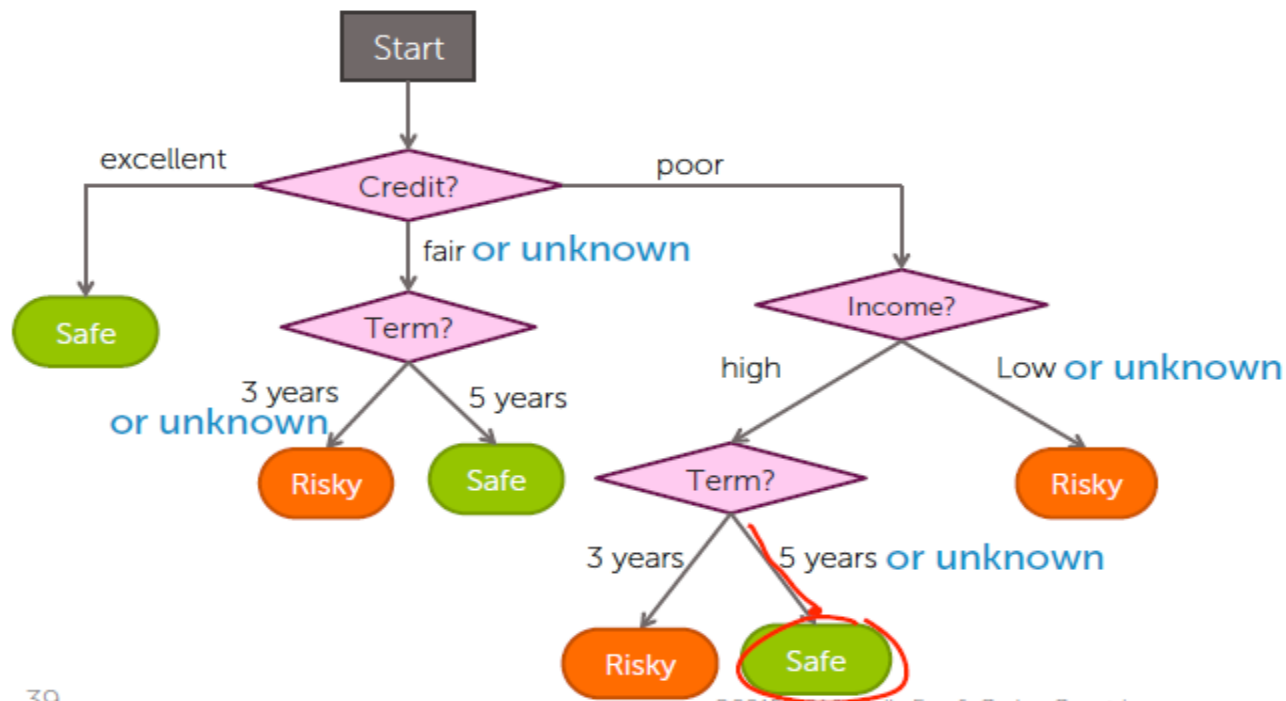
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Strategy 3: adapt algorithm

228

Prediction with missing values becomes simple

$x_i = (\text{Credit} = \text{poor}, \text{Income} = \text{high}, \text{Term} = ?)$



39

Strategy 3: adapt algorithm

229

Explicitly handling missing data by learning algorithm: Pros and Cons

Pros

- Addresses training and prediction time
- More accurate predictions

Cons

- Requires modification of learning algorithm
 - Very simple for decision trees

Feature split selection with missing data

230

Greedy decision tree learning

- **Step 1:** Start with an empty tree
- **Step 2:** Select a feature to split data
- For each split of the tree:
 - **Step 3:** If nothing more to, make predictions
 - **Step 4:** Otherwise, go to **Step 2** & continue (recurse) on this split

Pick feature split leading to lowest classification error

Must select feature & branch for missing values!

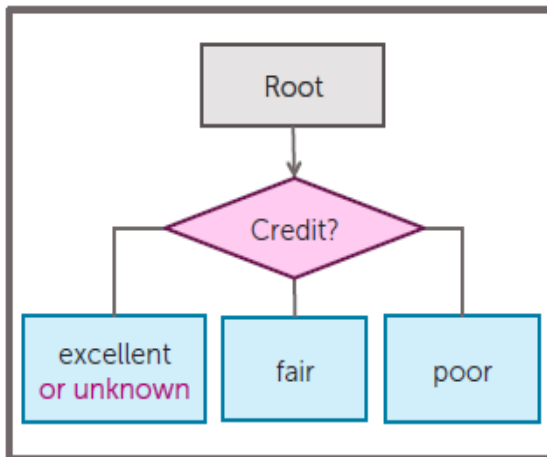
Feature split selection with missing data

231

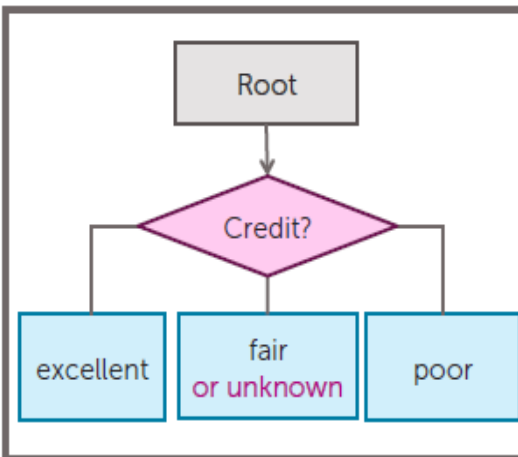
Should **missing** go left, right, or middle?

Choose branch that leads to lowest classification error!

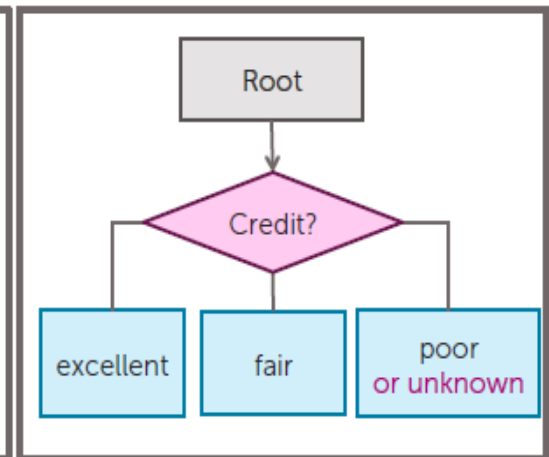
Choice 1: Missing values go with Credit=excellent



Choice 2: Missing values go with Credit=fair



Choice 3: Missing values go with Credit=poor



Feature split selection with missing data

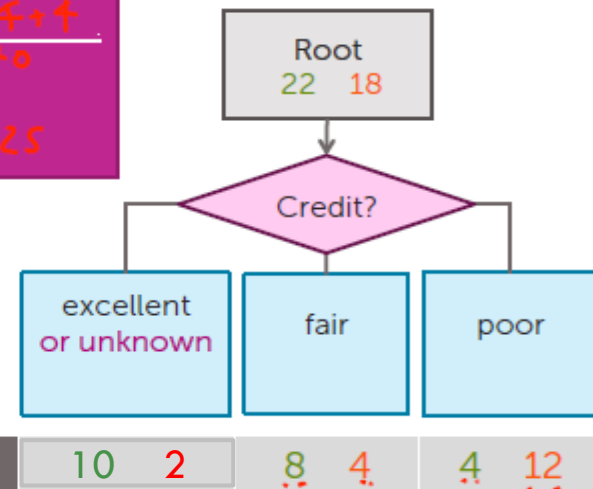
232

Computing classification error of decision stump with missing data

N = 40, 3 features

Credit	Term	Income	y
excellent	3 yrs	high	safe
?	5 yrs	low	risky
fair	3 yrs	high	safe
poor	5 yrs	high	risky
?	3 yrs	low	risky
?	5 yrs	low	safe
poor	3 yrs	high	risky
poor	5 yrs	low	safe
fair	3 yrs	high	safe
...

$$\text{Error} = \frac{2+4+4}{40} = 0.25$$



Feature split selection with missing data

233

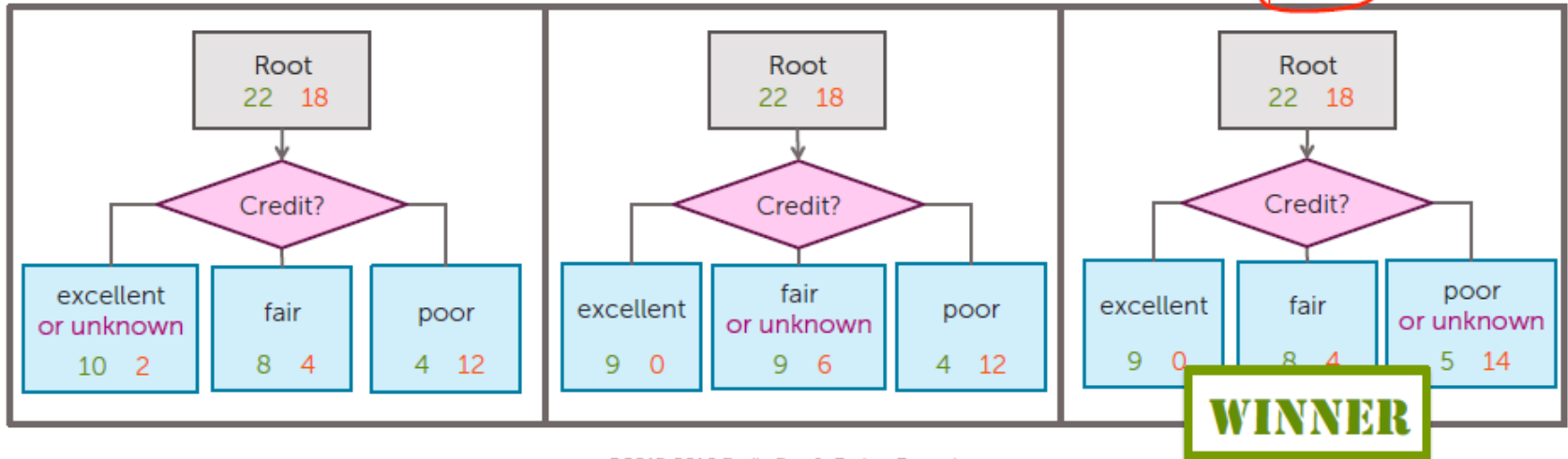
Use classification error to decide

Best choice → assign "unknown" to Credit = poor

Choice 1: error = 0.25

Choice 2: error = 0.25

Choice 3: error = 0.225



50

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Feature split selection with missing data

234

- Given a subset of data M (a node in a tree)
- For each feature $h_i(\mathbf{x})$:
 1. Split data points of M where $h_i(\mathbf{x})$ is *not* “unknown” according to feature $h_i(\mathbf{x})$
 2. Consider assigning data points with “unknown” value for $h_i(\mathbf{x})$ to each branch
 - A. Compute classification error split & branch assignment of “unknown” values
- Chose feature $h^*(\mathbf{x})$ & branch assignment of “unknown” with lowest classification error

What can you do now

235

Describe common ways to handling missing data:

1. Skip all rows with any missing values
2. Skip features with many missing values
3. Impute missing values using other data points

Modify learning algorithm (**decision trees**) to handle missing data:

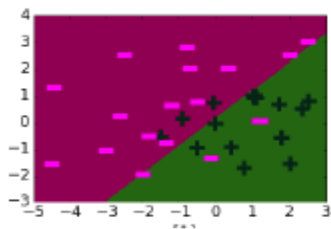
1. Missing values get added to one branch of split
2. Use classification error to determine where missing values go

Ensemble classifiers and boosting

Simple classifiers

237

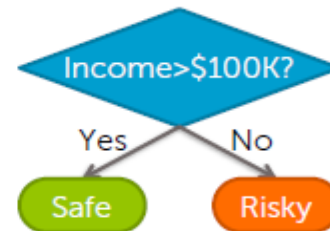
Simple (weak) classifiers are good!



