Elementary Particle Physics: theory and experiments

Electron-proton elastic scattering

Follow the course/slides from M. A. Thomson lectures at Cambridge University

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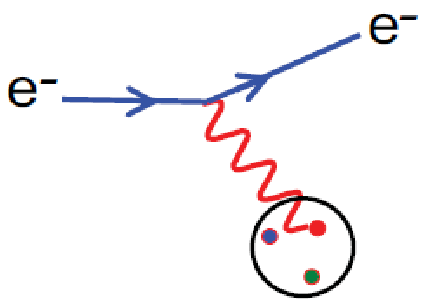
Electron-Proton Scattering

- In this handout aiming towards a study of electron-proton scattering as a probe of the structure of the proton
- Two main topics:
  - $e^-p \to e^-p$ elastic scattering
  - $e^-p \to e^-X$ deep inelastic scattering

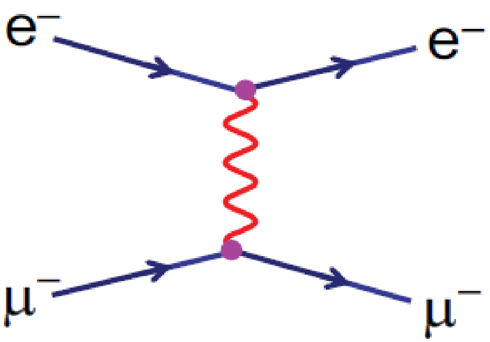


- But first consider scattering from a point-like particle e.g.

  $e^-\mu^- \to e^-\mu^-$

  i.e. the QED part of

  $(e^-q \to e^-q)$



- Two ways to proceed:
  - perform QED calculation from scratch

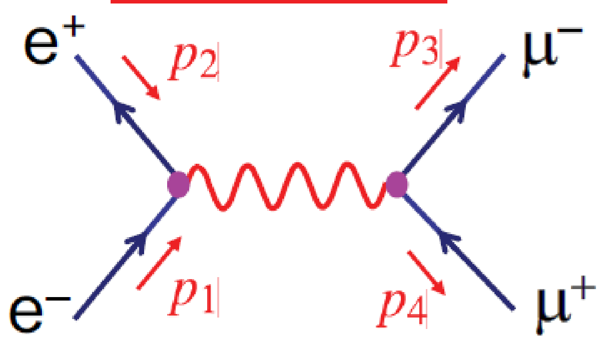
$$\langle|M_{fi}|^2\rangle = \frac{8e^4}{(p_1 - p_3)^4}\left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3)\right] \quad (1)$$

  - take results from $e^+e^- \to \mu^+\mu^-$ and use **"Crossing Symmetry"** to obtain the matrix element for $e^-\mu^- \to e^-\mu^-$

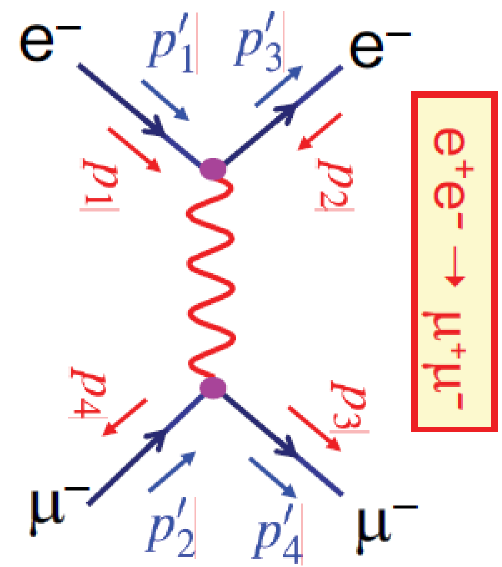
Crossing Symmetry

★ Having derived the Lorentz invariant matrix element for $e^+e^- \rightarrow \mu^+\mu^-$ "rotate" the diagram to correspond to $e^-\mu^- \rightarrow e^-\mu^-$ and apply the principle of crossing symmetry to write down the matrix element !

$e^+e^- \rightarrow \mu^+\mu^-$



$e^-\mu^- \rightarrow e^-\mu^-$



★ The transformation:

$$p_1 \rightarrow p_1'; \; p_2 \rightarrow -p_3'; \; p_3 \rightarrow p_4'; \; p_4 \rightarrow -p_2'$$

Changes the spin averaged matrix element for

$e^- e^+ \rightarrow \mu^- \mu^+$ ($p_1 \; p_2 \quad p_3 \; p_4$) ➡ $e^- \mu^- \rightarrow e^- \mu^-$ ($p_1' \; p_2' \quad p_3' \; p_4'$)

- Take ME for $e^+e^- \to \mu^+\mu^-$ (and apply crossing symmetry:

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1.p_3)^2 + (p_1.p_4)^2}{(p_1.p_2)^2} \quad \Rightarrow \quad \boxed{\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p'_1.p'_4)^2 + (p'_1.p'_2)^2}{(p'_1.p'_3)^2}} \quad (1)$$

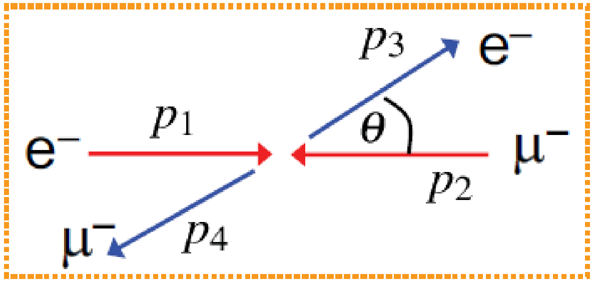
Electron-Proton Scattering

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1.p_4)^2 + (p_1.p_2)^2}{(p_1.p_3)^2} \quad \textbf{(2)}$$

