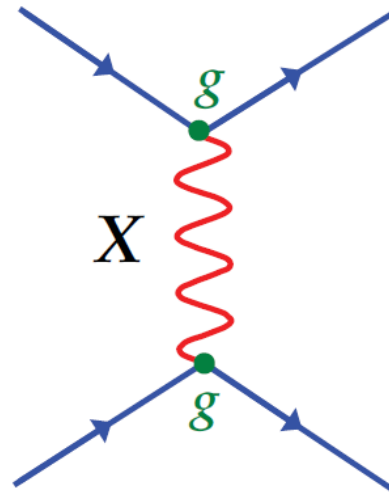
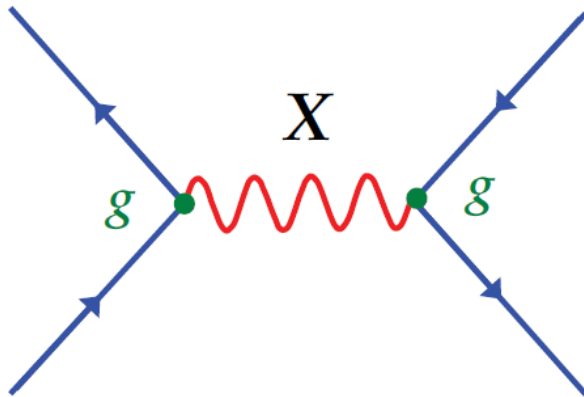


Elementary Particle Physics: theory and experiments

Interaction by Particle Exchange and QED



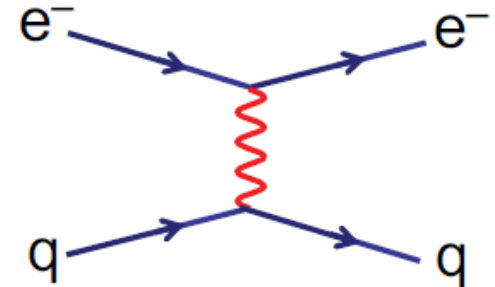
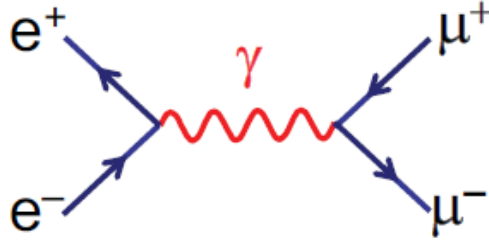
Follow the course/slides from M. A. Thomson lectures at Cambridge University

Recap

★ Working towards a proper calculation of decay and scattering processes

Initially concentrate on:

- $e^+e^- \rightarrow \mu^+\mu^-$
- $e^-q \rightarrow e^-q$



▲ covered the relativistic calculation of particle decay rates and cross sections

$$\sigma \propto \frac{|M|^2}{\text{flux}} \times (\text{phase space})$$

▲ covered relativistic treatment of spin-half particles

Dirac Equation

▲ concentrate on the **Lorentz Invariant Matrix Element**

- Interaction by particle exchange
- Introduction to Feynman diagrams
- The Feynman rules for QED

Interaction by Particle Exchange

- Calculate transition rates from Fermi's Golden Rule

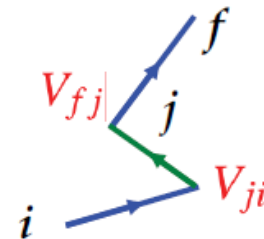
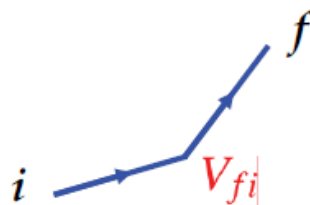
$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

where T_{fi} is perturbation expansion for the Transition Matrix Element

$$T_{fi} = \langle f|V|i\rangle + \sum_{j \neq i} \frac{\langle f|V|j\rangle \langle j|V|i\rangle}{E_i - E_j} + \dots$$

- For particle scattering, the first two terms in the perturbation series can be viewed as:

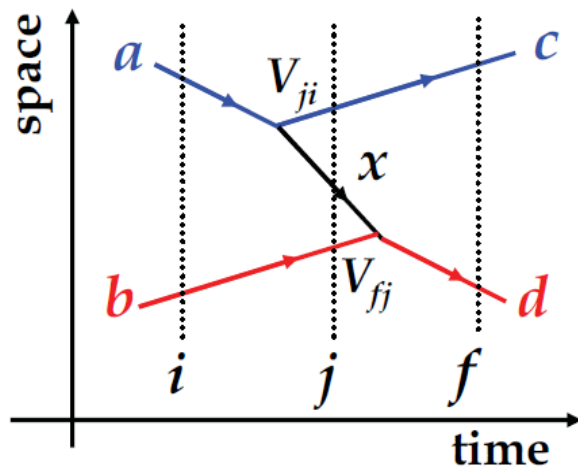
“scattering in a potential”



“scattering via an intermediate state”

- “Classical picture” – particles act as sources for fields which give rise a potential in which other particles scatter – “action at a distance”
- “Quantum Field Theory picture” – forces arise due to the exchange of virtual particles. No action at a distance + **forces** between particles now **due to particles**

- Consider the particle interaction $a + b \rightarrow c + d$ which occurs via an intermediate state corresponding to the exchange of particle x
- One possible space-time picture of this process is:



Initial state i : $a + b$

Final state f : $c + d$

Intermediate state j : $c + b + x$

- This time-ordered diagram corresponds to a "emitting" x and then b absorbing x

- The corresponding term in the perturbation expansion is:

$$T_{fi} = \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j}$$

$$T_{fi}^{ab} = \frac{\langle d|V|x+b\rangle\langle c+x|V|a\rangle}{(E_a + E_b) - (E_c + E_x + E_b)}$$

- T_{fi}^{ab} refers to the time-ordering where a emits x before b absorbs it

- Need an expression for $\langle c + x|V|a \rangle$ in non-invariant matrix element T_{fi}
- Ultimately aiming to obtain Lorentz Invariant ME
- Recall T_{fi} is related to the invariant matrix element by

$$T_{fi} = \prod_k (2E_k)^{-1/2} M_{fi}$$

where k runs over all particles in the matrix element

- Here we have

$$\langle c + x|V|a \rangle = \frac{M_{(a \rightarrow c+x)}}{(2E_a 2E_c 2E_x)^{1/2}}$$

$M_{(a \rightarrow c+x)}$ is the “**Lorentz Invariant**” matrix element for $a \rightarrow c + x$

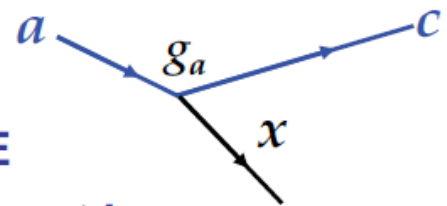
- ★ The simplest Lorentz Invariant quantity is a scalar, in this case

$$\langle c + x|V|a \rangle = \frac{g_a}{(2E_a 2E_c 2E_x)^{1/2}}$$

g_a is a measure of the strength of the interaction $a \rightarrow c + x$

Note : the matrix element is only LI in the sense that it is defined in terms of LI wave-function normalisations and that the form of the coupling is LI

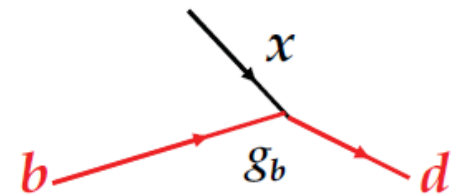
Note : in this “illustrative” example g is not dimensionless.



Similarly $\langle d|V|x+b\rangle = \frac{g_b}{(2E_b 2E_d 2E_x)^{1/2}}$

Giving $T_{fi}^{ab} = \frac{\langle d|V|x+b\rangle \langle c+x|V|a\rangle}{(E_a + E_b) - (E_c + E_x + E_b)}$

$$= \frac{1}{2E_x} \cdot \frac{1}{(2E_a 2E_b 2E_c 2E_d)^{1/2}} \cdot \frac{g_a g_b}{(E_a - E_c - E_x)}$$



★ The “Lorentz Invariant” matrix element for the **entire** process is

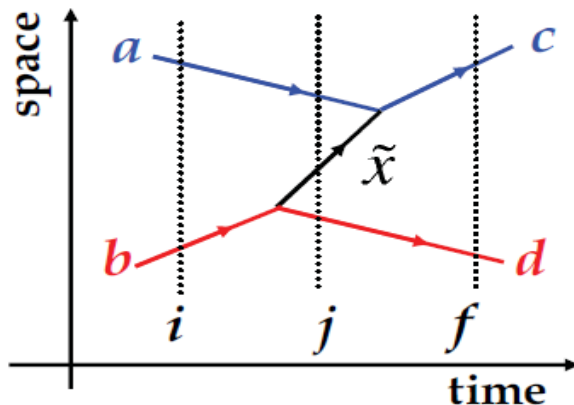
$$M_{fi}^{ab} = (2E_a 2E_b 2E_c 2E_d)^{1/2} T_{fi}^{ab}$$

$$= \frac{1}{2E_x} \cdot \frac{g_a g_b}{(E_a - E_c - E_x)}$$

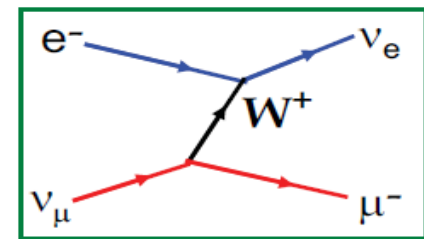
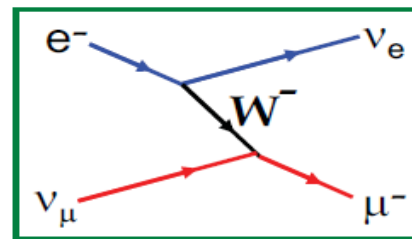
Note:

- ♦ M_{fi}^{ab} refers to the time-ordering where a emits x before b absorbs it
It is **not Lorentz invariant**, order of events in time depends on frame
- ♦ Momentum is conserved at each interaction vertex but not energy
 $E_j \neq E_i$
- ♦ Particle x is “on-mass shell” i.e. $E_x^2 = \vec{p}_x^2 + m^2$

★ But need to consider also the other time ordering for the process



- This time-ordered diagram corresponds to b “emitting” \tilde{x} and then a absorbing \tilde{x}
- \tilde{x} is the anti-particle of x e.g.