Logistic regression
w. simple
features



Shallow
decision trees



Decision
stumps

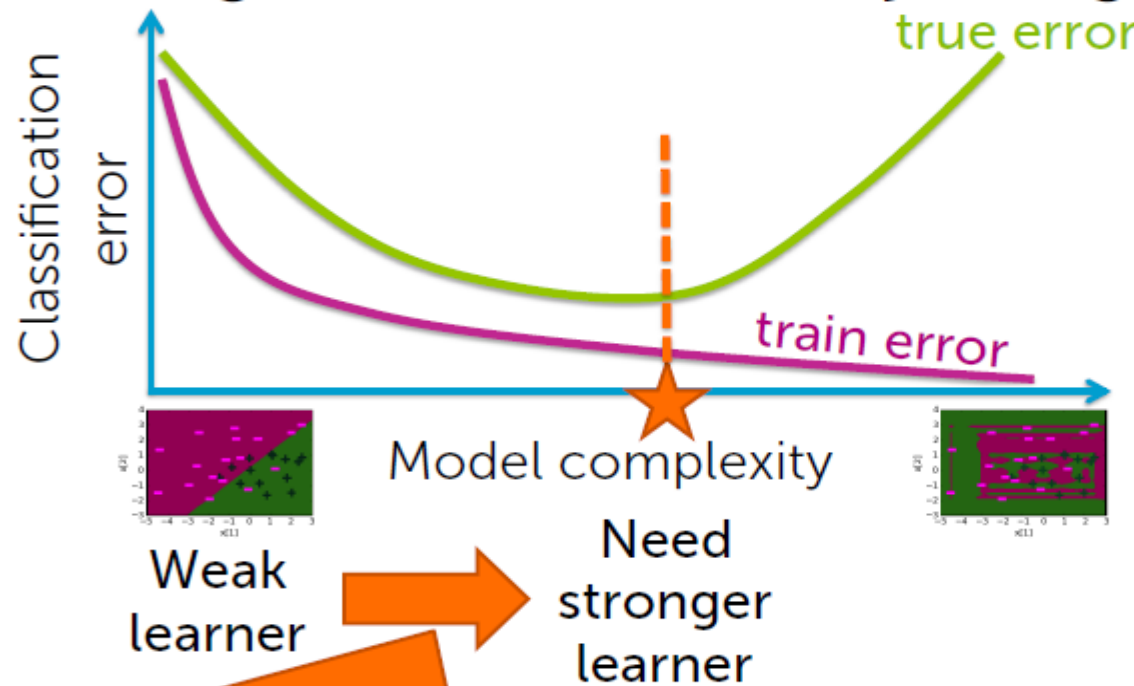
Low variance. Learning is fast!

But high bias...

Simple classifiers

238

Finding a classifier that's just right



Option 1: add more features or depth
Option 2: ?????

Can they be combined?

239

Boosting question

"Can a set of weak learners be combined to create a stronger learner?" *Kearns and Valiant (1988)*



Yes! *Schapire (1990)*



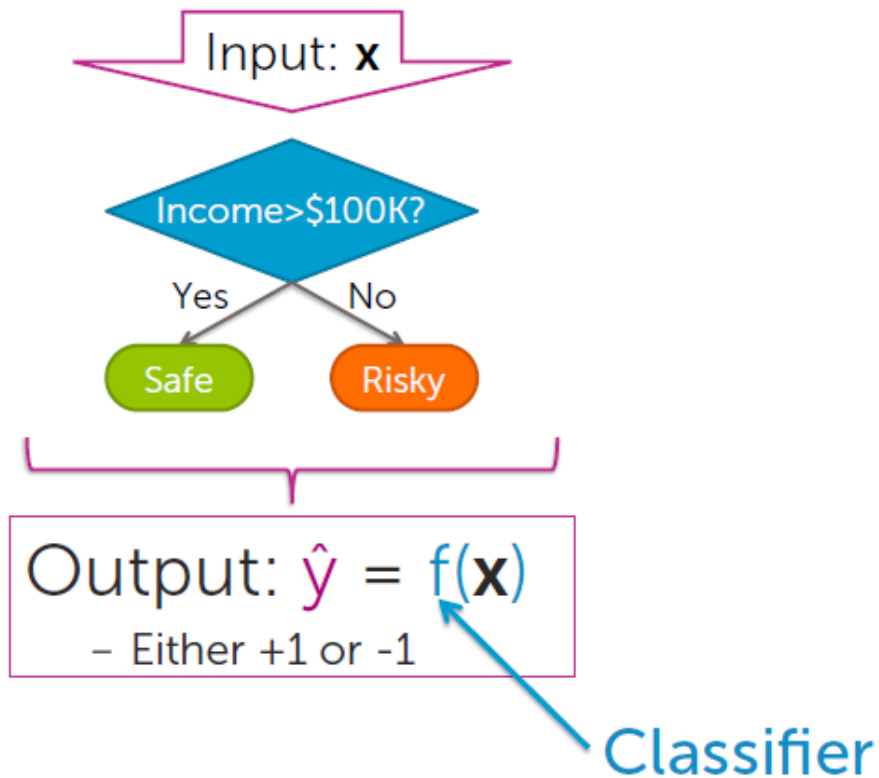
Boosting



Amazing impact: • simple approach • widely used in industry • wins most Kaggle competitions

A single classifier

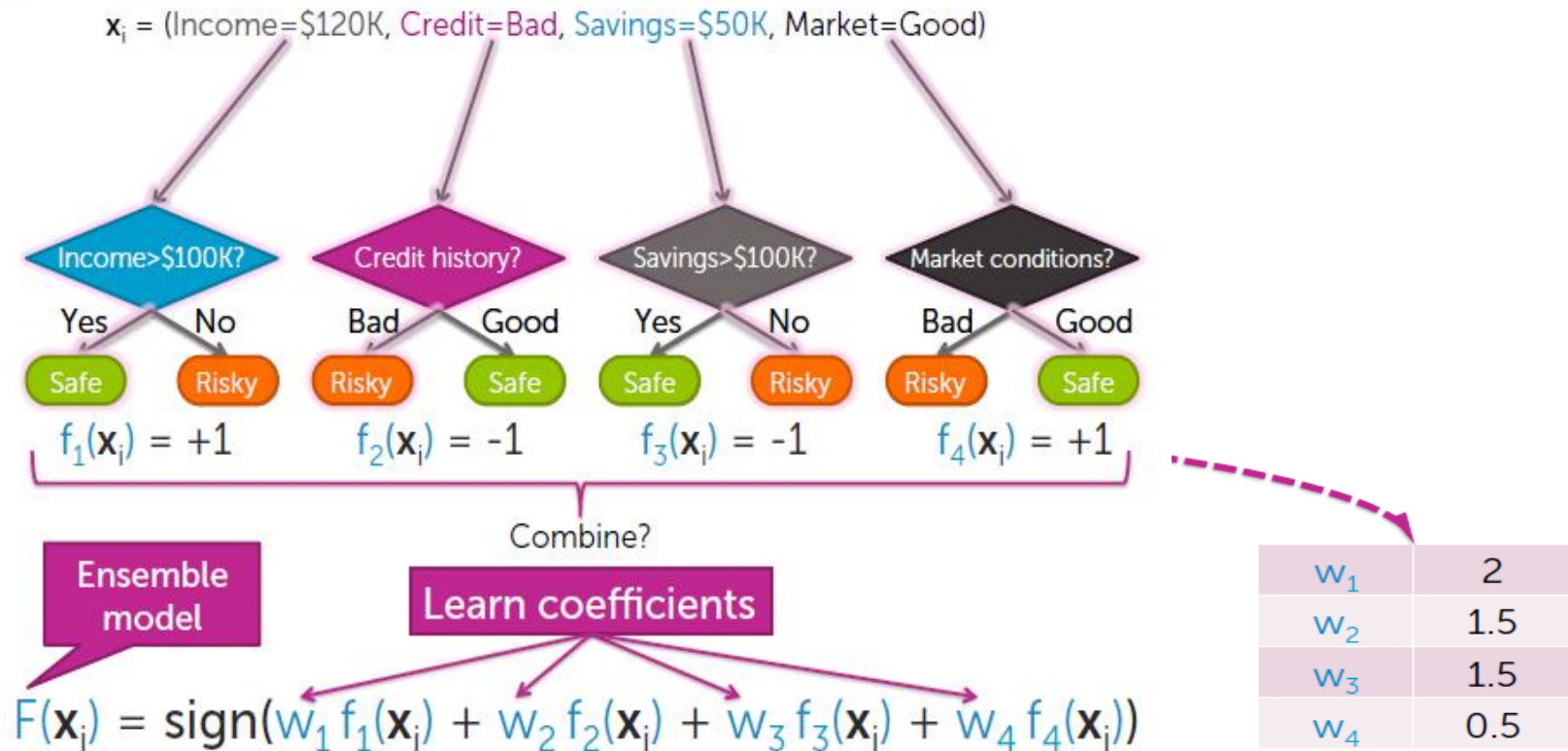
240



Ensemble methods

241

Each classifier "votes" on prediction



Ensemble classifier

242

- Goal:
 - Predict output y
 - Either +1 or -1
 - From input \mathbf{x}
- Learn ensemble model:
 - Classifiers: $f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_T(\mathbf{x})$
 - Coefficients: $\hat{w}_1, \hat{w}_2, \dots, \hat{w}_T$
- Prediction:

$$\hat{y} = \text{sign} \left(\sum_{t=1}^T \hat{w}_t f_t(\mathbf{x}) \right)$$

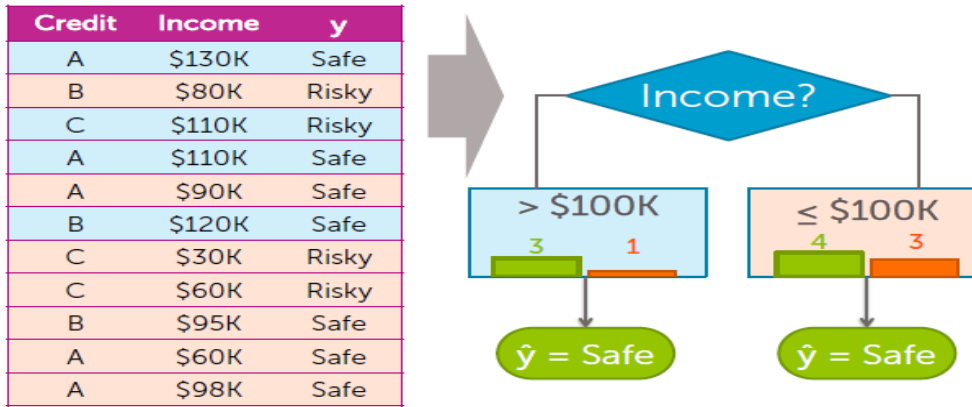
Boosting

243

Training a classifier



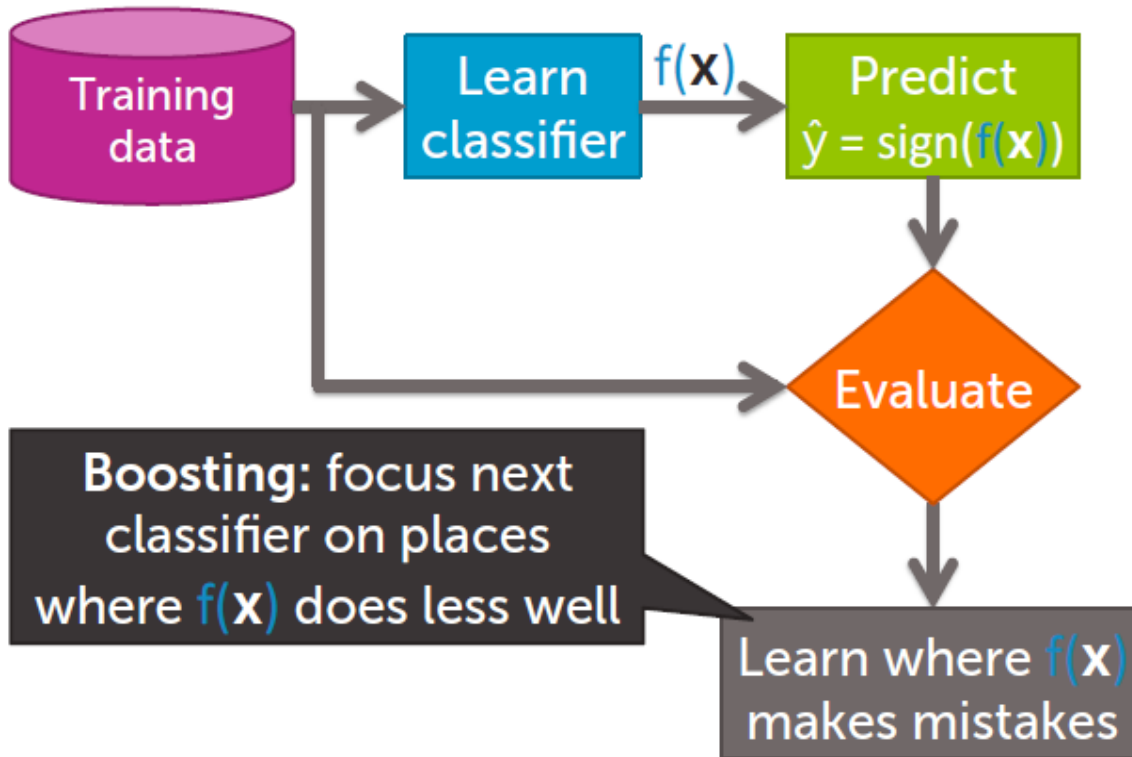
Learning decision stump



Boosting

244

Boosting = Focus learning on “hard” points



Weighted data

245

Learning on weighted data:

More weight on “hard” or more important points

- Weighted dataset:
 - Each \mathbf{x}_i, y_i weighted by α_i
 - More important point = higher weight α_i
- Learning:
 - Data point j counts as α_j data points
 - E.g., $\alpha_j = 2 \rightarrow$ count point twice

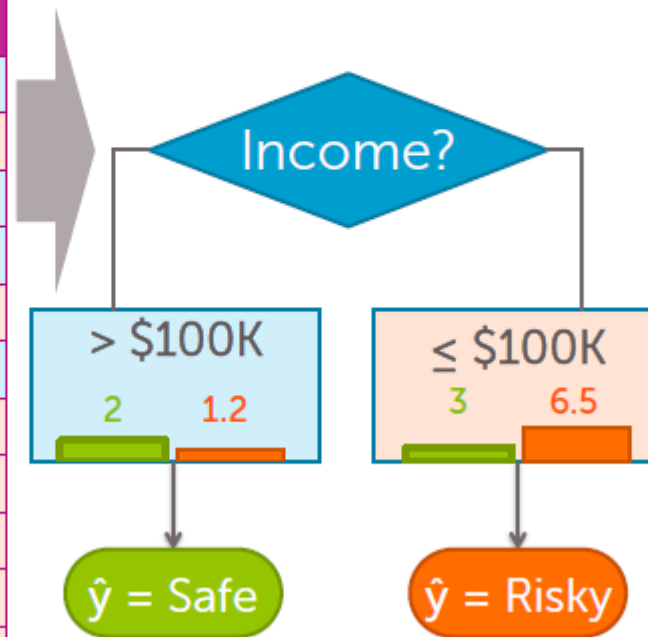
Weighted data

246

Learning a decision stump on weighted data

Increase weight α of harder/
misclassified points

Credit	Income	y	Weight α
A	\$130K	Safe	0.5
B	\$80K	Risky	1.5
C	\$110K	Risky	1.2
A	\$110K	Safe	0.8
A	\$90K	Safe	0.6
B	\$120K	Safe	0.7
C	\$30K	Risky	3
C	\$60K	Risky	2
B	\$95K	Safe	0.8
A	\$60K	Safe	0.7
A	\$98K	Safe	0.9



*Use sum over
weights of the
data points*

Weighted data

247

Learning from weighted data in general

- Usually, learning from weighted data
 - Data point i counts as α_i data points
- E.g., gradient ascent for logistic regression:

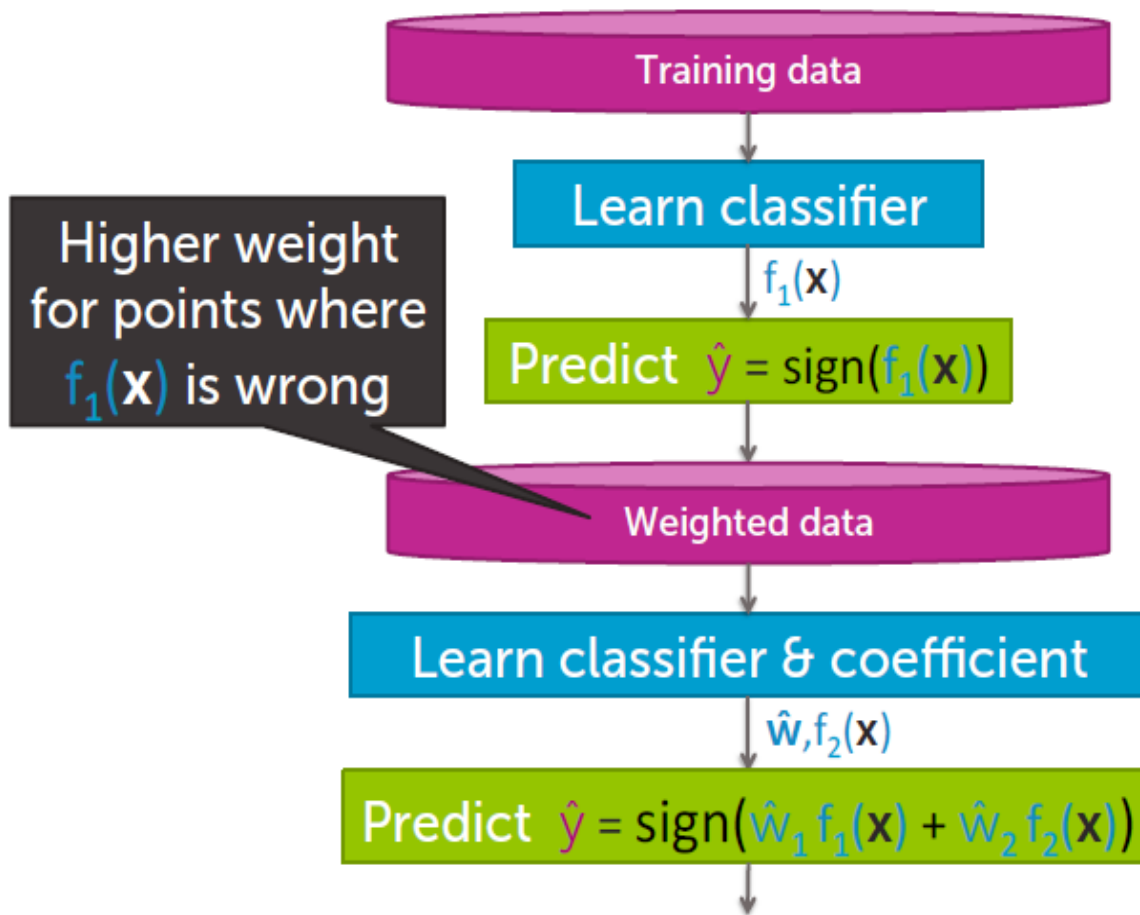
$$\mathbf{w}_j^{(t+1)} \leftarrow \mathbf{w}_j^{(t)} + \eta \sum_{i=1}^N \alpha_i \mathbf{x}_i \left(\mathbf{1}[y_i = +1] - P(y = +1 \mid \mathbf{x}_i, \mathbf{w}^{(t)}) \right)$$

Sum over data points

Weigh each point by α_i

Boosting = greedy learning ensembles from data

248



AdaBoost: learning ensemble

[Freund & Schapire 1999]

249

- Start same weight for all points: $\alpha_j = 1/N$

- For $t = 1, \dots, T$

- Learn $f_t(\mathbf{x})$ with data weights α_j

- Compute coefficient \hat{w}_t

- Recompute weights α_j

Problem 1: How much do I trust f_t ?

Problem 2: weigh mistakes more?