$$\equiv 2e^4 \left(\frac{s^2 + u^2}{t^2} \right)$$

- **Work in the C.o.M:**

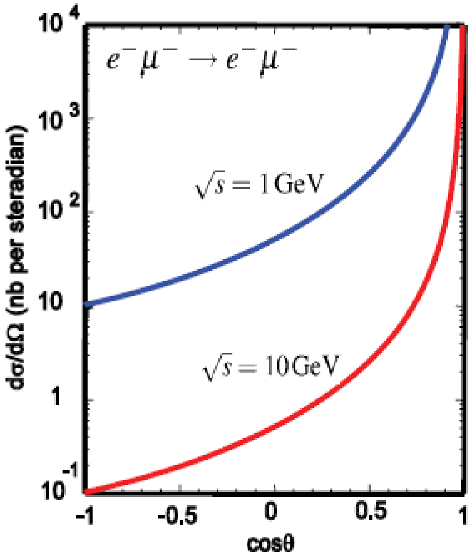
$$p_1 = (E,0,0,E) \qquad p_2 = (E,0,0,-E)$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta)$$
$$p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$



giving $p_1.p_2 = 2E^2; \quad p_1.p_3 = E^2(1-\cos\theta); \quad p_1.p_4 = E^2(1+\cos\theta)$

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{E^4(1+\cos\theta)^2 + 4E^4}{E^4(1-\cos\theta)^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\langle |M_{fi}|^2 \rangle = \frac{e^4}{8\pi^2 s} \frac{\left[1 + \frac{1}{4}(1+\cos\theta)^2\right]}{(1-\cos\theta)^2}$$



- **The denominator arises from the propagator** $-ig_{\mu\nu}/q^2$

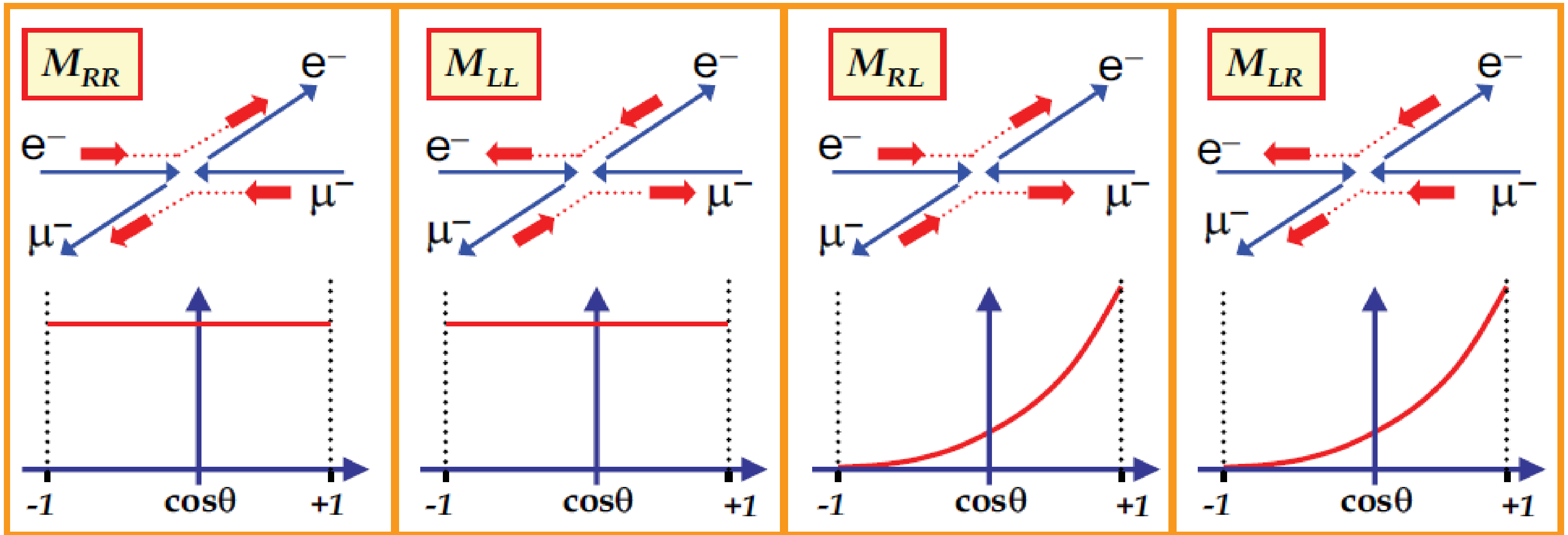
  here $q^2 = (p_1 - p_3)^2 = E^2(1-\cos\theta)$

  as $q^2 \to 0$ **the cross section tends to infinity.**

- What about the angular dependence of the numerator ?

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{8\pi^2 s}\frac{[1+\frac{1}{4}(1+\cos\theta)^2]}{(1-\cos\theta)^2}$$

- The factor $1+\frac{1}{4}(1+\cos\theta)^2$ reflects helicity (really chiral) structure of QED
- Of the 16 possible helicity combinations only 4 are non-zero:



$S_z = 0$

$$\frac{d\sigma}{d\Omega} \propto 1$$

i.e. no preferred polar angle

$S_z = +1$ $S_z = -1$

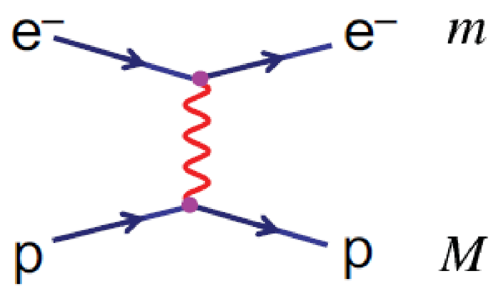
$$\frac{d\sigma}{d\Omega} \propto \frac{1}{4}(1+\cos\theta)^2$$

spin 1 rotation again

- The cross section calculated above is appropriate for the scattering of two spin half Dirac (i.e. point-like) particles in the ultra-relativistic limit (where both electron and muon masses can be neglected). In this case

$$\langle|M_{fi}|^2\rangle = 2e^4 \frac{(p_1.p_4)^2 + (p_1.p_2)^2}{(p_1.p_3)^2}$$