• The Lorentz invariant matrix element for this time ordering is:

$$M_{fi}^{ba} = \frac{1}{2E_x} \cdot \frac{g_a g_b}{(E_b - E_d - E_x)}$$

★ In QM need to sum over matrix elements corresponding to same final

state: $M_{fi} = M_{fi}^{ab} + M_{fi}^{ba}$

$$= \frac{g_a g_b}{2E_x} \cdot \left(\frac{1}{E_a - E_c - E_x} + \frac{1}{E_b - E_d - E_x} \right)$$

$$= \frac{g_a g_b}{2E_x} \cdot \left(\frac{1}{E_a - E_c - E_x} - \frac{1}{E_a - E_c + E_x} \right)$$

Energy conservation:
($E_a + E_b = E_c + E_d$)

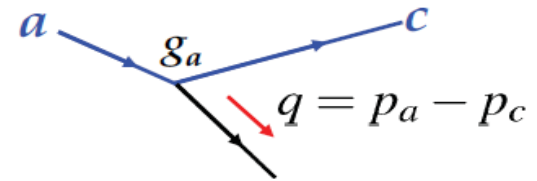
- Which gives
$$M_{fi} = \frac{g_a g_b}{2E_x} \cdot \frac{2E_x}{(E_a - E_c)^2 - E_x^2}$$

$$= \frac{g_a g_b}{(E_a - E_c)^2 - E_x^2}$$

- From 1st time ordering $E_x^2 = \vec{p}_x^2 + m_x^2 = (\vec{p}_a - \vec{p}_c)^2 + m_x^2$

giving
$$M_{fi} = \frac{g_a g_b}{(E_a - E_c)^2 - (\vec{p}_a - \vec{p}_c)^2 - m_x^2}$$

$$= \frac{g_a g_b}{(p_a - p_c)^2 - m_x^2}$$

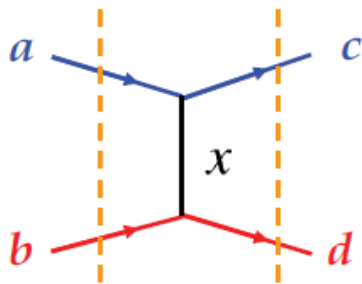
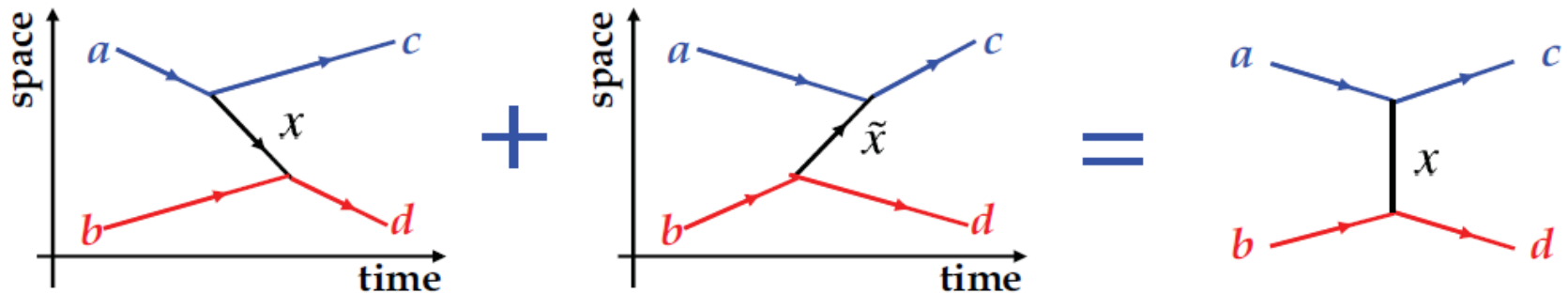


➔
$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- After summing over all possible time orderings, M_{fi} is (as anticipated) **Lorentz invariant**. This is a remarkable result – the sum over all time orderings gives a frame independent matrix element.
- Exactly the same result would have been obtained by considering the annihilation process

Feynman diagrams

- The sum over all possible time-orderings is represented by a **FEYNMAN diagram**

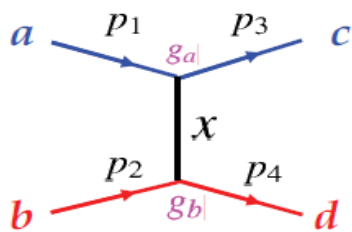


In a Feynman diagram:

- the LHS represents the initial state
 - the RHS is the final state
 - everything in between is “how the interaction happened”
- It is important to remember that **energy and momentum** are conserved at each interaction vertex in the diagram.
 - The factor $1/(q^2 - m_x^2)$ is the propagator; it arises naturally from the above discussion of interaction by particle exchange

★ The matrix element: $M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$ depends on:

- The fundamental strength of the interaction at the two vertices g_a, g_b
- The four-momentum, q , carried by the (virtual) particle which is determined from energy/momentum conservation at the vertices. Note q^2 can be either positive or negative.



Here $q = p_1 - p_3 = p_4 - p_2 = t$

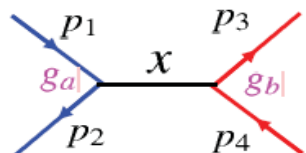
“t-channel”

For **elastic scattering**: $p_1 = (E, \vec{p}_1)$; $p_3 = (E, \vec{p}_3)$

$$q^2 = (E - E)^2 - (\vec{p}_1 - \vec{p}_3)^2$$

$$q^2 < 0$$

termed “space-like”



Here $q = p_1 + p_2 = p_3 + p_4 = s$

“s-channel”

In CoM: $p_1 = (E, \vec{p})$; $p_2 = (E, -\vec{p})$

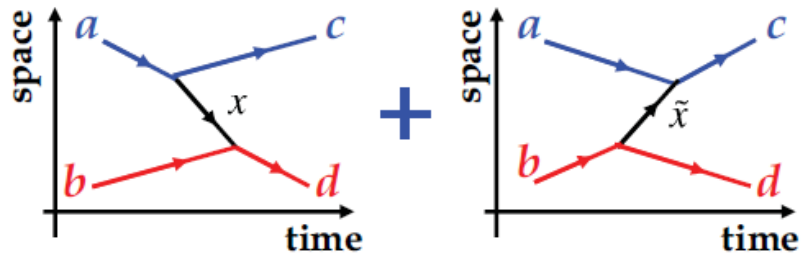
$$q^2 = (E + E)^2 - (\vec{p} - \vec{p})^2 = 4E^2$$

$$q^2 > 0$$

termed “time-like”

Virtual Particles

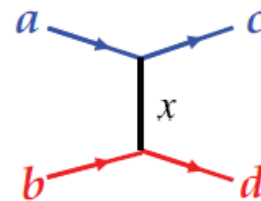
"Time-ordered QM"



- Momentum conserved at vertices
- Energy **not** conserved at vertices
- Exchanged particle **"on mass shell"**

$$E_x^2 - |\vec{p}_x|^2 = m_x^2$$

Feynman diagram



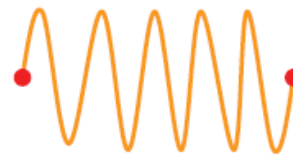
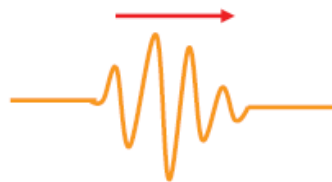
$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- Momentum **AND** energy conserved at interaction vertices
- Exchanged particle **"off mass shell"**

$$E_x^2 - |\vec{p}_x|^2 = q^2 \neq m_x^2$$

VIRTUAL PARTICLE

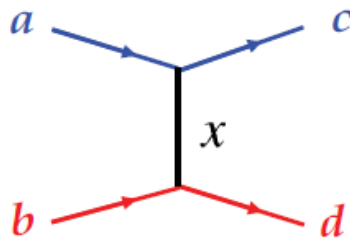
- Can think of observable **"on mass shell"** particles as propagating waves and unobservable virtual particles as normal modes between the source particles:



Aside: $V(r)$ from Particle Exchange

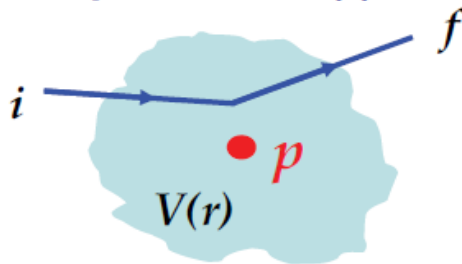
★ Can view the scattering of an electron by a proton at rest in two ways:

- Interaction by particle exchange in 2nd order perturbation theory.



$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- Could also evaluate the same process in first order perturbation theory treating proton as a fixed source of a field which gives rise to a potential $V(r)$



$$M = \langle \psi_f | V(r) | \psi_i \rangle$$

Obtain same expression for M_{fi} using

$$V(r) = g_a g_b \frac{e^{-mr}}{r}$$

**YUKAWA
potential**

- ★ In this way can relate potential and forces to the particle exchange picture
- ★ However, scattering from a fixed potential $V(r)$ is not a relativistic invariant view

Quantum Electrodynamics (QED)

- ★ Now consider the interaction of an electron and tau lepton by the exchange of a photon. Although the general ideas we applied previously still hold, we now have to account for the **spin of the electron/tau-lepton** and also the **spin (polarization) of the virtual photon**.