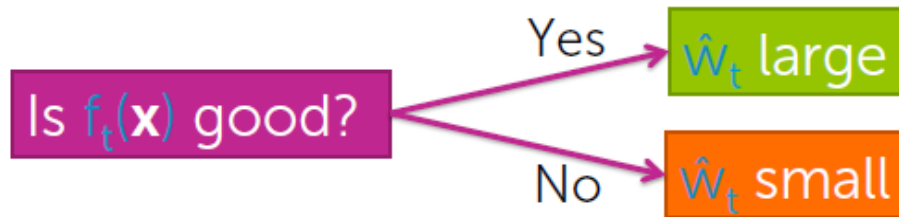
- Final model predicts by:

$$\hat{y} = \text{sign} \left(\sum_{t=1}^T \hat{w}_t f_t(\mathbf{x}) \right)$$

coefficients

AdaBoost: Computing coefficients w_t

250



- $f_t(\mathbf{x})$ is good $\rightarrow f_t$ has low training error
- Measuring error in weighted data?
 - Just weighted # of misclassified points

Weighted classification error

251

- Total weight of mistakes:

$$= \sum_{i=1}^n \alpha_i \mathbb{1}(\hat{y}_i \neq y_i)$$

mistake?

- Total weight of all points:

$$= \sum_{i=1}^n \alpha_i$$

- Weighted error measures fraction of weight of mistakes:

$$\text{weighted_error} = \frac{\text{Total weight of mistakes}}{\text{Total weight of all data points}}$$

- Best possible value is 0.0 \rightarrow worst 1.0 \rightarrow Random classifier = 0.5

AdaBoost formula

252

AdaBoost: Formula for computing coefficient \hat{w}_t of classifier $f_t(x)$

$$\hat{w}_t = \frac{1}{2} \ln \left(\frac{1 - \text{weighted_error}(f_t)}{\text{weighted_error}(f_t)} \right)$$

Is $f_t(x)$ good?

	weighted_error(f_t) on training data	$\frac{1 - \text{weighted_error}(f_t)}{\text{weighted_error}(f_t)}$	\hat{w}_t
Yes	0.01	$\frac{1 - 0.01}{0.01} = 99$	$\frac{1}{2} \ln 99 = 2.3$
No	0.5	$\frac{1 - 0.5}{0.5} = 1$	0
	0.99	$\frac{1 - 0.99}{0.99} = 0.01$	-2.3

Terrible classifier, but $1 - f_t$ is awesome !!

AdaBoost: learning ensemble

253

- Start same weight for all points: $\alpha_i = 1/N$

- For $t = 1, \dots, T$

– Learn $f_t(\mathbf{x})$ with data weights α_i

\hat{w}_t – Compute coefficient \hat{w}_t

– Recompute weights α_i

$$\hat{w}_t = \frac{1}{2} \ln \left(\frac{1 - \text{weighted_error}(f_t)}{\text{weighted_error}(f_t)} \right)$$

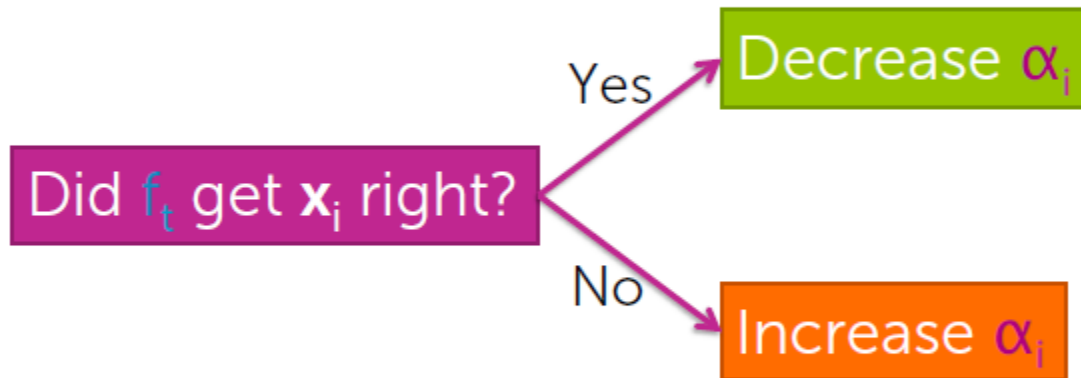
- Final model predicts by:

$$\hat{y} = \text{sign} \left(\sum_{t=1}^T \hat{w}_t f_t(\mathbf{x}) \right)$$

AdaBoost: updating weights α_i

254

Updating weights α_i based on where classifier $f_t(x)$ makes mistakes



AdaBoost: updating weights α_i

255

AdaBoost: Formula for updating weights α_i

$$\alpha_i \leftarrow \begin{cases} \alpha_i e^{-\hat{W}_t}, & \text{if } f_t(\mathbf{x}_i) = y_i \leftarrow \text{Correct} \\ \alpha_i e^{\hat{W}_t}, & \text{if } f_t(\mathbf{x}_i) \neq y_i \leftarrow \text{Mistake} \end{cases}$$

		$f_t(\mathbf{x}_i) = y_i ?$	\hat{W}_t	Multiply α_i by	Implication
Did f_t get \mathbf{x}_i right?	Yes	Correct	2.3	$e^{-2.3} = 0.1$	Decrease importance of \mathbf{x}_i, y_i
		Correct	0	$e^0 = 1$	Keep importance the same
	No	Mistake	2.3	$e^{2.3} = 9.98$	Increasing importance of \mathbf{x}_i, y_i
		Mistake	0	$e^0 = 1$	Keep importance the same

AdaBoost: learning ensemble

256

- Start same weight for all points: $\alpha_i = 1/N$

- For $t = 1, \dots, T$

- Learn $f_t(\mathbf{x})$ with data weights α_i

- Compute coefficient \hat{w}_t

- Recompute weights α_i

$$\hat{w}_t = \frac{1}{2} \ln \left(\frac{1 - \text{weighted_error}(f_t)}{\text{weighted_error}(f_t)} \right)$$

- Final model predicts by:

$$\hat{y} = \text{sign} \left(\sum_{t=1}^T \hat{w}_t f_t(\mathbf{x}) \right)$$

$$\alpha_i \leftarrow \begin{cases} \alpha_i e^{-\hat{w}_t}, & \text{if } f_t(\mathbf{x}_i) = y_i \\ \alpha_i e^{\hat{w}_t}, & \text{if } f_t(\mathbf{x}_i) \neq y_i \end{cases}$$

AdaBoost: normalizing weights α_i

257

x_i

If x_i often mistake,
weight α_i gets very
large

If x_i often correct,
weight α_i gets very
small

Can cause numerical instability
after many iterations

Normalize weights to
add up to 1 after every iteration

$$\alpha_i \leftarrow \frac{\alpha_i}{\sum_{j=1}^N \alpha_j}$$

AdaBoost: learning ensemble

258

- Start same weight for all points: $\alpha_i = 1/N$

$$\hat{w}_t = \frac{1}{2} \ln \left(\frac{1 - \text{weighted_error}(f_t)}{\text{weighted_error}(f_t)} \right)$$

- For $t = 1, \dots, T$

- Learn $f_t(\mathbf{x})$ with data weights α_i

- Compute coefficient \hat{w}_t

- Recompute weights α_i

- Normalize weights α_i

$$\alpha_i \leftarrow \begin{cases} \alpha_i e^{-\hat{w}_t}, & \text{if } f_t(\mathbf{x}_i) = y_i \\ \alpha_i e^{\hat{w}_t}, & \text{if } f_t(\mathbf{x}_i) \neq y_i \end{cases}$$

- Final model predicts by:

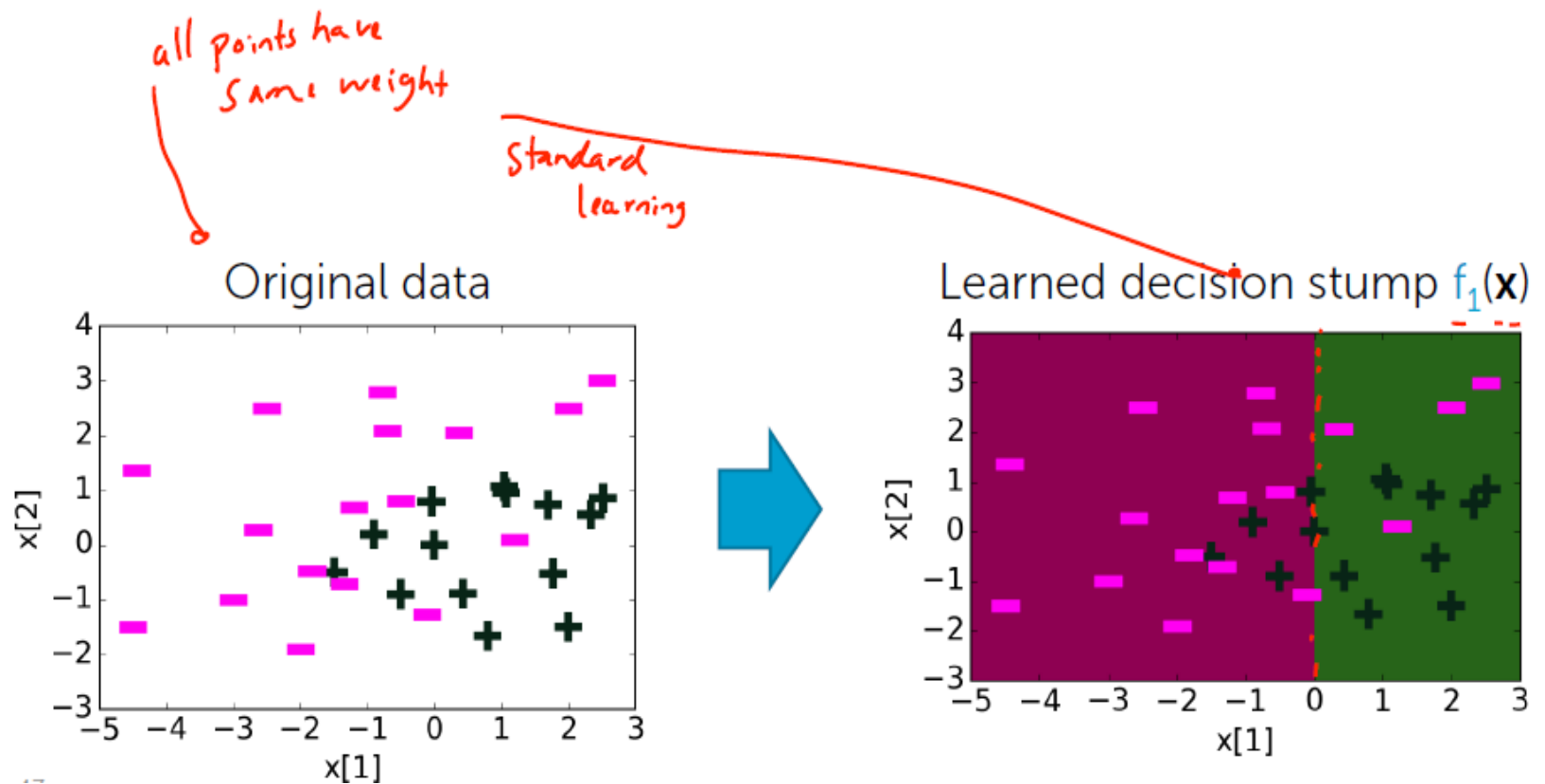
$$\hat{y} = \text{sign} \left(\sum_{t=1}^T \hat{w}_t f_t(\mathbf{x}) \right)$$