- We will use this again in the discussion of "Deep Inelastic Scattering" of electrons from the quarks within a proton
- Before doing so we will consider the scattering of electrons from the composite proton - i.e. how do we know the proton isn't fundamental "point-like" particle ?
- In this discussion we will not be able to use the relativistic limit and require the general expression for the matrix element

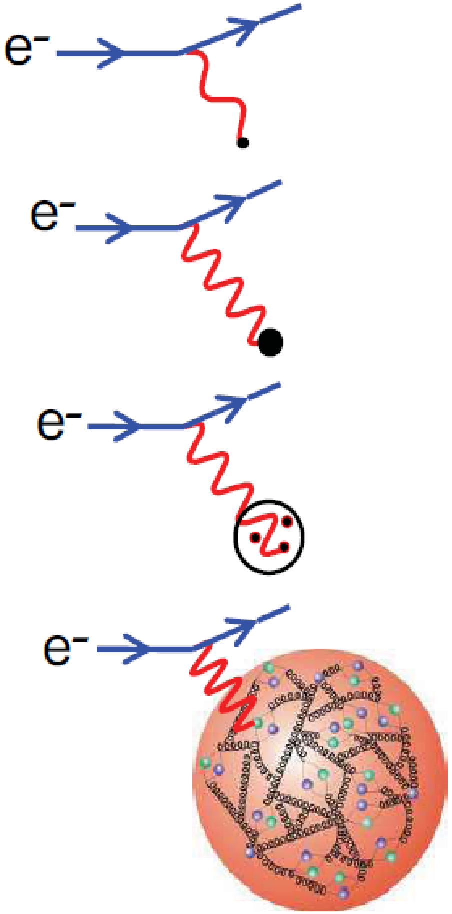


$$\langle|M_{fi}|^2\rangle = \frac{8e^4}{(p_1 - p_3)^4}\left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 - (p_1.p_4)m^2 + 2m^2M^2\right] \quad (3)$$

Probing the structure of the proton

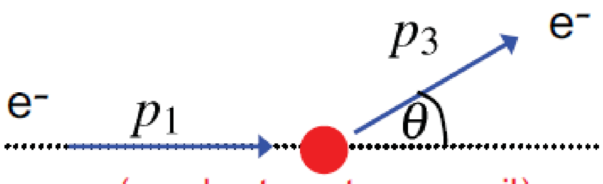
★ In $e^-p \to e^-p$ scattering the nature of the interaction of the virtual photon with the proton depends strongly on wavelength

- At very low electron energies $\lambda \gg r_p$: the scattering is equivalent to that from a "point-like" spin-less object
- At low electron energies $\lambda \sim r_p$: the scattering is equivalent to that from a extended charged object
- At high electron energies $\lambda < r_p$: the wavelength is sufficiently short to resolve sub-structure. Scattering from constituent quarks
- At very high electron energies $\lambda \ll r_p$: the proton appears to be a sea of quarks and gluons.



Rutherford Scattering Revisited

★ Rutherford scattering is the low energy limit where the recoil of the proton can be neglected and the electron is non-relativistic



- Start from RH and LH Helicity particle spinors

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix}$$

$$N = \sqrt{E+m}; \qquad s = \sin(\theta/2); \quad c = \cos(\theta/2)$$

- Now write in terms of: $\alpha = \dfrac{|\vec{p}|}{E+m_e}$

Non-relativistic limit: $\alpha \to 0$

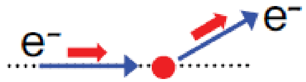
Ultra-relativistic limit: $\alpha \to 1$

$$\Rightarrow \quad u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \alpha c \\ \alpha e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \alpha s \\ -\alpha e^{i\phi}c \end{pmatrix}$$

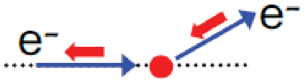
and the possible initial and final state electron spinors are:

$$u_\uparrow(p_1) = N_e\begin{pmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{pmatrix} \quad u_\downarrow(p_1) = N_e\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\alpha \end{pmatrix} \quad u_\uparrow(p_3) = N_e\begin{pmatrix} c \\ s \\ \alpha c \\ \alpha s \end{pmatrix} \quad u_\downarrow(p_3) = N_e\begin{pmatrix} -s \\ c \\ \alpha s \\ -\alpha c \end{pmatrix}$$

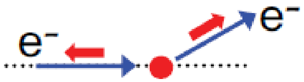
- Consider all four possible electron currents, i.e. Helicities **R→R, L→L, L→R, R→L**



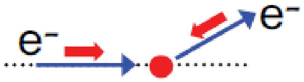
$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \quad (4)$$



$$\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \quad (5)$$



$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(1-\alpha^2)s, 0, 0, 0\right] \quad (6)$$



$$\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(\alpha^2-1)s, 0, 0, 0\right] \quad (7)$$

- In the relativistic limit ($\alpha = 1$), i.e. $E \gg m$

  (6) and (7) are identically zero; only R→R and L→L combinations non-zero

- In the non-relativistic limit, $|\vec{p}| \ll E$ we have $\alpha = 0$

$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = \bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (2m_e)\left[c, 0, 0, 0\right]$$

$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = -\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (2m_e)\left[s, 0, 0, 0\right]$$

All four electron helicity combinations have non-zero Matrix Element

i.e. Helicity eigenstates ≠ Chirality eigenstates

- The initial and final state **proton** spinors (assuming no recoil) are:

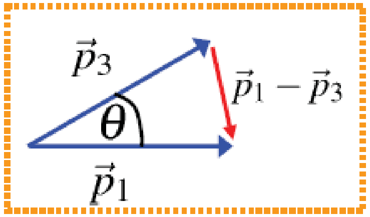
$$u_\uparrow(0) = \sqrt{2M_p}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad u_\downarrow(0) = \sqrt{2M_p}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Solutions of Dirac equation for a particle at rest

giving the proton currents:

$$j_{p\uparrow\uparrow} = j_{p\downarrow\downarrow} = 2M_p\,(1,0,0,0)$$
$$j_{p\uparrow\downarrow} = j_{p\downarrow\uparrow} = 0$$