- The basic interaction between a photon and a charged particle can be introduced by making the minimal substitution

$$\vec{p} \rightarrow \vec{p} - q\vec{A}; \quad E \rightarrow E - q\phi$$

In QM:

$$\vec{p} = -i\vec{\nabla}; \quad E = i\partial/\partial t$$

(here $q = \text{charge}$)

Therefore make substitution: $i\partial_\mu \rightarrow i\partial_\mu - qA_\mu$

where $A_\mu = (\phi, -\vec{A}); \quad \partial_\mu = (\partial/\partial t, +\vec{\nabla})$

- The Dirac equation:

$$\gamma^\mu \partial_\mu \psi + im\psi = 0 \quad \rightarrow \quad \gamma^\mu \partial_\mu \psi + iq\gamma^\mu A_\mu \psi + im\psi = 0$$

$$(\times i) \quad \rightarrow \quad i\gamma^0 \frac{\partial \psi}{\partial t} + i\vec{\gamma} \cdot \vec{\nabla} \psi - q\gamma^\mu A_\mu \psi - m\psi = 0$$

$$i\gamma^0 \frac{\partial \psi}{\partial t} = \gamma^0 \hat{H} \psi = m\psi - i\vec{\gamma} \cdot \vec{\nabla} \psi + q\gamma^\mu A_\mu \psi$$

$$\times \gamma^0 : \quad \hat{H} \psi = \underbrace{(\gamma^0 m - i\gamma^0 \vec{\gamma} \cdot \vec{\nabla})}_{\text{Combined rest mass + K.E.}} \psi + \underbrace{q\gamma^0 \gamma^\mu A_\mu}_{\text{Potential energy}} \psi$$

- We can identify the potential energy of a charged spin-half particle in an electromagnetic field as:

$$\hat{V}_D = q\gamma^0 \gamma^\mu A_\mu$$

(note the A_0 term is just: $q\gamma^0 \gamma^0 A_0 = q\phi$)

- The final complication is that we have to account for the photon polarization states.

$$A_\mu = \epsilon_\mu^{(\lambda)} e^{i(\vec{p} \cdot \vec{r} - Et)}$$

e.g. for a real photon propagating in the z direction we have two orthogonal transverse polarization states

$$\epsilon^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \epsilon^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Could equally have chosen circularly polarized states

- Previously with the example of a simple spin-less interaction we had:

$$M = \langle \psi_c | V | \psi_a \rangle \frac{1}{q^2 - m_x^2} \langle \psi_d | V | \psi_b \rangle$$

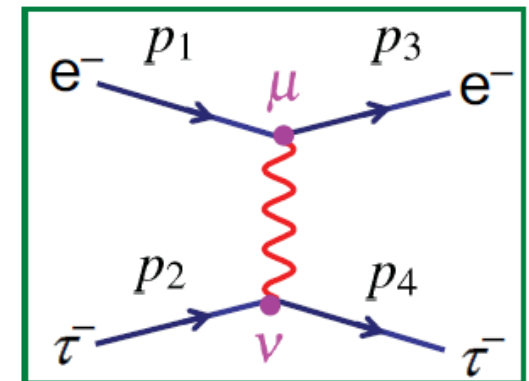
- ★ In QED we could again go through the procedure of summing the time-orderings using Dirac spinors and the expression for \hat{V}_D . If we were to do this, remembering to sum over all photon polarizations, we would obtain:

$$M = [u_e^\dagger(p_3) q_e \gamma^0 \gamma^\mu u_e(p_1)] \sum_\lambda \frac{\epsilon_\mu^\lambda (\epsilon_\nu^\lambda)^*}{q^2} [u_\tau^\dagger(p_4) q_\tau \gamma^0 \gamma^\nu u_\tau(p_2)]$$

Interaction of e^- with photon

Massless photon propagator summing over polarizations

Interaction of τ^- with photon



- All the physics of **QED** is in the above expression !

- The sum over the polarizations of the **VIRTUAL** photon has to include longitudinal and scalar contributions, i.e. 4 polarisation states

$$\epsilon^{(0)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \epsilon^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \epsilon^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \epsilon^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

and gives:

$$\sum_{\lambda} \epsilon_{\mu}^{\lambda} (\epsilon_{\nu}^{\lambda})^* = -g_{\mu\nu}$$

This is not obvious - for the moment just take it on trust

and the invariant matrix element becomes:

$$M = [u_e^{\dagger}(p_3) q_e \gamma^0 \gamma^{\mu} u_e(p_1)] \frac{-g_{\mu\nu}}{q^2} [u_{\tau}^{\dagger}(p_4) q_{\tau} \gamma^0 \gamma^{\nu} u_{\tau}(p_2)]$$

- Using the definition of the adjoint spinor $\bar{\psi} = \psi^{\dagger} \gamma^0$

$$M = [\bar{u}_e(p_3) q_e \gamma^{\mu} u_e(p_1)] \frac{-g_{\mu\nu}}{q^2} [\bar{u}_{\tau}(p_4) q_{\tau} \gamma^{\nu} u_{\tau}(p_2)]$$

- ★ This is a remarkably simple expression ! It is shown in Appendix V of Lecture 2 that $\bar{u}_1 \gamma^{\mu} u_2$ transforms as a four vector. Writing

$$j_e^{\mu} = \bar{u}_e(p_3) \gamma^{\mu} u_e(p_1) \quad j_{\tau}^{\nu} = \bar{u}_{\tau}(p_4) \gamma^{\nu} u_{\tau}(p_2)$$

$$M = -q_e q_{\tau} \frac{j_e \cdot j_{\tau}}{q^2} \quad \text{showing that } M \text{ is Lorentz Invariant}$$

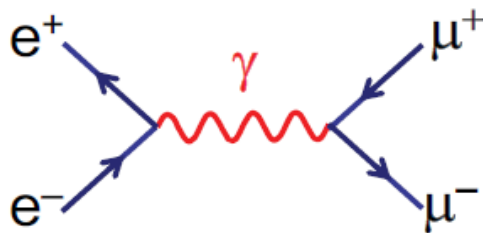
Feynman rules for QED

- It should be remembered that the expression

$$M = [\bar{u}_e(p_3)q_e\gamma^\mu u_e(p_1)] \frac{-g^{\mu\nu}}{q^2} [\bar{u}_\tau(p_4)q_\tau\gamma^\nu u_\tau(p_2)]$$

hides a lot of complexity. We have summed over all possible **time-orderings** and summed over all **polarization states** of the virtual photon. If we are then presented with a new Feynman diagram we don't want to go through the full calculation again.

Fortunately this isn't necessary – can just write down matrix element using a set of simple rules









Basic Feynman Rules:



- Propagator factor for each internal line
(i.e. each internal virtual particle)
- Dirac Spinor for each external line
(i.e. each real incoming or outgoing particle)
- Vertex factor for each vertex

Basic rules for QED

External Lines

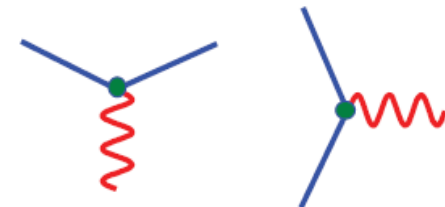
spin 1/2	{	incoming particle	$u(p)$	
		outgoing particle	$\bar{u}(p)$	
		incoming antiparticle	$\bar{v}(p)$	
		outgoing antiparticle	$v(p)$	
spin 1	{	incoming photon	$\varepsilon^\mu(p)$	
		outgoing photon	$\varepsilon^\mu(p)^*$	

Internal Lines (propagators)

spin 1	photon	$-\frac{ig_{\mu\nu}}{q^2}$	
spin 1/2	fermion	$\frac{i(\gamma^\mu q_\mu + m)}{q^2 - m^2}$	

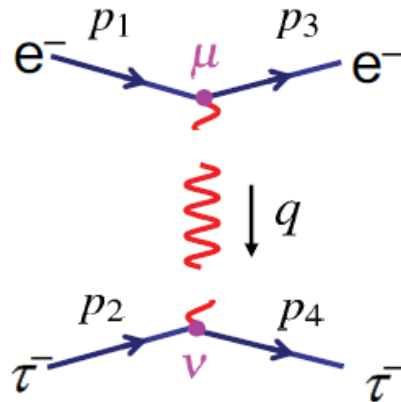
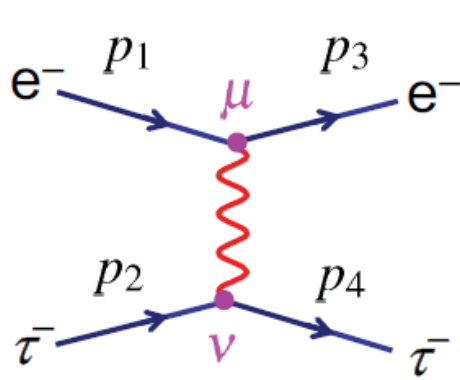
Vertex Factors

spin 1/2	fermion (charge $- e $)	$ie\gamma^\mu$
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Matrix Element $-iM =$ product of all factors

e.g.



$$\bar{u}_e(p_3)[ie\gamma^\mu]u_e(p_1)$$

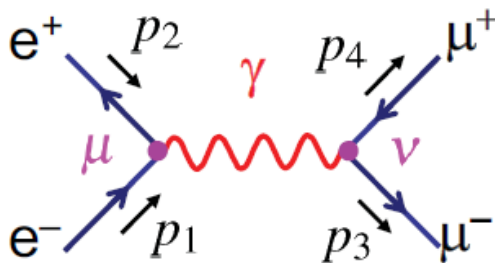
$$\frac{-ig_{\mu\nu}}{q^2}$$

$$\bar{u}_\tau(p_4)[ie\gamma^\nu]u_\tau(p_2)$$

$$-iM = [\bar{u}_e(p_3)ie\gamma^\mu u_e(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}_\tau(p_4)ie\gamma^\nu u_\tau(p_2)]$$

- Which is the same expression as we obtained previously

e.g.



$$-iM = [\bar{v}(p_2)ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}(p_3)ie\gamma^\nu v(p_4)]$$

Note:

- ♦ At each vertex the adjoint spinor is written first
- ♦ Each vertex has a different index
- ♦ The $g_{\mu\nu}$ of the propagator connects the indices at the vertices

Summary

- ★ Interaction by particle exchange naturally gives rise to **Lorentz Invariant Matrix Element** of the form

$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- ★ Derived the basic interaction in **QED** taking into account the spins of the fermions and polarization of the virtual photons:

$$-iM = [\bar{u}(p_3)ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}(p_4)ie\gamma^\nu u(p_2)]$$

- ★ **We now have all the elements to perform proper calculations in QED !**

Electron-positron annihilation

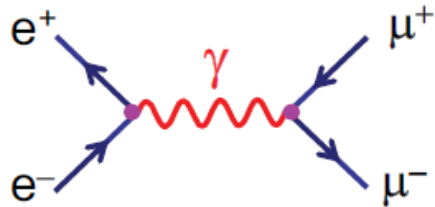


QED calculations

- How to calculate a cross section using QED (e.g. $e^+e^- \rightarrow \mu^+\mu^-$):

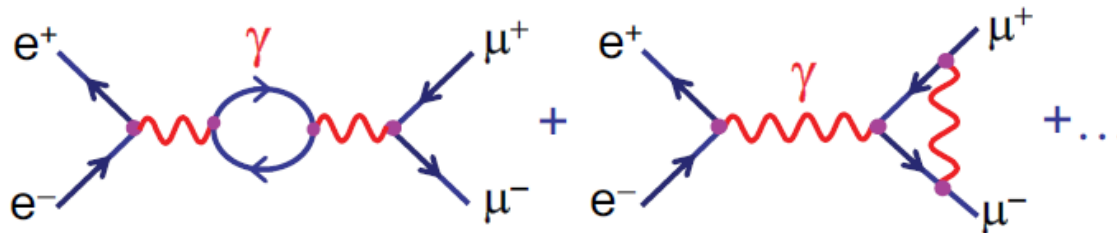
- ① Draw all possible Feynman Diagrams

- For $e^+e^- \rightarrow \mu^+\mu^-$ there is just one lowest order diagram



$$M \propto e^2 \propto \alpha_{em}$$

+ many **second order** diagrams + ...