$$\alpha_i \leftarrow \frac{\alpha_i}{\sum_{j=1}^N \alpha_j}$$

AdaBoost: example

259

$t=1$: Just learn a classifier on original data

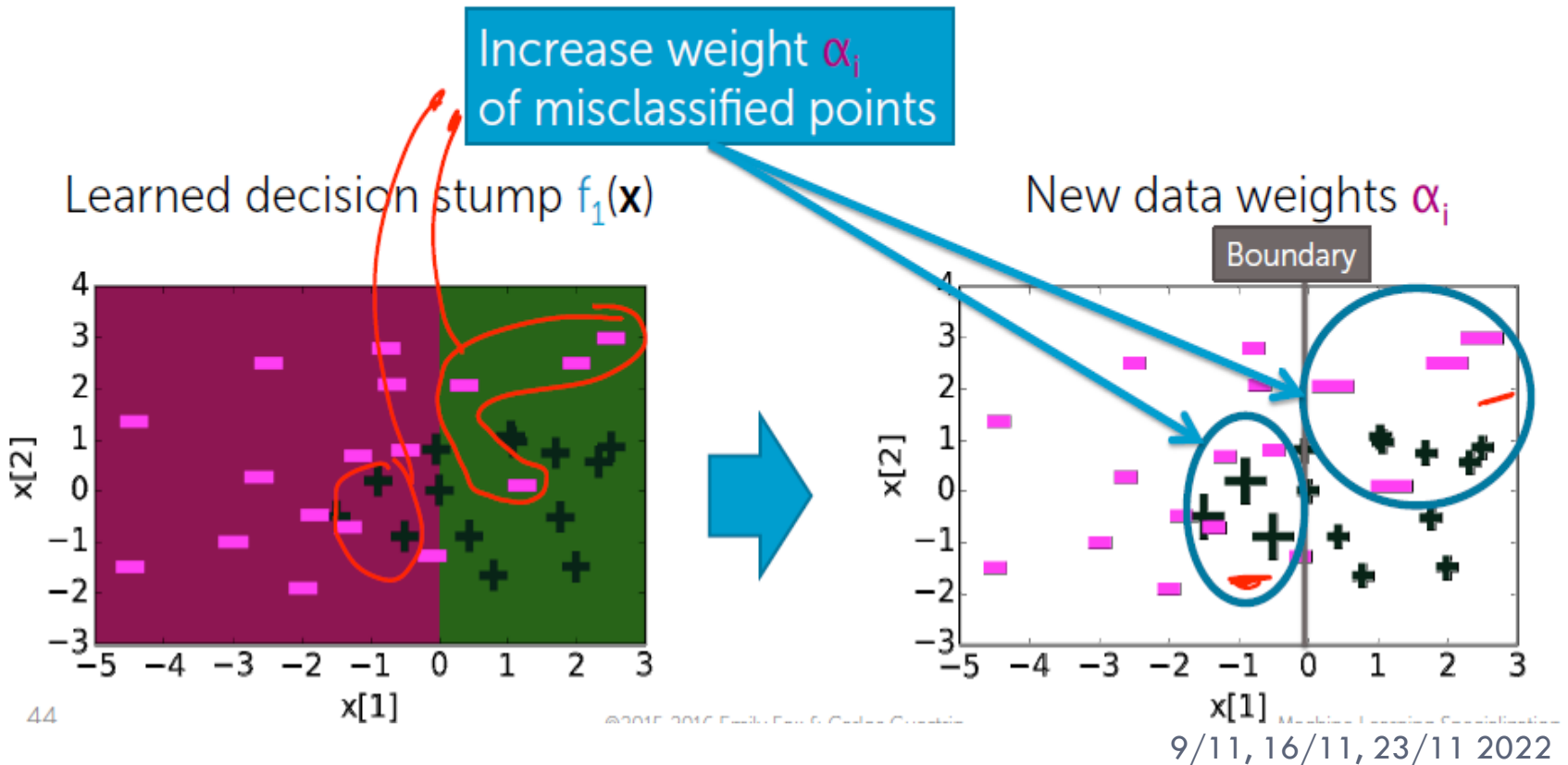


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AdaBoost: example

260

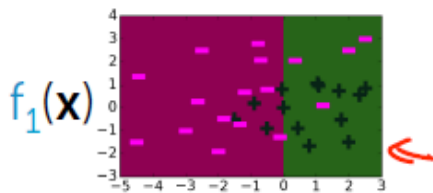
Updating weights α_i



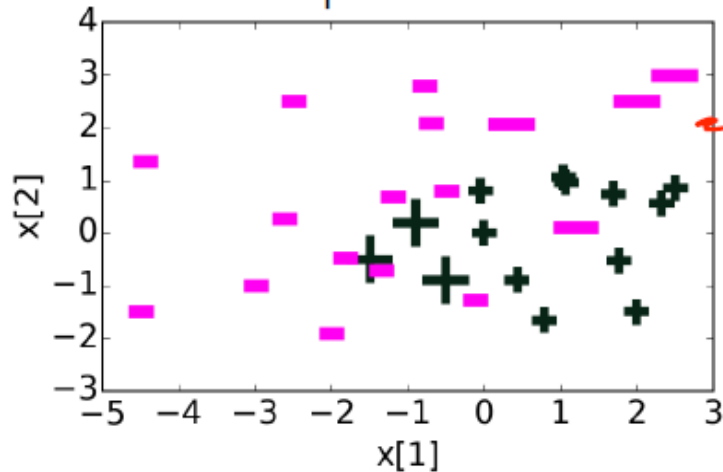
AdaBoost: example

261

t=2: Learn classifier on weighted data

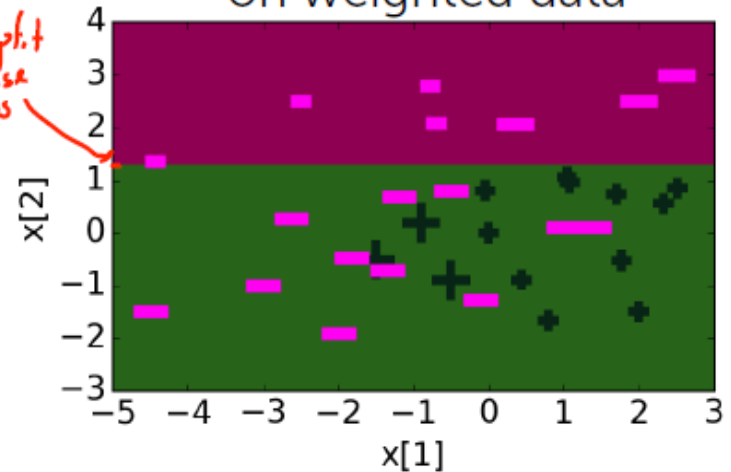


Weighted data: using α_i
chosen in previous iteration



*better split
for these
weights*

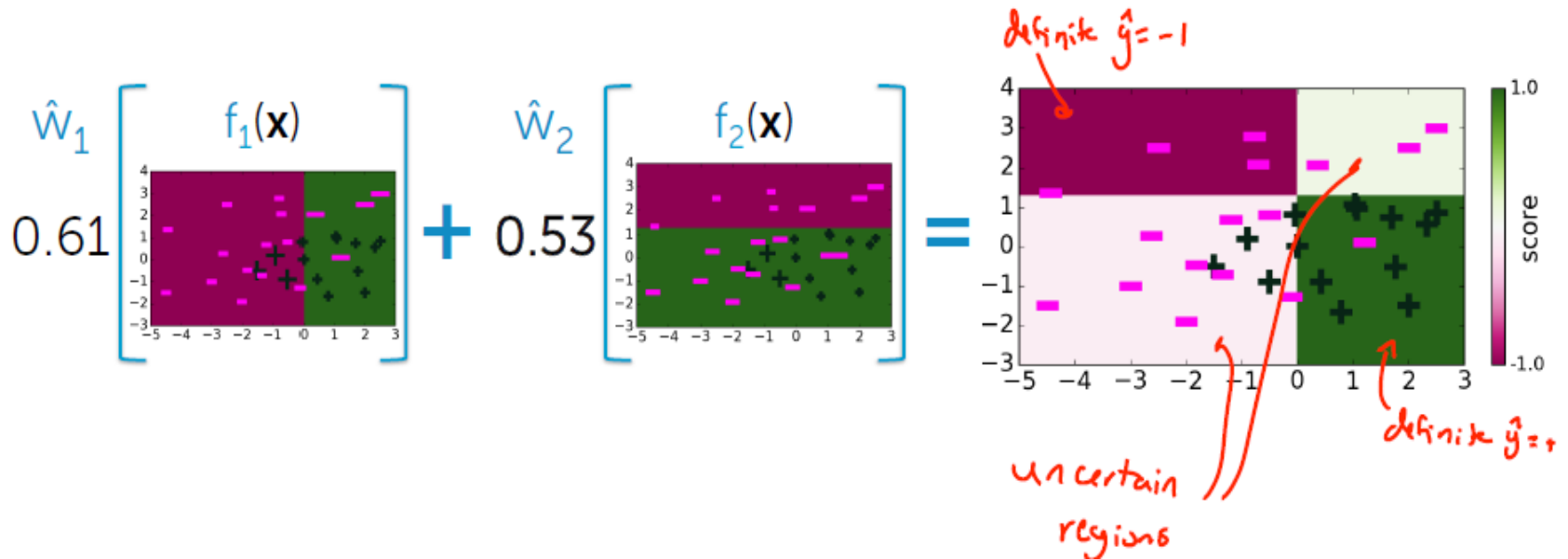
Learned decision stump $f_2(x)$
on weighted data



AdaBoost: example

262

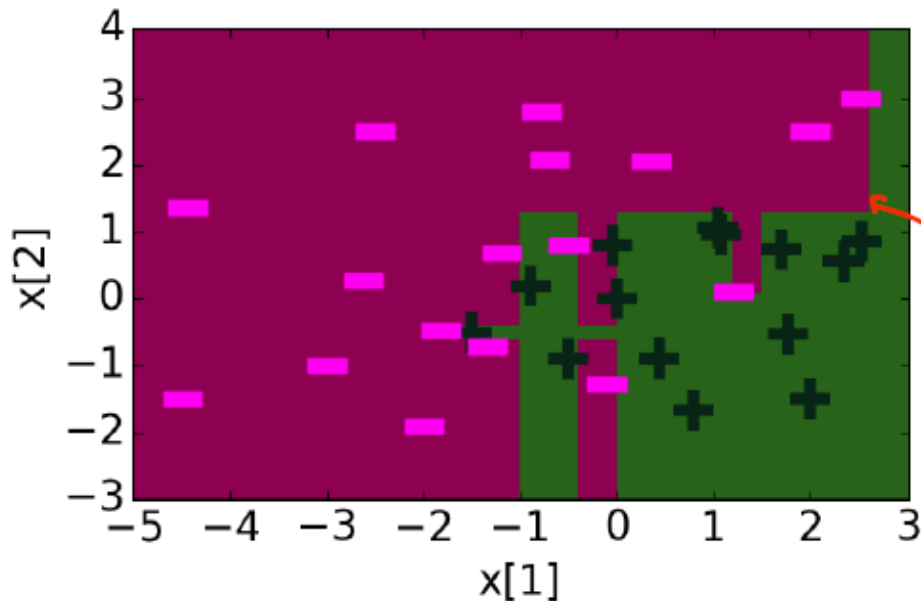
Ensemble becomes weighted sum of learned classifiers