- The spin-averaged ME summing over the 8 allowed helicity states



$$\langle|M_{fi}^2|\rangle = \frac{1}{4}\frac{e^4}{q^4}(16M_p^2 m_e^2)\underline{(4c^2+4s^2)} = \frac{16M_p^2 m_e^2 e^4}{q^4}$$

where $q^2 = (p_1 - p_3)^2 = (0, \vec{p}_1 - \vec{p}_3)^2 = -4|\vec{p}|^2\sin^2(\theta/2)$

$$\langle|M_{fi}^2|\rangle = \frac{M_p^2 m_e^2 e^4}{|\vec{p}|^4\sin^4(\theta/2)}$$

Note: in this limit all angular dependence is in the propagator

- The formula for the differential cross-section in the lab. frame was derived in handout 1:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2 \qquad (8)$$

- Here the electron is non-relativistic so $E \sim m_e \ll M_p$ and we can neglect E_1 in the denominator of equation (8)

$$\rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 M_p^2}|M_{fi}|^2 = \frac{m_e^2 e^4}{64\pi^2 |\vec{p}|^4 \sin^4(\theta/2)}$$

- Writing $e^2 = 4\pi\alpha$ and the kinetic energy of the electron as $E_K = p^2/2m_e$

$$\rightarrow \quad \left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{\alpha^2}{16 E_K^2 \sin^4 \theta/2} \qquad (9)$$

★ This is the normal expression for the Rutherford cross section. It could have been derived by considering the scattering of a non-relativistic particle in the static Coulomb potential of the proton $V(\vec{r})$, without any consideration of the interaction due to the intrinsic magnetic moments of the electron or proton. From this we can conclude, that in this non-relativistic limit only the interaction between the electric charges of the particles matters.

The Mott Scattering Cross Section

- For Rutherford scattering we are in the limit where the target recoil is neglected and the scattered particle is non-relativistic $E_K \ll m_e$
- The limit where the target recoil is neglected and the scattered particle is relativistic (i.e. just neglect the electron mass) is called Mott Scattering
- In this limit the electron currents, equations (4) and (6), become:

$$\overline{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\uparrow}(p_1) = 2E\,[c, s, -is, c] \qquad \overline{u}_{\uparrow}(p_3)\gamma^{\mu}u_{\downarrow}(p_1) = E\,[0,0,0,0]$$

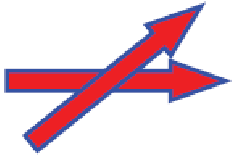
Relativistic ➡ Electron "helicity conserved"

- It is then straightforward to obtain the result:

$$\rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2 \sin^4\theta/2} \cos^2\frac{\theta}{2} \qquad (10)$$

Rutherford formula with $E_K = E \quad (E \gg m_e)$

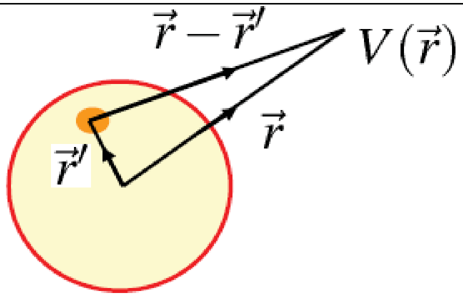
Overlap between initial/final state electron wave-functions. Just QM of spin ½



★ NOTE: we could have derived this expression from scattering of electrons in a static potential from a fixed point in space $V(\vec{r})$.
The interaction is ELECTRIC rather than magnetic (spin-spin) in nature.

★ Still haven't taken into account the charge distribution of the proton.....

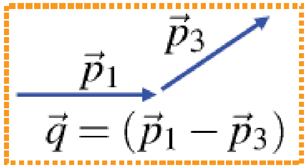
Form Factors



- Consider the scattering of an electron in the static potential due to an extended charge distribution.
- The potential at $\vec{r}$ from the centre is given by:

$$V(\vec{r}) = \int \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} d^3\vec{r}' \quad \text{with} \quad \int \rho(\vec{r})\mathrm{d}^3\vec{r} = 1$$

- In first order perturbation theory the matrix element is given by:



$$M_{fi} = \langle \psi_f | V(\vec{r}) | \psi_i \rangle = \int e^{-i\vec{p}_3.\vec{r}} V(\vec{r}) e^{i\vec{p}_1.\vec{r}} \mathrm{d}^3\vec{r}$$

$$= \int\int e^{i\vec{q}.\vec{r}} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} \mathrm{d}^3\vec{r}'\mathrm{d}^3\vec{r} = \int\int e^{i\vec{q}.(\vec{r}-\vec{r}')} e^{i\vec{q}.\vec{r}'} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} \mathrm{d}^3\vec{r}'\mathrm{d}^3\vec{r}$$

- Fix $\vec{r}'$ and integrate over $\mathrm{d}^3\vec{r}$ with substitution $\vec{R} = \vec{r} - \vec{r}'$

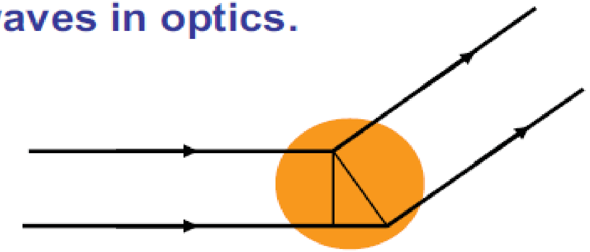
$$M_{fi} = \int e^{i\vec{q}.\vec{R}} \frac{Q}{4\pi|\vec{R}|} \mathrm{d}^3\vec{R} \int \rho(\vec{r}') e^{i\vec{q}.\vec{r}'} \mathrm{d}^3\vec{r}' = (M_{fi})_{point} F(\vec{q}^2)$$