$$M \propto e^4 \propto \alpha_{em}^2$$

- ② For each diagram calculate the matrix element using Feynman rules

- ③ Sum the individual matrix elements (i.e. sum the amplitudes)

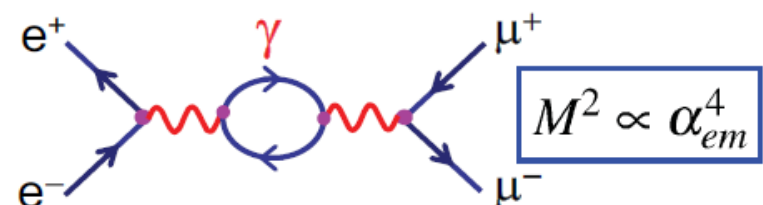
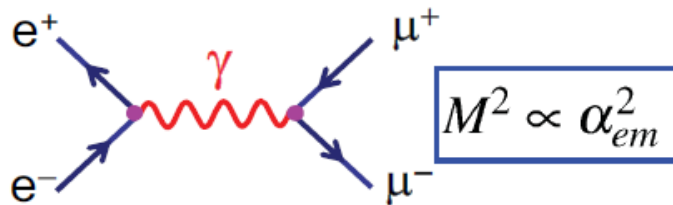
$$M_{fi} = M_1 + M_2 + M_3 + \dots$$

- Note: summing amplitudes therefore different diagrams for the same final state can interfere either positively or negatively!

and then square $|M_{fi}|^2 = (M_1 + M_2 + M_3 + \dots)(M_1^* + M_2^* + M_3^* + \dots)$

➔ this gives the full perturbation expansion in α_{em}

- For QED $\alpha_{em} \sim 1/137$ the lowest order diagram dominates and for most purposes it is sufficient to **neglect** higher order diagrams.



④ Calculate decay rate/cross section using formulae

- e.g. for a decay

$$\Gamma = \frac{P^*}{32\pi^2 m_a^2} \int |M_{fi}|^2 d\Omega$$

- For scattering in the centre-of-mass frame

$$\frac{d\sigma}{d\Omega^*} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} |M_{fi}|^2 \quad (1)$$

- For scattering in lab. frame (neglecting mass of scattered particle)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{ME_1} \right)^2 |M_{fi}|^2$$

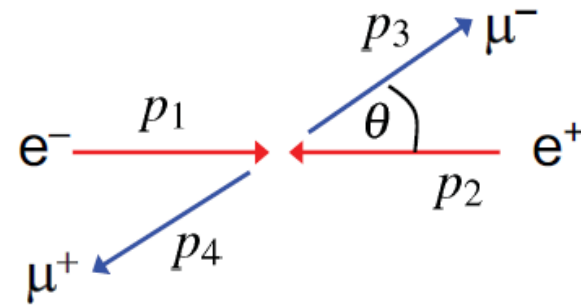
Electron Positron Annihilation

★ Consider the process: $e^+e^- \rightarrow \mu^+\mu^-$

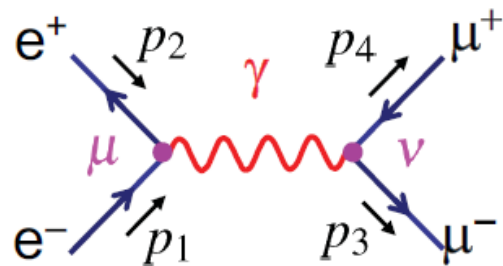
- Work in C.o.M. frame (this is appropriate for most e^+e^- colliders).

$$p_1 = (E, 0, 0, p) \quad p_2 = (E, 0, 0, -p)$$

$$p_3 = (E, \vec{p}_f) \quad p_4 = (E, -\vec{p}_f)$$



- Only consider the lowest order Feynman diagram:



- Feynman rules give:

$$-iM = [\bar{v}(p_2)ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}(p_3)ie\gamma^\nu v(p_4)]$$

- NOTE:**
- Incoming anti-particle \bar{v}
 - Incoming particle u
 - Adjoint spinor written first

- In the C.o.M. frame have

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f|}{|\vec{p}_i|} |M_{fi}|^2 \quad \text{with} \quad s = (p_1 + p_2)^2 = (E + E)^2 = 4E^2$$

Electron and Muon Currents

- Here $q^2 = (p_1 + p_2)^2 = s$ and matrix element

$$-iM = [\bar{v}(p_2)ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\bar{u}(p_3)ie\gamma^\nu v(p_4)]$$

$$\rightarrow M = -\frac{e^2}{s} g_{\mu\nu} [\bar{v}(p_2)\gamma^\mu u(p_1)][\bar{u}(p_3)\gamma^\nu v(p_4)]$$

- In Lecture 2 introduced the **four-vector** current

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

which has same form as the two terms in [] in the matrix element

- The matrix element can be written in terms of the electron and muon currents

$$(j_e)^\mu = \bar{v}(p_2)\gamma^\mu u(p_1) \quad \text{and} \quad (j_\mu)^\nu = \bar{u}(p_3)\gamma^\nu v(p_4)$$

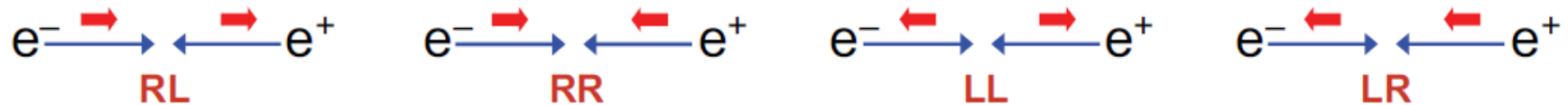
$$\rightarrow M = -\frac{e^2}{s} g_{\mu\nu} (j_e)^\mu (j_\mu)^\nu$$

$$M = -\frac{e^2}{s} j_e \cdot j_\mu$$

- Matrix element is a four-vector scalar product – confirming it is **Lorentz Invariant**

Spin in e^+e^- Annihilation

- In general the electron and positron will not be polarized, i.e. there will be equal numbers of positive and negative helicity states
- There are four possible combinations of spins in the **initial state** !



- Similarly there are four possible helicity combinations in the final state
- In total there are **16** combinations e.g. **RL**→RR, **RL**→RL, ...
- To account for these states we need to **sum over all 16 possible helicity combinations** and then **average over the number of initial helicity states**:

$$\langle |M|^2 \rangle = \frac{1}{4} \sum_{\text{spins}} |M_i|^2 = \frac{1}{4} (|M_{LL \rightarrow LL}|^2 + |M_{LL \rightarrow LR}|^2 + \dots)$$

- ★ i.e. need to evaluate:

$$M = -\frac{e^2}{s} j_e \cdot j_\mu$$

for all 16 helicity combinations !

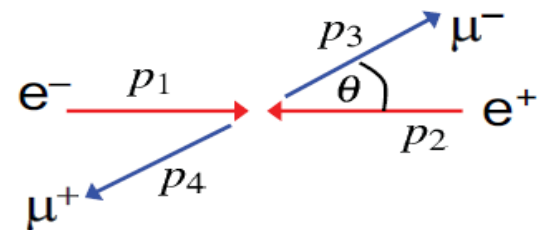
- ★ Fortunately, in the limit $E \gg m_\mu$ only 4 helicity combinations give non-zero matrix elements - we will see that this is an important feature of QED/QCD

- In the C.o.M. frame in the limit $E \gg m$

$$p_1 = (E, 0, 0, E); \quad p_2 = (E, 0, 0, -E)$$

$$p_3 = (E, E \sin \theta, 0, E \cos \theta);$$

$$p_4 = (E, -\sin \theta, 0, -E \cos \theta)$$



- Left- and right-handed helicity spinors (Lecture 2) for particles/anti-particles are:

$$u_{\uparrow} = N \begin{pmatrix} c \\ e^{i\phi} s \\ \frac{|\vec{p}|}{E+m} c \\ \frac{|\vec{p}|}{E+m} e^{i\phi} s \end{pmatrix} \quad u_{\downarrow} = N \begin{pmatrix} -s \\ e^{i\phi} c \\ \frac{|\vec{p}|}{E+m} s \\ -\frac{|\vec{p}|}{E+m} e^{i\phi} c \end{pmatrix} \quad v_{\uparrow} = N \begin{pmatrix} \frac{|\vec{p}|}{E+m} s \\ -\frac{|\vec{p}|}{E+m} e^{i\phi} c \\ -s \\ e^{i\phi} c \end{pmatrix} \quad v_{\downarrow} = N \begin{pmatrix} \frac{|\vec{p}|}{E+m} c \\ \frac{|\vec{p}|}{E+m} e^{i\phi} s \\ c \\ e^{i\phi} s \end{pmatrix}$$

where $s = \sin \frac{\theta}{2}$; $c = \cos \frac{\theta}{2}$ and $N = \sqrt{E+m}$

- In the limit $E \gg m$ these become:

$$u_{\uparrow} = \sqrt{E} \begin{pmatrix} c \\ s e^{i\phi} \\ c \\ s e^{i\phi} \end{pmatrix}; \quad u_{\downarrow} = \sqrt{E} \begin{pmatrix} -s \\ c e^{i\phi} \\ s \\ -c e^{i\phi} \end{pmatrix}; \quad v_{\uparrow} = \sqrt{E} \begin{pmatrix} s \\ -c e^{i\phi} \\ -s \\ c e^{i\phi} \end{pmatrix}; \quad v_{\downarrow} = \sqrt{E} \begin{pmatrix} c \\ s e^{i\phi} \\ c \\ s e^{i\phi} \end{pmatrix}$$

- The incident electron can either be in a left- or right-handed helicity state

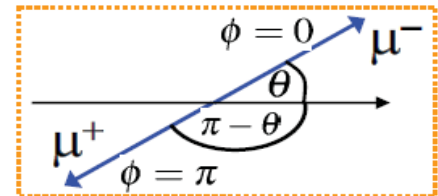
$$u_{\uparrow}(p_1) = \sqrt{E} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad u_{\downarrow}(p_1) = \sqrt{E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix};$$