AdaBoost: example

263

Decision boundary of ensemble classifier
after 30 iterations



training_error = 0

Decision boundary is
crazy!!
ü

probably
overfitting

AdaBoost: learning ensemble

264

- Start same weight for all points: $\alpha_i = 1/N$

$$\hat{w}_t = \frac{1}{2} \ln \left(\frac{1 - \text{weighted_error}(f_t)}{\text{weighted_error}(f_t)} \right)$$

- For $t = 1, \dots, T$

- Learn $f_t(\mathbf{x})$ with data weights α_i

- Compute coefficient \hat{w}_t

- Recompute weights α_i

- Normalize weights α_i

$$\alpha_i \leftarrow \begin{cases} \alpha_i e^{-\hat{w}_t}, & \text{if } f_t(\mathbf{x}_i) = y_i \\ \alpha_i e^{\hat{w}_t}, & \text{if } f_t(\mathbf{x}_i) \neq y_i \end{cases}$$

- Final model predicts by:

$$\hat{y} = \text{sign} \left(\sum_{t=1}^T \hat{w}_t f_t(\mathbf{x}) \right)$$

$$\alpha_i \leftarrow \frac{\alpha_i}{\sum_{j=1}^N \alpha_j}$$

Boosted decision stumps

265

- Start same weight for all points: $\alpha_i = 1/N$
- For $t = 1, \dots, T$
 - Learn $f_t(\mathbf{x})$: pick decision stump with lowest weighted training error according to α_i
 - Compute coefficient \hat{w}_t
 - Recompute weights α_i
 - Normalize weights α_i

- Final model predicts by:

$$\hat{y} = \text{sign} \left(\sum_{t=1}^T \hat{w}_t f_t(\mathbf{x}) \right)$$

Boosted decision stumps

266

Finding best next decision stump $f_t(\mathbf{x})$

Consider splitting on each feature:



$$\hat{W}_t = \frac{1}{2} \ln \left(\frac{1 - \text{weighted_error}(f_t)}{\text{weighted_error}(f_t)} \right) = 0.69$$

Boosted decision stumps

267

- Start same weight for all points: $\alpha_i = 1/N$
- For $t = 1, \dots, T$
 - Learn $f_t(\mathbf{x})$: pick decision stump with lowest weighted training error according to α_i
 - Compute coefficient \hat{w}_t
 - Recompute weights α_i
 - Normalize weights α_i

- Final model predicts by:

$$\hat{y} = \text{sign} \left(\sum_{t=1}^T \hat{w}_t f_t(\mathbf{x}) \right)$$

Boosted decision stumps

268

Updating weights α_i



$$\alpha_i \leftarrow \begin{cases} \alpha_i e^{-\hat{v}_t} = \alpha_i e^{-0.69} = \alpha_i / 2, & \text{if } f_t(x_i) = y_i \\ \alpha_i e^{\hat{w}_t} = \alpha_i e^{0.69} = 2 \alpha_i, & \text{if } f_t(x_i) \neq y_i \end{cases}$$

Credit	Income	y	\hat{y}	Previous weight α	New weight α
A	\$130K	Safe	Safe	0.5	$0.5/2 = 0.25$
B	\$80K	Risky	Risky	1.5	0.75
C	\$110K	Risky	Safe	1.5	$2 * 1.5 = 3$
A	\$110K	Safe	Safe	2	1
A	\$90K	Safe	Risky	1	2
B	\$120K	Safe	Safe	2.5	1.25
C	\$30K	Risky	Risky	3	1.5
C	\$60K	Risky	Risky	2	1
B	\$95K	Safe	Risky	0.5	1
A	\$60K	Safe	Risky	1	2
A	\$98K	Safe	Risky	0.5	1

Boosting convergence & overfitting

269

Boosting question revisited

"Can a set of weak learners be combined to create a stronger learner?" *Kearns and Valiant (1988)*



Yes! *Schapire (1990)*

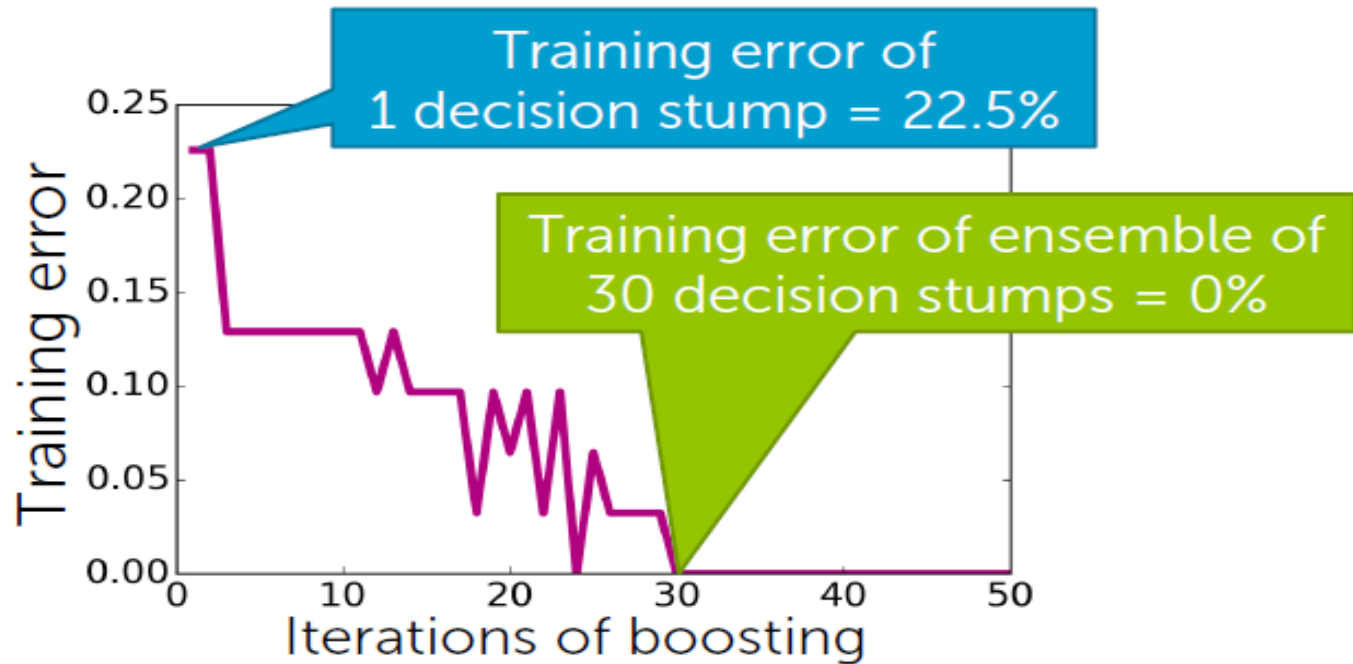


Boosting

Boosting convergence & overfitting

270

After some iterations,
training error of boosting goes to zero!!!



Boosted decision stumps on toy dataset

Boosting convergence & overfitting

271

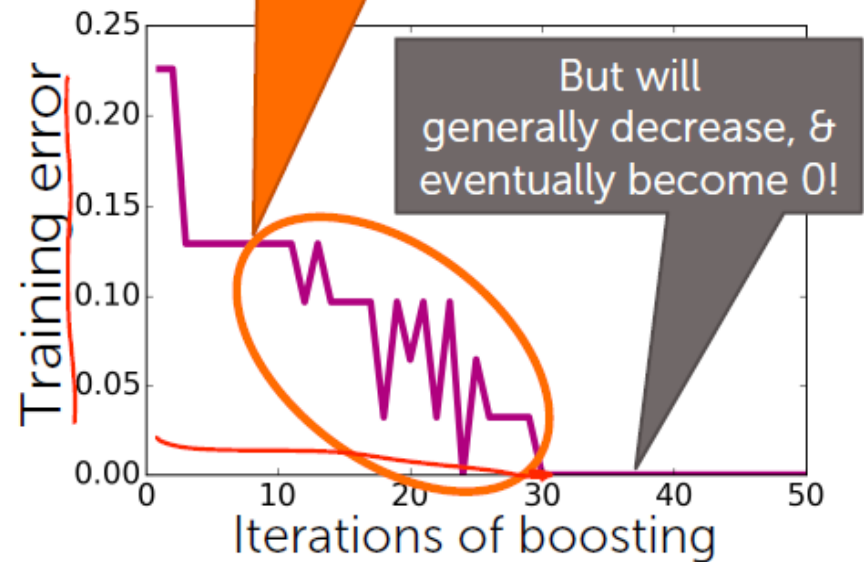
AdaBoost Theorem

Under some technical conditions...



Training error of
boosted classifier $\rightarrow 0$
as $T \rightarrow \infty$

May oscillate a bit



Boosting convergence & overfitting

272

Condition of AdaBoost Theorem

Under some technical conditions...