★ The resulting matrix element is equivalent to the matrix element for scattering from a point source multiplied by the form factor

$$F(\vec{q}^2) = \int \rho(\vec{r}) e^{i\vec{q}.\vec{r}} \mathrm{d}^3\vec{r}$$

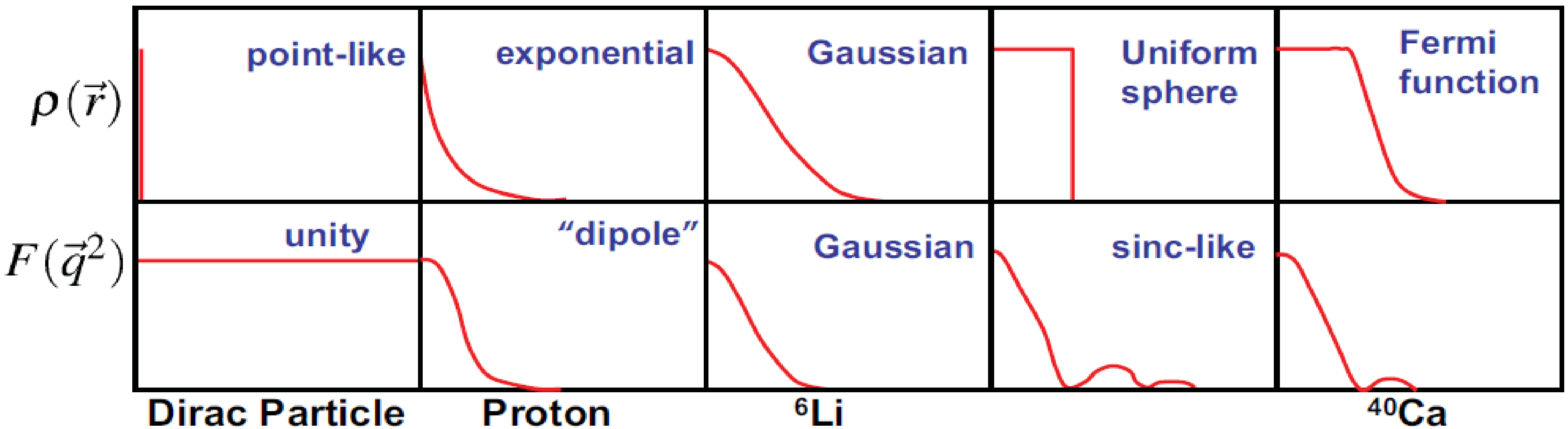
$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \rightarrow \frac{\alpha^2}{4E^2 \sin^4 \theta/2} \cos^2 \frac{\theta}{2} |F(\vec{q}^2)|^2$$

- There is nothing mysterious about form factors – similar to diffraction of plane waves in optics.



- The finite size of the scattering centre introduces a phase difference between plane waves "scattered from different points in space". If wavelength is long compared to size all waves in phase and $F(\vec{q}^2) = 1$

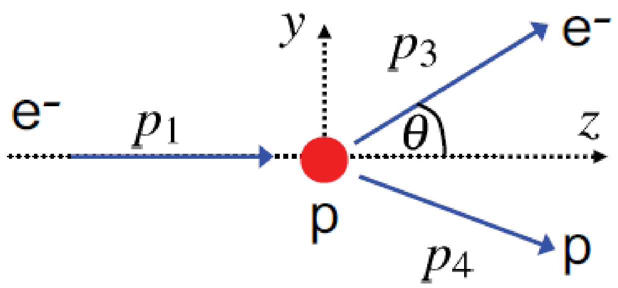
<u>For example:</u>



- NOTE that for a point charge the form factor is unity.

Point-like Electron-Proton Elastic Scattering

- So far have only considered the case we the proton does not recoil...
  For $E_1 \gg m_e$ the general case is



$$p_1 = (E_1, 0, 0, E_1)$$
$$p_2 = (M, 0, 0, 0)$$
$$p_3 = (E_3, 0, E_3 \sin\theta, E_3 \cos\theta)$$
$$p_4 = (E_4, \vec{p}_4)$$

- From Eqn. (2) with $m = m_e = 0$ the matrix element for this process is:

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 \right] \quad (11)$$

- Experimentally observe scattered electron so eliminate p_4
- The scalar products not involving p_4 are:

$$p_1.p_2 = E_1 M \qquad p_1.p_3 = E_1 E_3 (1 - \cos\theta) \qquad p_2.p_3 = E_3 M$$

- From momentum conservation can eliminate p_4 : $\quad p_4 = p_1 + p_2 - p_3$

$$p_3.p_4 = p_3.p_1 + p_3.p_2 - p_3.p_3 = E_1 E_3 (1 - \cos\theta) + E_3 M$$
$$p_1.p_4 = p_1.p_1 + p_1.p_2 - p_1.p_3 = E_1 M - E_1 E_3 (1 - \cos\theta)$$

$$p_1.p_1 = E_1^2 - |\vec{p}_1|^2 = m_e^2 \approx 0 \qquad \text{i.e. neglect } m_e$$

- Substituting these scalar products in Eqn. (11) gives

$$\langle|M_{fi}|^2\rangle = \frac{8e^4}{(p_1-p_3)^4}ME_1E_3\left[(E_1-E_3)(1-\cos\theta)+M(1+\cos\theta)\right]$$

$$= \frac{8e^4}{(p_1-p_3)^4}2ME_1E_3\left[(E_1-E_3)\sin^2(\theta/2)+M\cos^2(\theta/2)\right] \quad (12)$$

- Now obtain expressions for $q^4=(p_1-p_3)^4$ and (E_1-E_3)

$$q^2=(p_1-p_3)^2=p_1^2+p_3^2-2p_1.p_3=-2E_1E_3(1-\cos\theta) \quad (13)$$

$$=-4E_1E_3\sin^2\theta/2 \quad (14)$$

NOTE: $q^2<0$ Space-like

- For (E_1-E_3) start from

$$q.p_2=(p_1-p_3).p_2=M(E_1-E_3)$$

and use

$$q=(p_1-p_3)=(p_4-p_2)$$

$$(q+p_2)^2 = p_4^2$$

$$q^2+p_2^2+2q.p_2 = p_4^2$$

$$q^2+M^2+2q.p_2 = M^2$$

$$\Rightarrow \quad q.p_2 = -q^2/2$$

- Hence the energy transferred to the proton:

$$E_1 - E_3 = -\frac{q^2}{2M} \tag{15}$$

Because q^2 is always negative $E_1 - E_3 > 0$ and the scattered electron is always lower in energy than the incoming electron