- For the initial state positron ($\theta = \pi$) can have either:

$$v_{\uparrow}(p_2) = \sqrt{E} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}; v_{\downarrow}(p_2) = \sqrt{E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

- Similarly for the final state μ^- which has polar angle θ and choosing $\phi = 0$

$$u_{\uparrow}(p_3) = \sqrt{E} \begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix}; u_{\downarrow}(p_3) = \sqrt{E} \begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix};$$

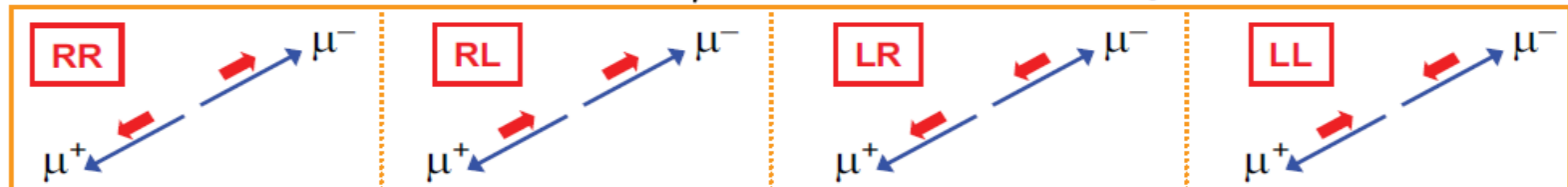


- And for the final state μ^+ replacing $\theta \rightarrow \pi - \theta$; $\phi \rightarrow \pi$

$$v_{\uparrow}(p_4) = \sqrt{E} \begin{pmatrix} c \\ s \\ -c \\ -s \end{pmatrix}; v_{\downarrow}(p_4) = \sqrt{E} \begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}; \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \text{ using } \begin{array}{l} \sin\left(\frac{\pi-\theta}{2}\right) = \cos\frac{\theta}{2} \\ \cos\left(\frac{\pi-\theta}{2}\right) = \sin\frac{\theta}{2} \\ e^{i\pi} = -1 \end{array}$$

- Wish to calculate the matrix element $M = -\frac{e^2}{s} j_e \cdot j_{\mu}$

- ★ first consider the muon current j_{μ} for 4 possible helicity combinations



The muon current

- Want to evaluate $(j_\mu)^\nu = \bar{u}(p_3)\gamma^\nu v(p_4)$ for all four helicity combinations
- For arbitrary spinors ψ, ϕ with it is straightforward to show that the components of $\bar{\psi}\gamma^\mu\phi$ are

$$\bar{\psi}\gamma^0\phi = \psi^\dagger\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 \quad (3)$$

$$\bar{\psi}\gamma^1\phi = \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \quad (4)$$

$$\bar{\psi}\gamma^2\phi = \psi^\dagger\gamma^0\gamma^2\phi = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) \quad (5)$$

$$\bar{\psi}\gamma^3\phi = \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2 \quad (6)$$

- Consider the $\mu_R^- \mu_L^+$ combination using $\psi = u_\uparrow, \phi = v_\downarrow$

with $v_\downarrow = \sqrt{E} \begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}; u_\uparrow = \sqrt{E} \begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix};$

$$\bar{u}_\uparrow(p_3)\gamma^0 v_\downarrow(p_4) = E(cs - sc + cs - sc) = 0$$

$$\bar{u}_\uparrow(p_3)\gamma^1 v_\downarrow(p_4) = E(-c^2 + s^2 - c^2 + s^2) = 2E(s^2 - c^2) = -2E \cos \theta$$

$$\bar{u}_\uparrow(p_3)\gamma^2 v_\downarrow(p_4) = -iE(-c^2 - s^2 - c^2 - s^2) = 2iE$$

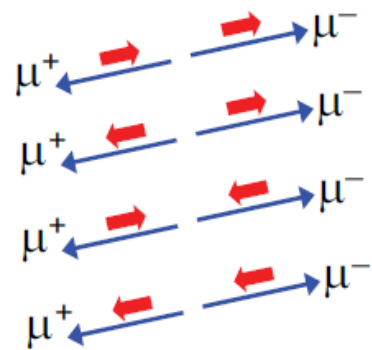
$$\bar{u}_\uparrow(p_3)\gamma^3 v_\downarrow(p_4) = E(cs + sc + cs + sc) = 4Esc = 2E \sin \theta$$



- Hence the four-vector muon current for the **RL** combination is

$$\bar{u}_{\uparrow}(p_3)\gamma^{\nu}v_{\downarrow}(p_4) = 2E(0, -\cos\theta, i, \sin\theta)$$

- The results for the 4 helicity combinations (obtained in the same manner) are:



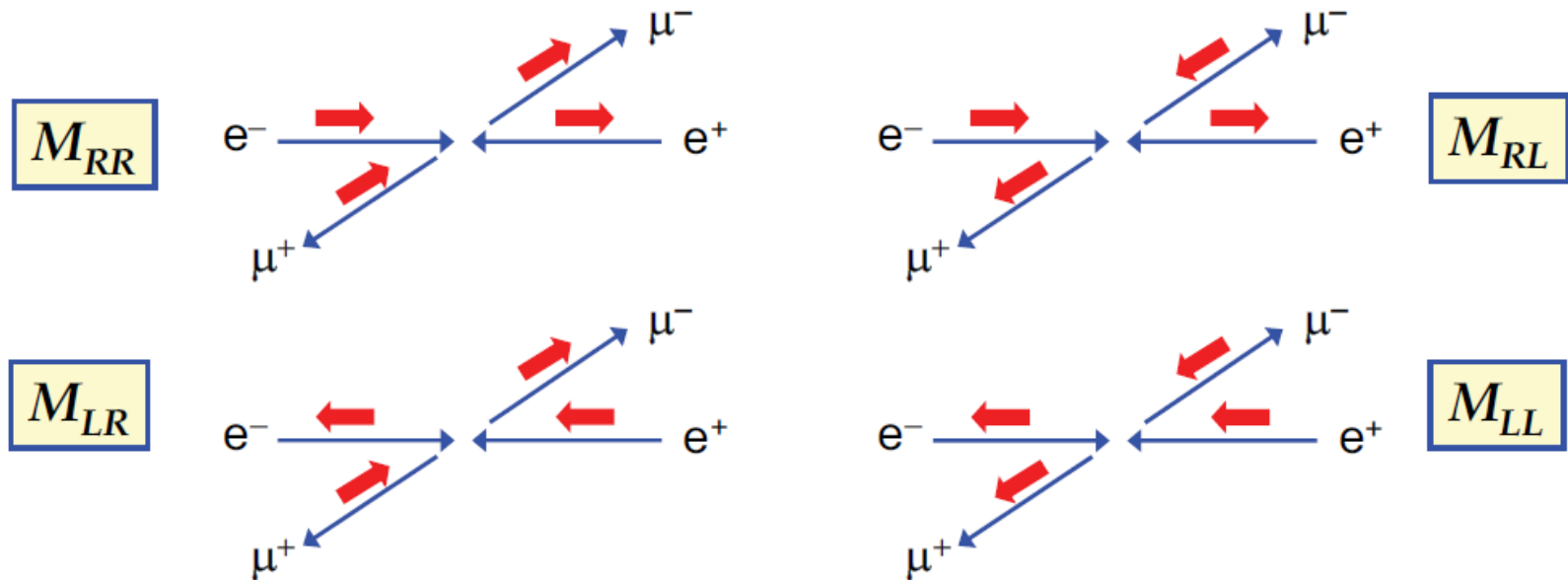
$\bar{u}_{\uparrow}(p_3)\gamma^{\nu}v_{\downarrow}(p_4)$	$= 2E(0, -\cos\theta, i, \sin\theta)$	RL
$\bar{u}_{\uparrow}(p_3)\gamma^{\nu}v_{\uparrow}(p_4)$	$= (0, 0, 0, 0)$	RR
$\bar{u}_{\downarrow}(p_3)\gamma^{\nu}v_{\downarrow}(p_4)$	$= (0, 0, 0, 0)$	LL
$\bar{u}_{\downarrow}(p_3)\gamma^{\nu}v_{\uparrow}(p_4)$	$= 2E(0, -\cos\theta, -i, \sin\theta)$	LR

★ **IN THE LIMIT $E \gg m$ only two helicity combinations are non-zero !**

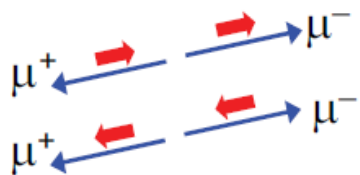
- This is an important feature of QED. It applies equally to QCD.
- In the Weak interaction only one helicity combination contributes.
- The origin of this will be discussed in the last part of this lecture
- But as a consequence of the 16 possible helicity combinations only four given non-zero matrix elements

Electron Positron Annihilation cont.

★ For $e^+e^- \rightarrow \mu^+\mu^-$ now only have to consider the 4 matrix elements:



• Previously we derived the muon currents for the allowed helicities:



$$\begin{aligned} \mu_R^- \mu_L^+ &: \quad \bar{u}_\uparrow(p_3) \gamma^\nu v_\downarrow(p_4) = 2E(0, -\cos \theta, i, \sin \theta) \\ \mu_L^- \mu_R^+ &: \quad \bar{u}_\downarrow(p_3) \gamma^\nu v_\uparrow(p_4) = 2E(0, -\cos \theta, -i, \sin \theta) \end{aligned}$$

• Now need to consider the electron current

The electron current

- The incoming electron and positron spinors (L and R helicities) are:

$$u_{\uparrow} = \sqrt{E} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \quad u_{\downarrow} = \sqrt{E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}; \quad v_{\uparrow} = \sqrt{E} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}; \quad v_{\downarrow} = \sqrt{E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

- The electron current can either be obtained from equations (3)-(6) as before or it can be obtained directly from the expressions for the muon current.

$$(j_e)^{\mu} = \bar{v}(p_2) \gamma^{\mu} u(p_1) \qquad (j_{\mu})^{\mu} = \bar{u}(p_3) \gamma^{\mu} v(p_4)$$

- Taking the Hermitian conjugate of the muon current gives

$$\begin{aligned} [\bar{u}(p_3) \gamma^{\mu} v(p_4)]^{\dagger} &= [u(p_3)^{\dagger} \gamma^0 \gamma^{\mu} v(p_4)]^{\dagger} \\ &= v(p_4)^{\dagger} \gamma^{\mu\dagger} \gamma^{0\dagger} u(p_3) && (AB)^{\dagger} = B^{\dagger} A^{\dagger} \\ &= v(p_4)^{\dagger} \gamma^{\mu\dagger} \gamma^0 u(p_3) && \gamma^{0\dagger} = \gamma^0 \\ &= v(p_4)^{\dagger} \gamma^0 \gamma^{\mu} u(p_3) && \gamma^{\mu\dagger} \gamma^0 = \gamma^0 \gamma^{\mu} \\ &= \bar{v}(p_4) \gamma^{\mu} u(p_3) \end{aligned}$$