Training error of
boosted classifier $\rightarrow 0$
as $T \rightarrow \infty$

Condition = At every t ,
can find a weak learner with
weighted_error(f_t) < 0.5



Not always
possible



Nonetheless, boosting often
yields great training error

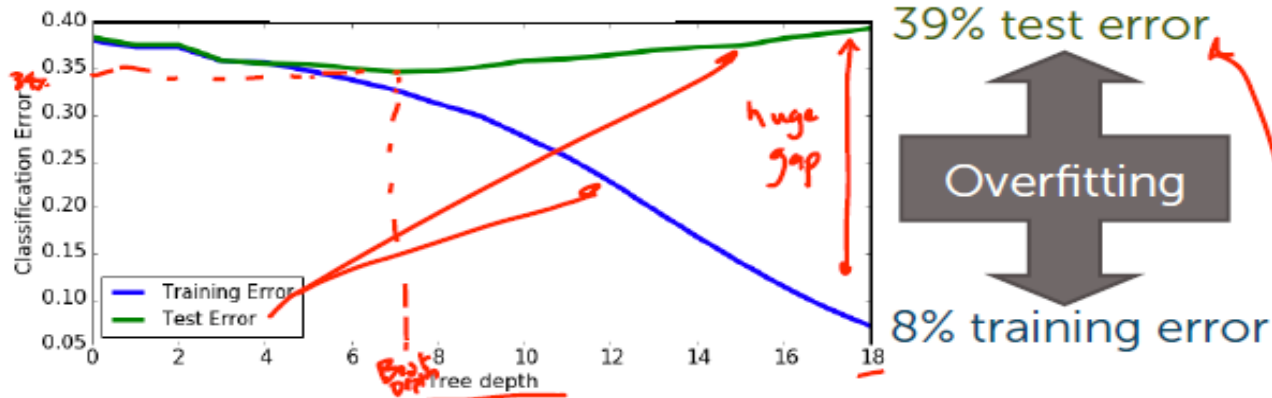
Extreme example:
No classifier can
separate a +1
on top of -1



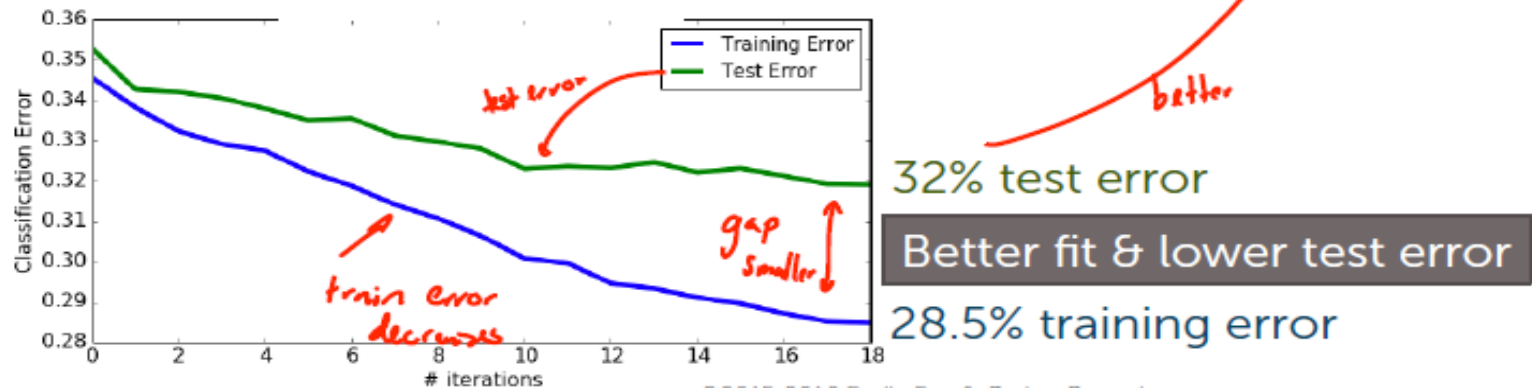
Example

273

Decision trees on loan data



Boosted decision stumps on loan data

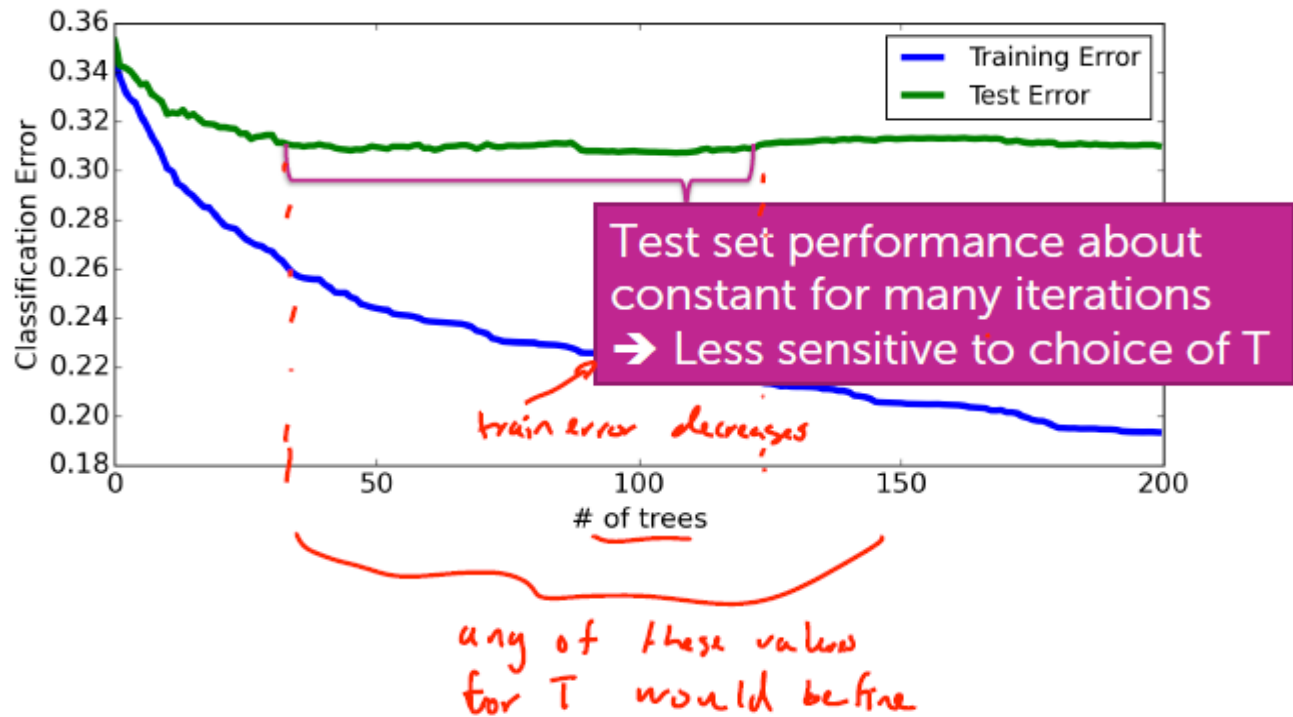


62

Example

274

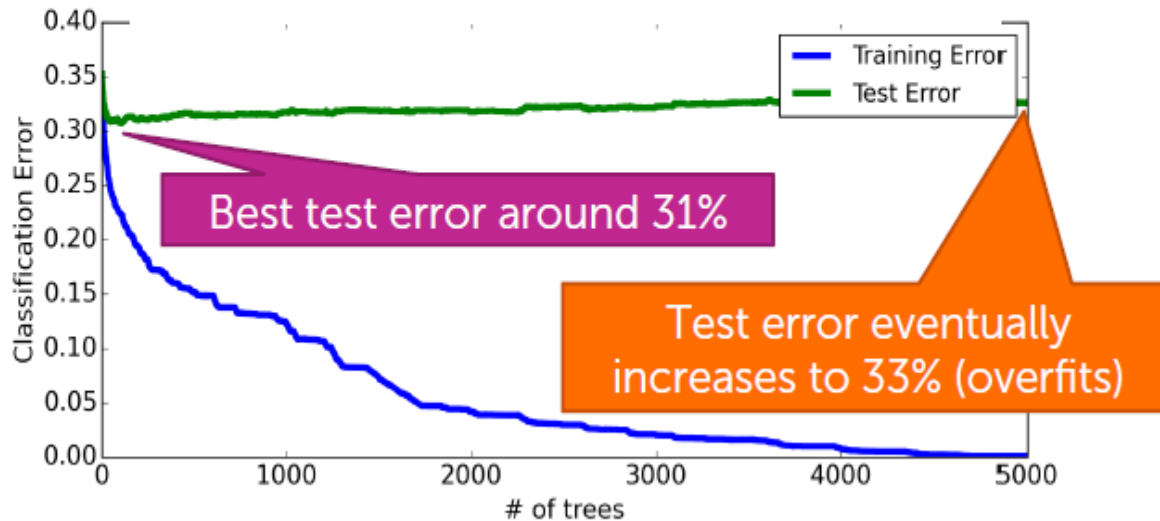
Boosting tends to be robust to overfitting



Example

275

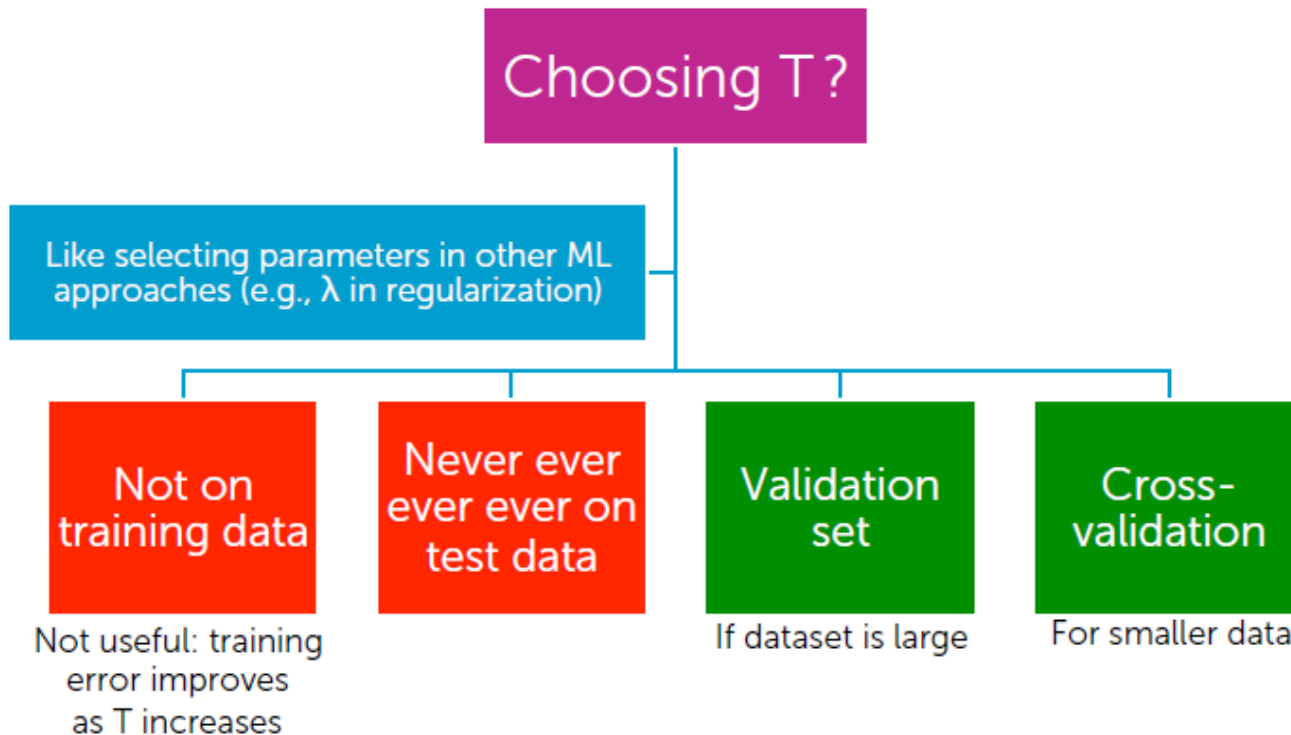
But boosting will eventually overfit,
so must choose max number of components T



Example

276

How do we decide when to stop boosting?