- Combining equations (11), (13) and (14):

$$\begin{aligned}\langle|M_{fi}|^2\rangle &= \frac{8e^4}{16E_1^2E_3^2\sin^4\theta/2}2ME_1E_3\left[M\cos^2\theta/2 - \frac{q^2}{2M}\sin^2\theta/2\right] \\ &= \frac{M^2e^4}{E_1E_3\sin^4\theta/2}\left[\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right]\end{aligned}$$

- For $E \gg m_e$ we have (see handout 1)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2|M_{fi}|^2$$

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right) \tag{16}$$

Interpretation

- So far have derived the differential cross-section for $e^-p \to e^-p$ elastic scattering assuming point-like Dirac spin ½ particles. How should we interpret the equation?

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right)$$

- Compare with $$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2\sin^4\theta/2}\cos^2\frac{\theta}{2}$$

the important thing to note about the Mott cross-section is that it is equivalent to scattering of spin ½ electrons in a fixed electro-static potential. Here the term E_3/E_1 is due to the proton recoil.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \underbrace{\frac{q^2}{2M^2}\sin^2\theta/2}\right)$$

- the new term: $\propto \sin^2\frac{\theta}{2}$ ⟷ **Magnetic interaction : due to the spin-spin interaction**

- The above differential cross-section depends on a single parameter. For an electron scattering angle θ, both q^2 and the energy, E_3, are fixed by kinematics

- Equating (13) and (15)

$$-2M(E_1 - E_3) = -2E_1E_3(1-\cos\theta)$$

$$\Rightarrow \quad \frac{E_3}{E_1} = \frac{M}{M + E_1(1-\cos\theta)}$$

- Substituting back into (13):

$$\Rightarrow \quad q^2 = -\frac{2ME_1^2(1-\cos\theta)}{M + E_1(1-\cos\theta)}$$

- e.g. $e^-p \rightarrow e^-p$ at E_{beam} = 529.5 MeV, look at scattered electrons at θ = 75°

For elastic scattering expect:

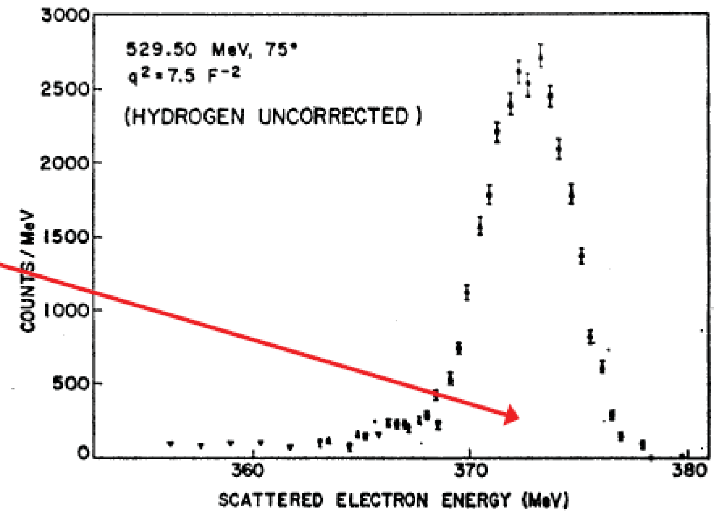
$$E_3 = \frac{ME_1}{M + E_1(1-\cos\theta)}$$

$$E_3 = \frac{938 \times 529}{938 + 529(1-\cos 75^\circ)} = 373 \text{ MeV}$$

The energy identifies the scatter as elastic.
Also know squared four-momentum transfer

$$|q^2| = \frac{2 \times 938 \times 529^2(1-\cos 75^\circ)}{938 + 529(1-\cos 75^\circ)} = 294 \, \text{MeV}^2$$

E.B.Hughes et al., Phys. Rev. 139 (1965) B458



Elastic scattering from a Finite Size Proton

★ **In general** the finite size of the proton can be accounted for by introducing two structure functions. One related to the charge distribution in the proton, $G_E(q^2)$ and the other related to the distribution of the magnetic moment of the proton, $G_M(q^2)$

- It can be shown that equation (16) generalizes to the **ROSENBLUTH FORMULA.**

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} \cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2} \right)$$

with the Lorentz Invariant quantity:

$$\tau = -\frac{q^2}{4M^2} > 0$$

- Unlike our previous discussion of form factors, here the form factors are a function of q^2 rather than $\vec{q}^2$ and cannot simply be considered in terms of the FT of the charge and magnetic moment distributions.

But $q^2 = (E_1 - E_3)^2 - \vec{q}^2$ and from eq (15) obtain

$$\Rightarrow \quad -\vec{q}^2 = q^2 \left[1 - \left(\frac{q}{2M} \right)^2 \right]$$

So for $\frac{q^2}{4M^2} \ll 1$ we have $q^2 \approx -\vec{q}^2$ and $G(q^2) \approx G(\vec{q}^2)$

- Hence in the limit $q^2/4M^2 \ll 1$ we can interpret the structure functions in terms of the Fourier transforms of the charge and magnetic moment distributions

$$G_E(q^2) \approx G_E(\vec{q}^2) = \int e^{i\vec{q}.\vec{r}} \rho(\vec{r}) \mathrm{d}^3\vec{r}$$

$$G_M(q^2) \approx G_M(\vec{q}^2) = \int e^{i\vec{q}.\vec{r}} \mu(\vec{r}) \mathrm{d}^3\vec{r}$$

- Note in deriving the Rosenbluth formula we assumed that the proton was a spin-half Dirac particle, i.e.

$$\vec{\mu} = \frac{e}{M}\vec{S}$$

- However, the experimentally measured value of the proton magnetic moment is larger than expected for a point-like Dirac particle:

$$\vec{\mu} = 2.79\frac{e}{M}\vec{S}$$

So for the proton expect

$$G_E(0) = \int \rho(\vec{r}) \mathrm{d}^3\vec{r} = 1 \qquad G_M(0) = \int \mu(\vec{r}) \mathrm{d}^3\vec{r} = \mu_p = +2.79$$

- Of course the anomalous magnetic moment of the proton is already evidence that it is not point-like !