- Taking the complex conjugate of the muon currents for the two non-zero helicity configurations:

$$\bar{\nu}_\downarrow(p_4)\gamma^\mu u_\uparrow(p_3) = [\bar{u}_\uparrow(p_3)\gamma^\nu \nu_\downarrow(p_4)]^* = 2E(0, -\cos\theta, -i, \sin\theta)$$

$$\bar{\nu}_\uparrow(p_4)\gamma^\mu u_\downarrow(p_3) = [\bar{u}_\downarrow(p_3)\gamma^\nu \nu_\uparrow(p_4)]^* = 2E(0, -\cos\theta, i, \sin\theta)$$

To obtain the electron currents we simply need to set $\theta = 0$

$$e^- \xrightarrow{\text{red arrow}} \xleftarrow{\text{red arrow}} e^+$$

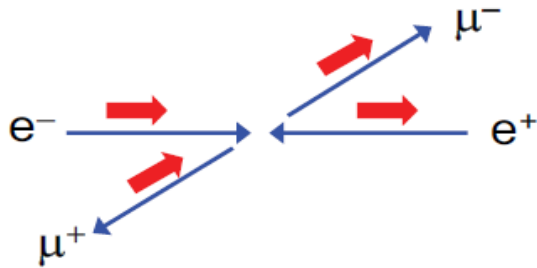
$$e^- \xleftarrow{\text{red arrow}} \xrightarrow{\text{red arrow}} e^+$$

$e_R^- e_L^+$:	$\bar{\nu}_\downarrow(p_2)\gamma^\nu u_\uparrow(p_1)$	=	$2E(0, -1, -i, 0)$
$e_L^- e_R^+$:	$\bar{\nu}_\uparrow(p_2)\gamma^\nu u_\downarrow(p_1)$	=	$2E(0, -1, i, 0)$

Matrix element calculation

- We can now calculate $M = -\frac{e^2}{s} j_e \cdot j_\mu$ for the four possible helicity combinations.

e.g. the matrix element for $e_R^- e_L^+ \rightarrow \mu_R^- \mu_L^+$ which will denote M_{RR}



Here the first subscript refers to the helicity of the e^- and the second to the helicity of the μ^- . Don't need to specify other helicities due to "helicity conservation", only certain chiral combinations are non-zero.

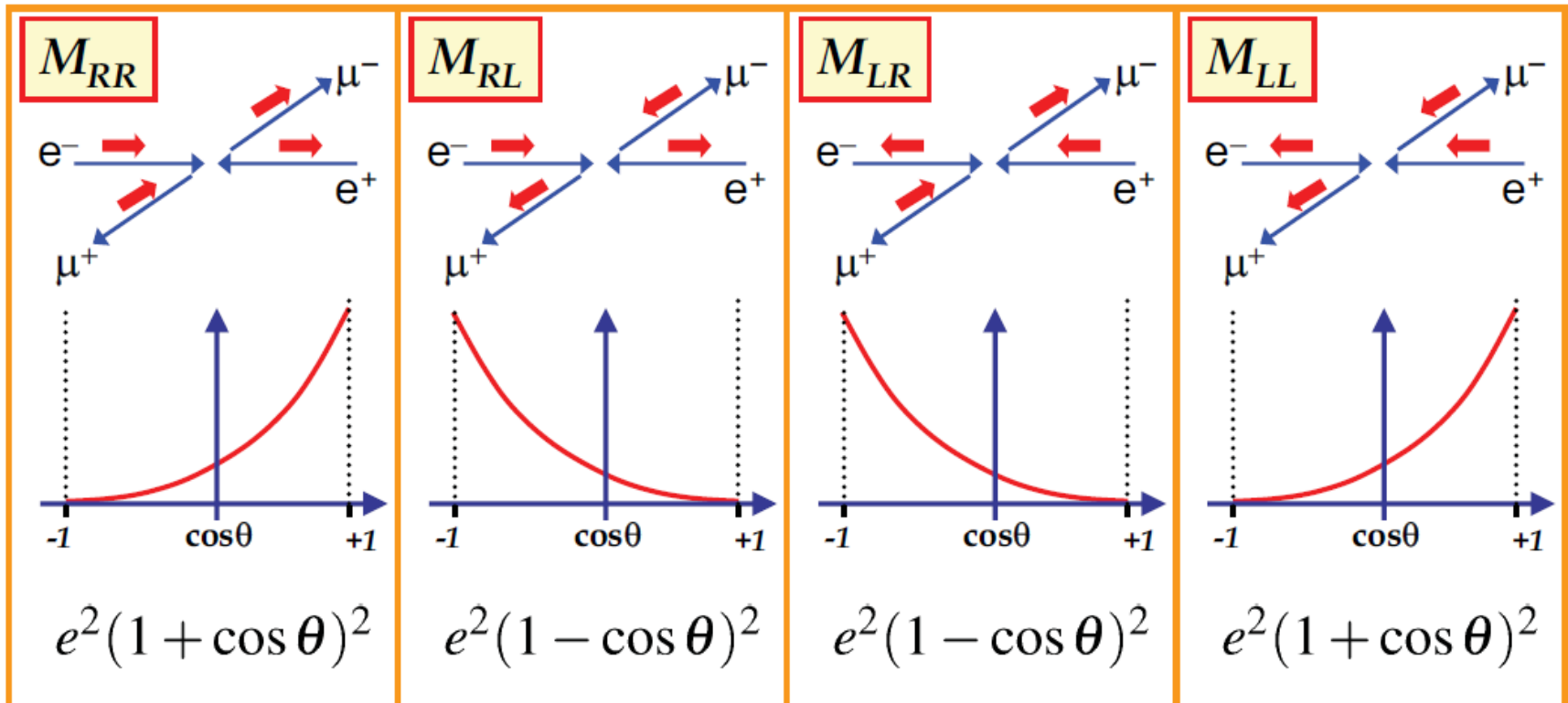
★ Using: $e_R^- e_L^+ : (j_e)^\mu = \bar{v}_\downarrow(p_2) \gamma^\mu u_\uparrow(p_1) = 2E(0, -1, -i, 0)$
 $\mu_R^- \mu_L^+ : (j_\mu)^\nu = \bar{u}_\uparrow(p_3) \gamma^\nu v_\downarrow(p_4) = 2E(0, -\cos \theta, i, \sin \theta)$

gives $M_{RR} = -\frac{e^2}{s} [2E(0, -1, -i, 0)] \cdot [2E(0, -\cos \theta, i, \sin \theta)]$
 $= -e^2(1 + \cos \theta)$
 $= -4\pi\alpha(1 + \cos \theta) \quad \text{where} \quad \alpha = e^2/4\pi \approx 1/137$

Similarly

$$|M_{RR}|^2 = |M_{LL}|^2 = (4\pi\alpha)^2(1 + \cos\theta)^2$$

$$|M_{RL}|^2 = |M_{LR}|^2 = (4\pi\alpha)^2(1 - \cos\theta)^2$$



- Assuming that the incoming electrons and positrons are **unpolarized**, all 4 possible initial helicity states are equally likely.

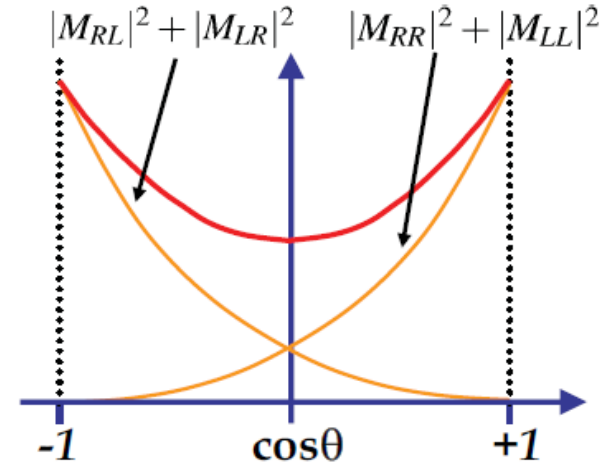
Differential cross-section

- The cross section is obtained by averaging over the initial spin states and summing over the final spin states:

$$\begin{aligned} \frac{d\sigma}{d\Omega} &= \frac{1}{4} \times \frac{1}{64\pi^2 s} (|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + |M_{LL}|^2) \\ &= \frac{(4\pi\alpha)^2}{256\pi^2 s} (2(1 + \cos\theta)^2 + 2(1 - \cos\theta)^2) \end{aligned}$$

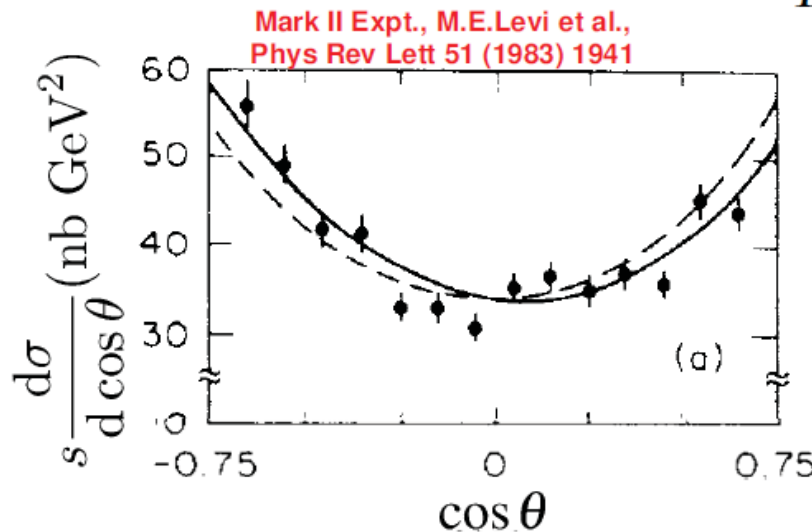


$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} (1 + \cos^2\theta)}$$



Example:

$e^+e^- \rightarrow \mu^+\mu^-$
 $\sqrt{s} = 29 \text{ GeV}$



--- pure QED, $O(\alpha^3)$
 — QED plus Z contribution

Angular distribution becomes slightly asymmetric in higher order QED or when Z contribution is included