Boosting: summary

277

Variants of boosting and related algorithms

There are hundreds of variants of boosting, most important:

Gradient boosting

- Like AdaBoost, but useful beyond basic classification

Many other approaches to learn ensembles, most important:

Random forests

- Bagging: Pick random subsets of the data
 - Learn a tree in each subset
 - Average predictions
- Simpler than boosting & easier to parallelize
- Typically higher error than boosting for same number of trees (# iterations T)

Boosting: summary

278

Impact of boosting (spoiler alert... *HUGE IMPACT*)

Amongst most useful
ML methods ever created

Extremely useful in
computer vision

- Standard approach for face detection, for example

Used by **most winners** of
ML competitions
(Kaggle, KDD Cup,...)

- Malware classification, credit fraud detection, ads click through rate estimation, sales forecasting, ranking webpages for search, Higgs boson detection,...

Most deployed ML systems
use model ensembles

- Coefficients chosen manually, with boosting, with bagging, or others

What you can do now

279

- Identify notion ensemble classifiers
- Formalize ensembles as the weighted combination of simpler classifiers
- Outline the boosting framework – sequentially learn classifiers on weighted data
- Describe the AdaBoost algorithm
 - Learn each classifier on weighted data
 - Compute coefficient of classifier
 - Recompute data weights
 - Normalize weights
- Implement AdaBoost to create an ensemble of decision stumps
- Discuss convergence properties of AdaBoost & how to pick the maximum number of iterations T

Details

- ▣ Derivative of likelihood for logistic regression

The log trick, often used in ML...

281

- Products become sums:
 $\ln a \cdot b = \ln a + \ln b$ | $\ln \frac{a}{b} = \ln a - \ln b$
- Doesn't change maximum!
 - If $\hat{\mathbf{w}}$ maximizes $f(\mathbf{w})$:

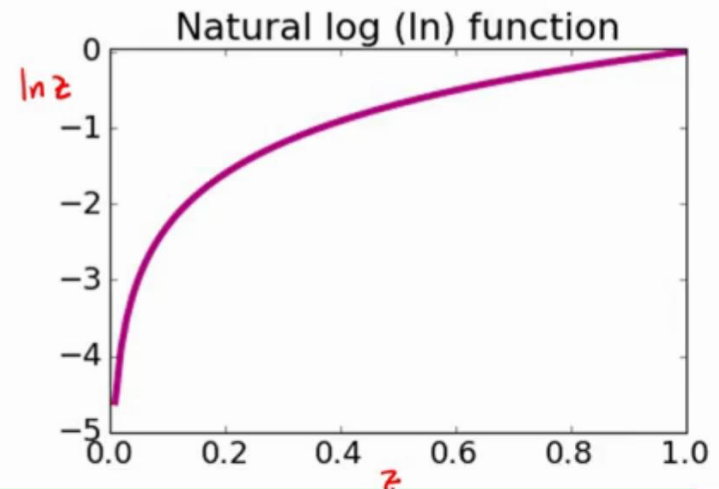
$$\hat{\mathbf{w}} = \underset{\mathbf{w}}{\operatorname{arg\,max}} f(\mathbf{w})$$

the \mathbf{w} that makes $f(\mathbf{w})$ largest

- Then $\hat{\mathbf{w}}_{\ln}$ maximizes $\ln(f(\mathbf{w}))$:

$$\hat{\mathbf{w}}_{\ln} = \underset{\mathbf{w}}{\operatorname{arg\,max}} \ln(f(\mathbf{w}))$$

$$\hat{\mathbf{w}} = \hat{\mathbf{w}}_{\ln}$$



Log-likelihood function

282

- Goal: choose coefficients \mathbf{w} maximizing likelihood:

$$\ell(\mathbf{w}) = \prod_{i=1}^N P(y_i | \mathbf{x}_i, \mathbf{w})$$

- Math simplified by using log-likelihood – taking (natural) log:

$$\ell\ell(\mathbf{w}) = \ln \prod_{i=1}^N P(y_i | \mathbf{x}_i, \mathbf{w})$$

natural log

Log-likelihood function

283

Using log to turn products into sums

$$\ln \prod_{i=1}^N f_i = \sum_{i=1}^N \ln f_i$$

- The log of the product of likelihoods becomes the sum of the logs:

$$\begin{aligned} \ell(\mathbf{w}) &= \ln \prod_{i=1}^N P(y_i | \mathbf{x}_i, \mathbf{w}) \\ &= \sum_{i=1}^N \ln P(y_i | \mathbf{x}_i, \mathbf{w}) \end{aligned}$$

Rewriting log-likelihood

284

- For simpler math, we'll rewrite likelihood with indicators:

$$\begin{aligned} \ell l(\mathbf{w}) &= \sum_{i=1}^N \ln P(y_i | \mathbf{x}_i, \mathbf{w}) \\ &= \sum_{i=1}^N [\mathbb{1}[y_i = +1] \ln P(y = +1 | \mathbf{x}_i, \mathbf{w}) + \mathbb{1}[y_i = -1] \ln P(y = -1 | \mathbf{x}_i, \mathbf{w})] \end{aligned}$$

Indicator function

if $y_i = +1$

if $y_i = -1$

✓

0

0

✓

Logistic regression

285

Logistic regression model: $P(y=-1|\mathbf{x}, \mathbf{w})$

- Probability model predicts $y=+1$:

$$P(y=+1|\mathbf{x}, \mathbf{w}) = \frac{1}{1 + e^{-\mathbf{w}^T h(\mathbf{x})}}$$

- Probability model predicts $y=-1$:

$$P(y=-1|\mathbf{x}, \mathbf{w}) = 1 - P(y=+1|\mathbf{x}, \mathbf{w}) = 1 - \frac{1}{1 + e^{-\mathbf{w}^T h(\mathbf{x})}}$$
$$\hookrightarrow = \frac{1 + e^{-\mathbf{w}^T h(\mathbf{x})} - 1}{1 + e^{-\mathbf{w}^T h(\mathbf{x})}} = \frac{e^{-\mathbf{w}^T h(\mathbf{x})}}{1 + e^{-\mathbf{w}^T h(\mathbf{x})}}$$

Logistic regression

286

Plugging in logistic function for 1 data point

$$P(y = +1 | \mathbf{x}, \mathbf{w}) = \frac{1}{1 + e^{-\mathbf{w}^\top h(\mathbf{x})}} \quad P(y = -1 | \mathbf{x}, \mathbf{w}) = \frac{e^{-\mathbf{w}^\top h(\mathbf{x})}}{1 + e^{-\mathbf{w}^\top h(\mathbf{x})}}$$

$$\ell\ell(\mathbf{w}) = \mathbb{1}[y_i = +1] \ln P(y = +1 | \mathbf{x}_i, \mathbf{w}) + \mathbb{1}[y_i = -1] \ln P(y = -1 | \mathbf{x}_i, \mathbf{w})$$

$$= \mathbb{1}[y_i = +1] \ln \frac{1}{1 + e^{-\mathbf{w}^\top h(\mathbf{x}_i)}} + (1 - \mathbb{1}[y_i = +1]) \ln \frac{e^{-\mathbf{w}^\top h(\mathbf{x}_i)}}{1 + e^{-\mathbf{w}^\top h(\mathbf{x}_i)}}$$

$$= -\mathbb{1}[y_i = +1] \ln(1 + e^{-\mathbf{w}^\top h(\mathbf{x}_i)}) + (1 - \mathbb{1}[y_i = +1]) [-\mathbf{w}^\top h(\mathbf{x}_i) - \ln(1 + e^{-\mathbf{w}^\top h(\mathbf{x}_i)})]$$

$$= - (1 - \mathbb{1}[y_i = +1]) \mathbf{w}^\top h(\mathbf{x}_i) - \ln(1 + e^{-\mathbf{w}^\top h(\mathbf{x}_i)})$$

Simpler form

$$\ln e^a = a$$

$$\mathbb{1}[y_i = -1] = 1 - \mathbb{1}[y_i = +1]$$

$$\ln \frac{1}{1 + e^{-\mathbf{w}^\top h(\mathbf{x}_i)}} = -\ln(1 + e^{-\mathbf{w}^\top h(\mathbf{x}_i)})$$

$$\ln \frac{e^{-\mathbf{w}^\top h(\mathbf{x}_i)}}{1 + e^{-\mathbf{w}^\top h(\mathbf{x}_i)}} = \frac{\ln e^{-\mathbf{w}^\top h(\mathbf{x}_i)}}{1 + e^{-\mathbf{w}^\top h(\mathbf{x}_i)}} = \frac{-\mathbf{w}^\top h(\mathbf{x}_i) - \ln(1 + e^{-\mathbf{w}^\top h(\mathbf{x}_i)})}{1 + e^{-\mathbf{w}^\top h(\mathbf{x}_i)}}$$

Logistic regression

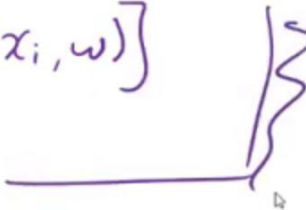
287

Gradient for 1 data point

$$\ell(\mathbf{w}) = -(1 - \mathbb{1}[y_i = +1])\mathbf{w}^\top h(\mathbf{x}_i) - \ln(1 + e^{-\mathbf{w}^\top h(\mathbf{x}_i)})$$

$$\frac{\partial \ell}{\partial w_j} = -(1 - \mathbb{1}[y_i = +1]) \frac{\partial}{\partial w_j} \mathbf{w}^\top h(\mathbf{x}_i) - \frac{\partial}{\partial w_j} \ln(1 + e^{-\mathbf{w}^\top h(\mathbf{x}_i)})$$

$$= -(1 - \mathbb{1}[y_i = +1]) h_j(\mathbf{x}_i) + h_j(\mathbf{x}_i) P(y = -1 | \mathbf{x}_i, \mathbf{w})$$

$$= h_j(\mathbf{x}_i) \left[\mathbb{1}[y_i = +1] - P(y = +1 | \mathbf{x}_i, \mathbf{w}) \right]$$


$$\begin{aligned} \frac{\partial}{\partial w_j} \mathbf{w}^\top h(\mathbf{x}_i) &= h_j(\mathbf{x}_i) \\ \hline \frac{\partial}{\partial w_j} \ln(1 + e^{-\mathbf{w}^\top h(\mathbf{x}_i)}) \\ &= -h_j(\mathbf{x}_i) \underbrace{\frac{e^{-\mathbf{w}^\top h(\mathbf{x}_i)}}{1 + e^{-\mathbf{w}^\top h(\mathbf{x}_i)}}}_{P(y = -1 | \mathbf{x}_i, \mathbf{w})} \end{aligned}$$

Logistic regression

288

Finally, gradient for all data points

- Gradient for one data point:

$$h_j(\mathbf{x}_i) \left(\mathbb{1}[y_i = +1] - P(y = +1 \mid \mathbf{x}_i, \mathbf{w}) \right)$$

- Adding over data points:

$$\frac{\partial \ell}{\partial w_j} = \sum_{i=1}^N h_j(\mathbf{x}_i) \left(\mathbb{1}[y_i = +1] - P(y = +1 \mid \mathbf{x}_i, \mathbf{w}) \right)$$