Measuring $G_E(q^2)$ and $G_M(q^2)$

- Express the Rosenbluth formula as:

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right)$$

where
$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \frac{E_3}{E_1} \cos^2\frac{\theta}{2}$$

i.e. the Mott cross-section including the proton recoil. It corresponds to scattering from a spin-0 proton.

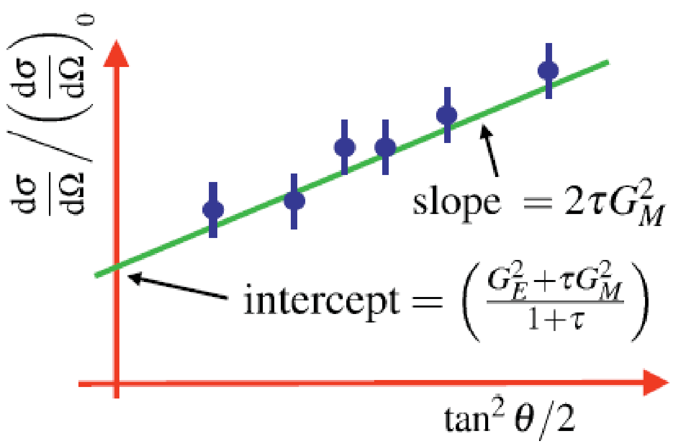
- At very low q^2: $\tau = -q^2/4M^2 \approx 0$

$$\frac{d\sigma}{d\Omega} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0 \approx G_E^2(q^2)$$

- At high q^2: $\tau \gg 1$

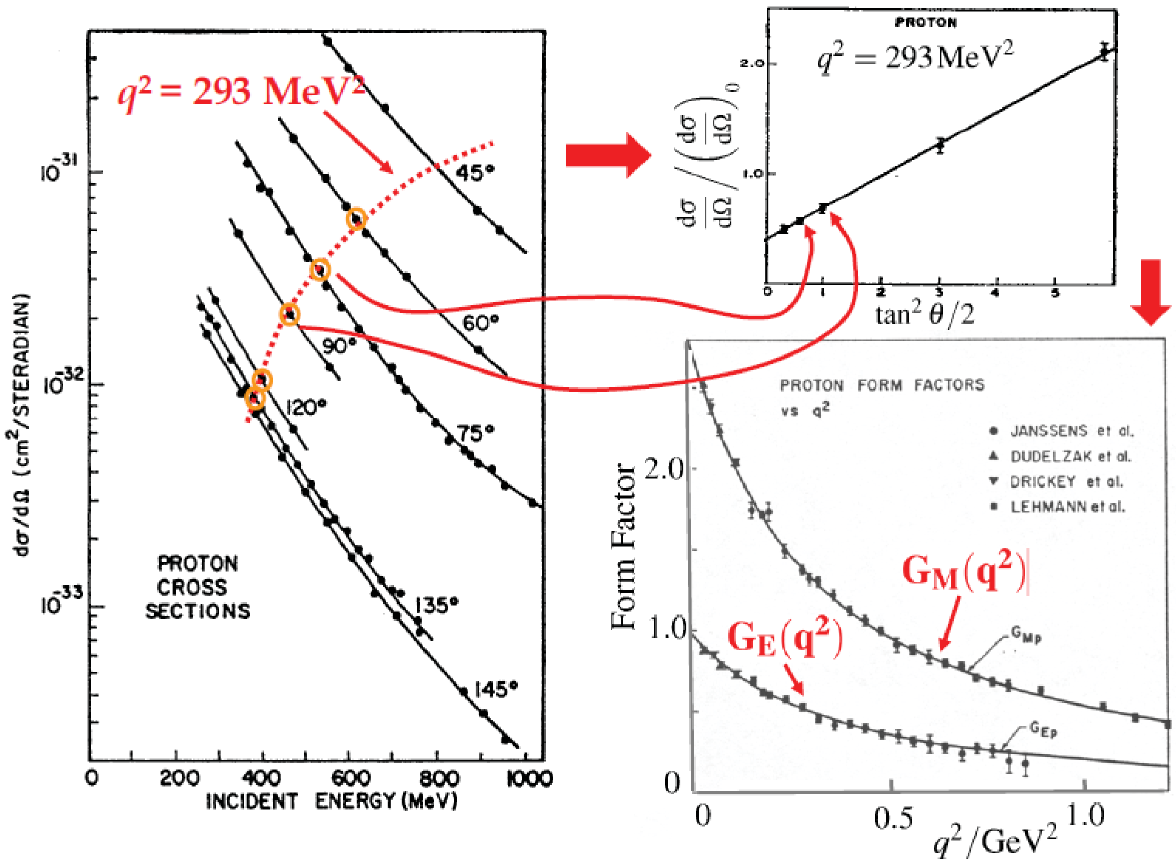
$$\frac{d\sigma}{d\Omega} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0 \approx \left(1 + 2\tau\tan^2\frac{\theta}{2}\right) G_M^2(q^2)$$

- In general we are sensitive to both structure functions! These can be resolved from the angular dependence of the cross section at **FIXED** q^2



- **EXAMPLE:** $e^-p \rightarrow e^-p$ **at** E_{beam} **= 529.5 MeV**
  - Electron beam energies chosen to give certain values of q^2
  - Cross sections measured to 2-3 %