- The total cross section is obtained by integrating over θ, ϕ using

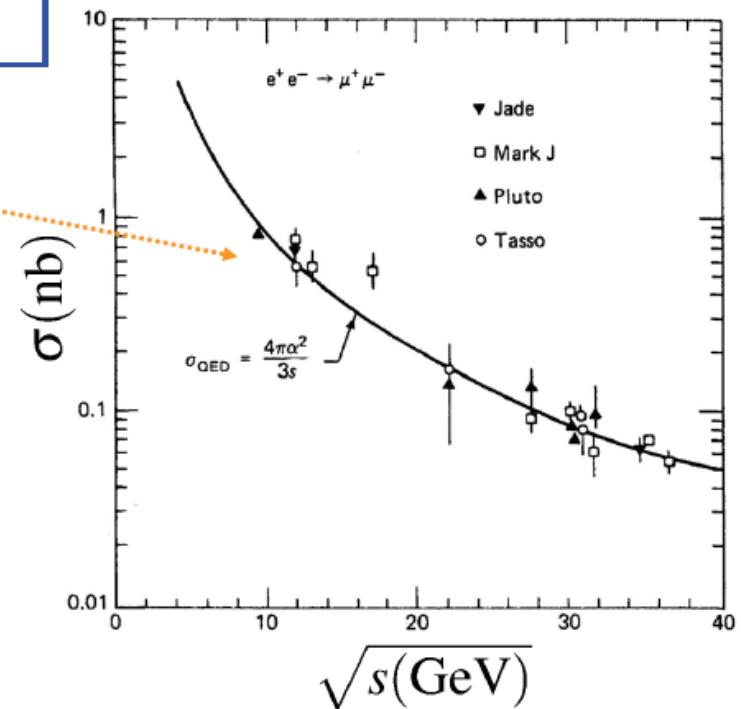
$$\int (1 + \cos^2 \theta) d\Omega = 2\pi \int_{-1}^{+1} (1 + \cos^2 \theta) d\cos \theta = \frac{16\pi}{3}$$

giving the **QED** total cross-section for the process $e^+e^- \rightarrow \mu^+\mu^-$

$$\sigma = \frac{4\pi\alpha^2}{3s}$$

★ Lowest order cross section calculation provides a good description of the data !

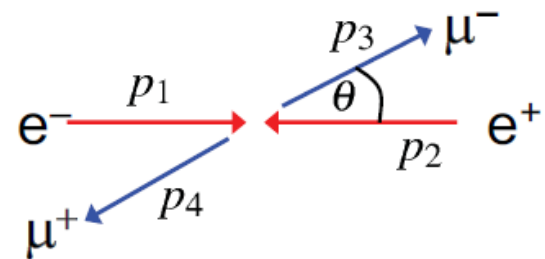
This is an impressive result. From first principles we have arrived at an expression for the electron-positron annihilation cross section which is good to **1%**



Lorentz Invariant form of Matrix Element

- Before concluding this discussion, note that the spin-averaged Matrix Element derived above is written in terms of the muon angle in the C.o.M. frame.

$$\begin{aligned}\langle |M_{fi}|^2 \rangle &= \frac{1}{4} \times (|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + |M_{LL}|^2) \\ &= \frac{1}{4} e^4 (2(1 + \cos \theta)^2 + 2(1 - \cos \theta)^2) \\ &= e^4 (1 + \cos^2 \theta)\end{aligned}$$



- The matrix element is **Lorentz Invariant** (scalar product of 4-vector currents) and it is desirable to write it in a frame-independent form, i.e. express in terms of Lorentz Invariant 4-vector scalar products

- In the C.o.M. $p_1 = (E, 0, 0, E)$ $p_2 = (E, 0, 0, -E)$
 $p_3 = (E, E \sin \theta, 0, E \cos \theta)$ $p_4 = (E, -E \sin \theta, 0, -E \cos \theta)$
 giving: $p_1 \cdot p_2 = 2E^2$; $p_1 \cdot p_3 = E^2(1 - \cos \theta)$; $p_1 \cdot p_4 = E^2(1 + \cos \theta)$

- Hence we can write

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1 \cdot p_3)^2 + (p_1 \cdot p_4)^2}{(p_1 \cdot p_2)^2}$$

★ Valid in any frame !

$$\equiv 2e^4 \left(\frac{t^2 + u^2}{s^2} \right)$$

Chirality

- The helicity eigenstates for a particle/anti-particle for $E \gg m$ are:

$$u_{\uparrow} = \sqrt{E} \begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}; \quad u_{\downarrow} = \sqrt{E} \begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix}; \quad v_{\uparrow} = \sqrt{E} \begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}; \quad v_{\downarrow} = \sqrt{E} \begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

where $s = \sin \frac{\theta}{2}$; $c = \cos \frac{\theta}{2}$

- Define the matrix

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

- In the limit $E \gg m$ the helicity states are also eigenstates of γ^5

$$\gamma^5 u_{\uparrow} = +u_{\uparrow}; \quad \gamma^5 u_{\downarrow} = -u_{\downarrow}; \quad \gamma^5 v_{\uparrow} = -v_{\uparrow}; \quad \gamma^5 v_{\downarrow} = +v_{\downarrow}$$

- ★ In general, define the eigenstates of γ^5 as **LEFT** and **RIGHT HANDED CHIRAL** states u_R ; u_L ; v_R ; v_L

i.e. $\gamma^5 u_R = +u_R$; $\gamma^5 u_L = -u_L$; $\gamma^5 v_R = -v_R$; $\gamma^5 v_L = +v_L$

- In the **LIMIT** $E \gg m$ (and **ONLY IN THIS LIMIT**):

$$u_R \equiv u_{\uparrow}; \quad u_L \equiv u_{\downarrow}; \quad v_R \equiv v_{\uparrow}; \quad v_L \equiv v_{\downarrow}$$

- ★ This is a subtle but important point: in general the **HELICITY** and **CHIRAL** eigenstates are not the same. It is **only** in the **ultra-relativistic limit** that the chiral eigenstates correspond to the helicity eigenstates.
- ★ Chirality is an important concept in the structure of QED, and any interaction of the form $\bar{u}\gamma^\nu u$

- In general, the eigenstates of the chirality operator are:

$$\gamma^5 u_R = +u_R; \quad \gamma^5 u_L = -u_L; \quad \gamma^5 v_R = -v_R; \quad \gamma^5 v_L = +v_L$$

- Define the **projection operators**:

$$P_R = \frac{1}{2}(1 + \gamma^5); \quad P_L = \frac{1}{2}(1 - \gamma^5)$$

- The projection operators, project out the chiral eigenstates

$$P_R u_R = u_R; \quad P_R u_L = 0; \quad P_L u_R = 0; \quad P_L u_L = u_L$$

$$P_R v_R = 0; \quad P_R v_L = v_L; \quad P_L v_R = v_R; \quad P_L v_L = 0$$

- Note P_R projects out **right-handed particle states** and **left-handed anti-particle states**
- We can then write any spinor in terms of its left and right-handed chiral components:

$$\psi = \psi_R + \psi_L = \frac{1}{2}(1 + \gamma^5)\psi + \frac{1}{2}(1 - \gamma^5)\psi$$

Chirality in QED

- In QED the basic interaction between a fermion and photon is:

$$ie\bar{\psi}\gamma^\mu\phi$$

- Can decompose the spinors in terms of **Left** and **Right**-handed chiral components:

$$\begin{aligned}ie\bar{\psi}\gamma^\mu\phi &= ie(\bar{\psi}_L + \bar{\psi}_R)\gamma^\mu(\phi_R + \phi_L) \\ &= ie(\bar{\psi}_R\gamma^\mu\phi_R + \bar{\psi}_R\gamma^\mu\phi_L + \bar{\psi}_L\gamma^\mu\phi_R + \bar{\psi}_L\gamma^\mu\phi_L)\end{aligned}$$

- Using the properties of γ^5

$$(\gamma^5)^2 = 1; \quad \gamma^{5\dagger} = \gamma^5; \quad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$$

it is straightforward to show

$$\bar{\psi}_R\gamma^\mu\phi_L = 0; \quad \bar{\psi}_L\gamma^\mu\phi_R = 0$$

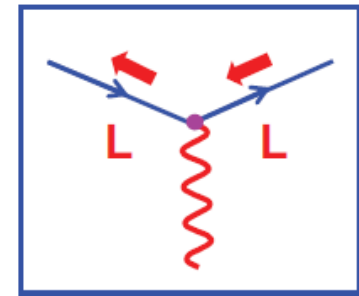
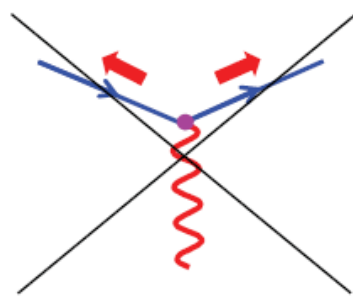
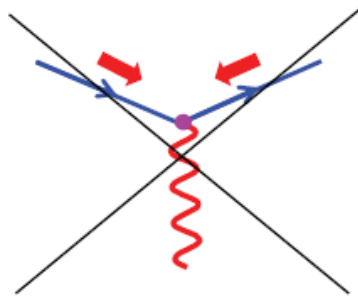
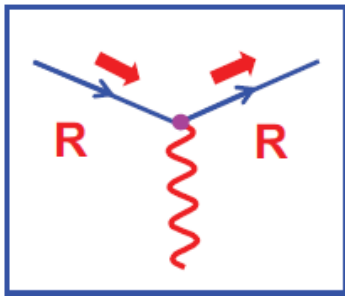
- ★ Hence only certain combinations of **chiral** eigenstates contribute to the interaction. This statement is **ALWAYS** true.
- For $E \gg m$, the chiral and helicity eigenstates are equivalent. This implies that for $E \gg m$ only certain helicity combinations contribute to the QED vertex ! This is why previously we found that for two of the four helicity combinations for the muon current were zero

Allowed QED Helicity Combinations

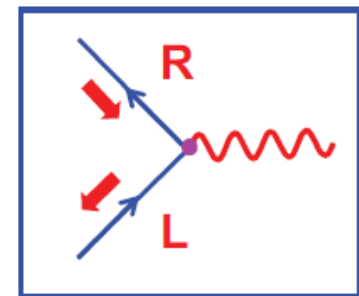
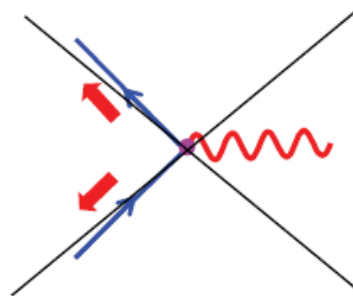
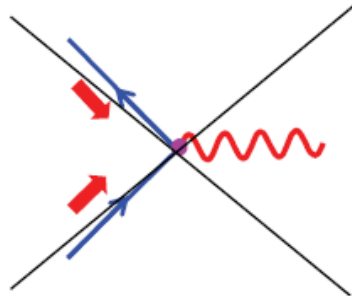
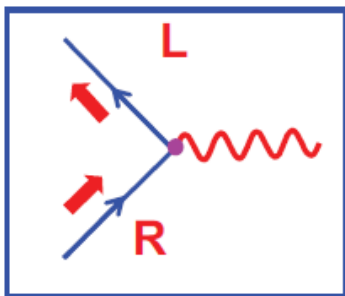
- ♦ In the ultra-relativistic limit the helicity eigenstates \equiv chiral eigenstates
- ♦ In this limit, the only non-zero helicity combinations in QED are:

Scattering:

"Helicity conservation"



Annihilation:



Summary

- ★ In the centre-of-mass frame the $e^+e^- \rightarrow \mu^+\mu^-$ differential cross-section is

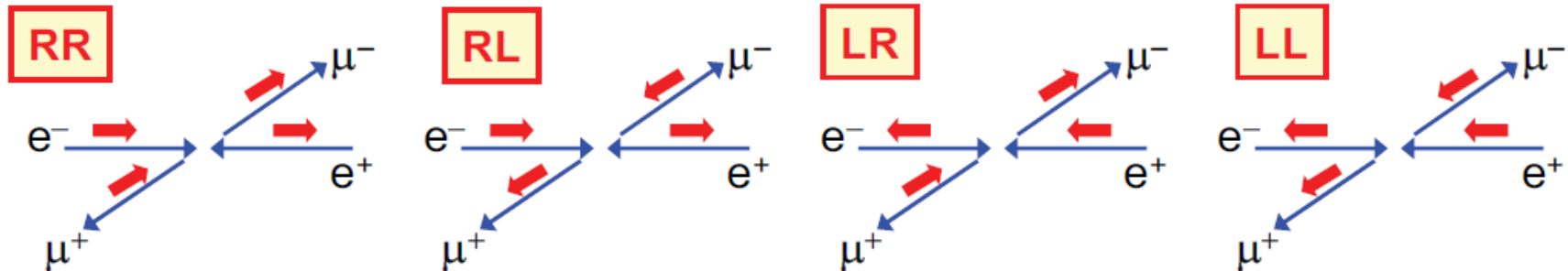
$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s} (1 + \cos^2 \theta)$$

NOTE: neglected masses of the muons, i.e. assumed $E \gg m_\mu$

- ★ In QED only certain combinations of **LEFT-** and **RIGHT-HANDED CHIRAL** states give non-zero matrix elements
- ★ **CHIRAL** states defined by chiral projection operators

$$P_R = \frac{1}{2}(1 + \gamma^5); \quad P_L = \frac{1}{2}(1 - \gamma^5)$$

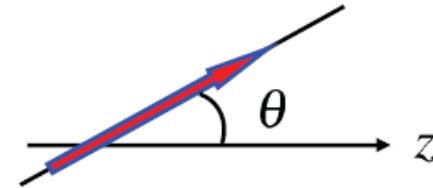
- ★ In limit $E \gg m$ the chiral eigenstates correspond to the **HELICITY** eigenstates and only certain **HELICITY** combinations give non-zero matrix elements



Appendix: Spin 1 Rotation Matrices

- Consider the spin-1 state with spin +1 along the axis defined by unit vector

$$\vec{n} = (\sin \theta, 0, \cos \theta)$$



- Spin state is an eigenstate of $\vec{n} \cdot \vec{S}$ with eigenvalue +1

$$(\vec{n} \cdot \vec{S})|\psi\rangle = +1|\psi\rangle \quad (\text{A1})$$

- Express in terms of linear combination of spin 1 states which are eigenstates of S_z

$$|\psi\rangle = \alpha|1, 1\rangle + \beta|1, 0\rangle + \gamma|1, -1\rangle$$

with

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

- (A1) becomes

$$(\sin \theta S_x + \cos \theta S_z)(\alpha|1, 1\rangle + \beta|1, 0\rangle + \gamma|1, -1\rangle) = \alpha|1, 1\rangle + \beta|1, 0\rangle + \gamma|1, -1\rangle \quad (\text{A2})$$

- Write S_x in terms of ladder operators $S_x = \frac{1}{2}(S_+ + S_-)$

where $S_+|1, 1\rangle = 0$ $S_+|1, 0\rangle = \sqrt{2}|1, 1\rangle$ $S_+|1, -1\rangle = \sqrt{2}|1, 0\rangle$

$S_-|1, 1\rangle = \sqrt{2}|1, 0\rangle$ $S_-|1, 0\rangle = \sqrt{2}|1, -1\rangle$ $S_-|1, -1\rangle = 0$

- from which we find

$$S_x|1, 1\rangle = \frac{1}{\sqrt{2}}|1, 0\rangle$$

$$S_x|1, 0\rangle = \frac{1}{\sqrt{2}}(|1, 1\rangle + |1, -1\rangle)$$

$$S_x|1, -1\rangle = \frac{1}{\sqrt{2}}|1, 0\rangle$$

- (A2) becomes

$$\sin \theta \left[\frac{\alpha}{\sqrt{2}}|1, 0\rangle + \frac{\beta}{\sqrt{2}}|1, -1\rangle + \frac{\beta}{\sqrt{2}}|1, 1\rangle + \frac{\gamma}{\sqrt{2}}|1, 0\rangle \right] + \alpha \cos \theta |1, 1\rangle - \gamma \cos \theta |1, -1\rangle = \alpha |1, 1\rangle + \beta |1, 0\rangle + \gamma |1, -1\rangle$$

- which gives

$$\left. \begin{aligned} \beta \frac{\sin \theta}{\sqrt{2}} + \alpha \cos \theta &= \alpha \\ (\alpha + \gamma) \frac{\sin \theta}{\sqrt{2}} &= \beta \\ \beta \frac{\sin \theta}{\sqrt{2}} - \gamma \cos \theta &= \gamma \end{aligned} \right\}$$

- using $\alpha^2 + \beta^2 + \gamma^2 = 1$ the above equations yield

$$\alpha = \frac{1}{\sqrt{2}}(1 + \cos \theta) \quad \beta = \frac{1}{\sqrt{2}} \sin \theta \quad \gamma = \frac{1}{\sqrt{2}}(1 - \cos \theta)$$

- hence

$$\psi = \frac{1}{2}(1 - \cos \theta)|1, -1\rangle + \frac{1}{\sqrt{2}} \sin \theta |1, 0\rangle + \frac{1}{2}(1 + \cos \theta)|1, +1\rangle$$

- The coefficients α, β, γ are examples of what are known as quantum mechanical **rotation matrices**. They express how an angular momentum eigenstate in a particular direction is expressed in terms of the eigenstates defined in a different direction

$$d_{m',m}^j(\theta)$$

- For spin-1 ($j = 1$) we have just shown that

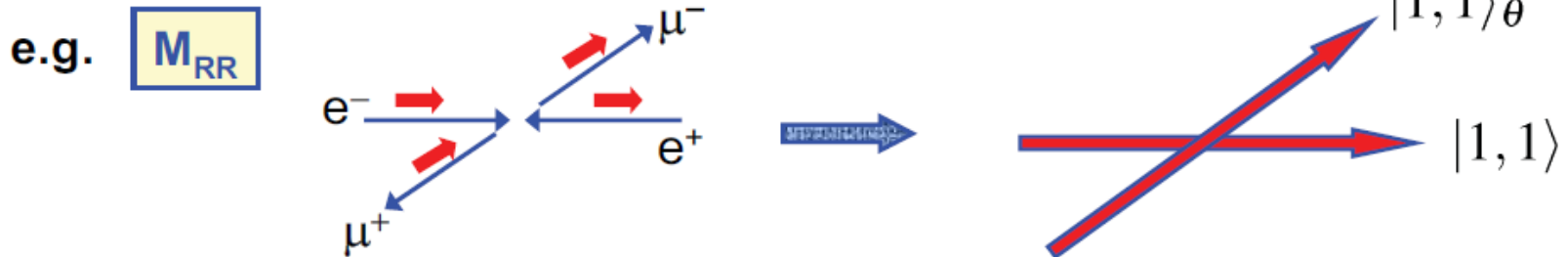
$$d_{1,1}^1(\theta) = \frac{1}{2}(1 + \cos \theta) \quad d_{0,1}^1(\theta) = \frac{1}{\sqrt{2}} \sin \theta \quad d_{-1,1}^1(\theta) = \frac{1}{2}(1 - \cos \theta)$$

- For spin-1/2 it is straightforward to show

$$d_{\frac{1}{2},\frac{1}{2}}^{\frac{1}{2}}(\theta) = \cos \frac{\theta}{2} \quad d_{-\frac{1}{2},\frac{1}{2}}^{\frac{1}{2}}(\theta) = \sin \frac{\theta}{2}$$

Spin considerations ($E \gg m$)

- ★ The angular dependence of the QED electron-positron matrix elements can be understood in terms of angular momentum
- Because of the allowed helicity states, the electron and positron interact in a spin state with $S_z = \pm 1$, i.e. in a total spin 1 state aligned along the z axis: $|1, +1\rangle$ or $|1, -1\rangle$
- Similarly the muon and anti-muon are produced in a total spin 1 state aligned along an axis with polar angle θ



- Hence $M_{RR} \propto \langle \psi | 1, 1 \rangle$ where ψ corresponds to the spin state, $|1, 1\rangle_\theta$, of the muon pair.
- To evaluate this need to express $|1, 1\rangle_\theta$ in terms of eigenstates of S_z
- In the appendix it is shown that:

$$|1, 1\rangle_\theta = \frac{1}{2}(1 - \cos \theta)|1, -1\rangle + \frac{1}{\sqrt{2}} \sin \theta |1, 0\rangle + \frac{1}{2}(1 + \cos \theta)|1, +1\rangle$$

- Using the wave-function for a spin 1 state along an axis at angle θ

$$\psi = |1, 1\rangle_{\theta} = \frac{1}{2}(1 - \cos \theta)|1, -1\rangle + \frac{1}{\sqrt{2}} \sin \theta |1, 0\rangle + \frac{1}{2}(1 + \cos \theta)|1, +1\rangle$$

can immediately understand the angular dependence