E.B.Hughes et al., Phys. Rev. 139 (1965) B458



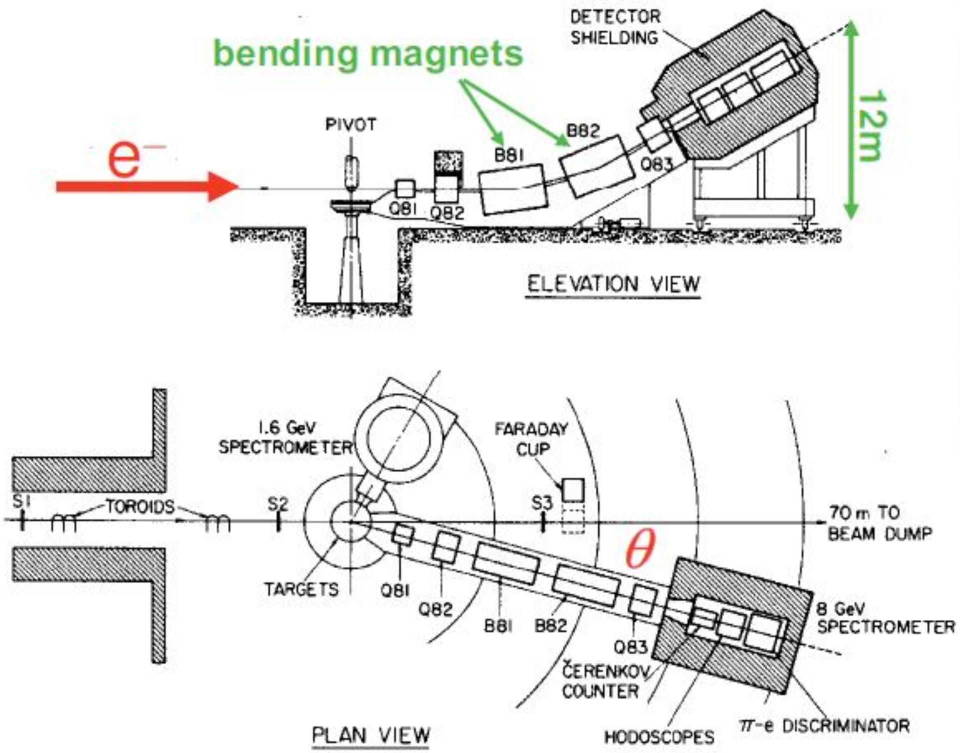
NOTE

Experimentally find $G_M(q^2) = 2.79 G_E(q^2)$, i.e. the electric and magnetic form factors have same distribution

Higher Energy Electron-Proton Scattering

★ Use electron beam from SLAC LINAC: $5 < E_{beam} < 20$ GeV

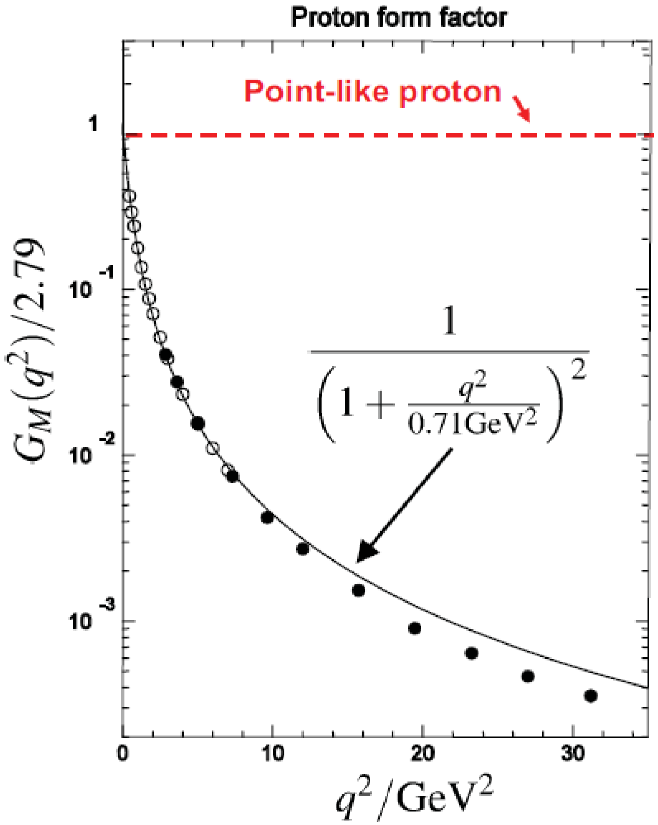
• Detect scattered electrons using the "8 GeV Spectrometer"



P.N.Kirk et al., Phys Rev D8 (1973) 63

High q^2 ➞ Measure $G_M(q^2)$

High q^2 Results



R.C.Walker et al., Phys. Rev. D49 (1994) 5671
A.F.Sill et al., Phys. Rev. D48 (1993) 29

★ Form factor falls rapidly with q^2
- Proton is not point-like
- Good fit to the data with "dipole form":

$$G_E^p(q^2) \approx \frac{G_M^p}{2.79} \approx \frac{1}{(1+q^2/0.71\text{GeV}^2)^2}$$

★ Taking FT find spatial charge and magnetic moment distribution

$$\rho(r) \approx \rho_0 e^{-r/a}$$

with $a \approx 0.24$ fm

- Corresponds to a rms charge radius

$$r_{rms} \approx 0.8 \text{ fm}$$

★ Although suggestive, does not imply proton is composite !

★ **Note:** so far have only considered ELASTIC scattering; Inelastic scattering is the subject of next handout

Summary: Elastic Scattering

★ For elastic scattering of relativistic electrons from a point-like Dirac proton:

$$\frac{d\sigma}{d\Omega} = \underbrace{\frac{\alpha^2}{4E_1^2 \sin^4 \theta/2}}_{\text{Rutherford}} \underbrace{\frac{E_3}{E_1}}_{\text{Proton recoil}} \left(\underbrace{\cos^2 \frac{\theta}{2}}_{\text{Electric/Magnetic scattering}} - \underbrace{\frac{q^2}{2M^2} \sin^2 \frac{\theta}{2}}_{\text{Magnetic term due to spin}} \right)$$

Rutherford

Proton recoil

Electric/ Magnetic scattering

Magnetic term due to spin

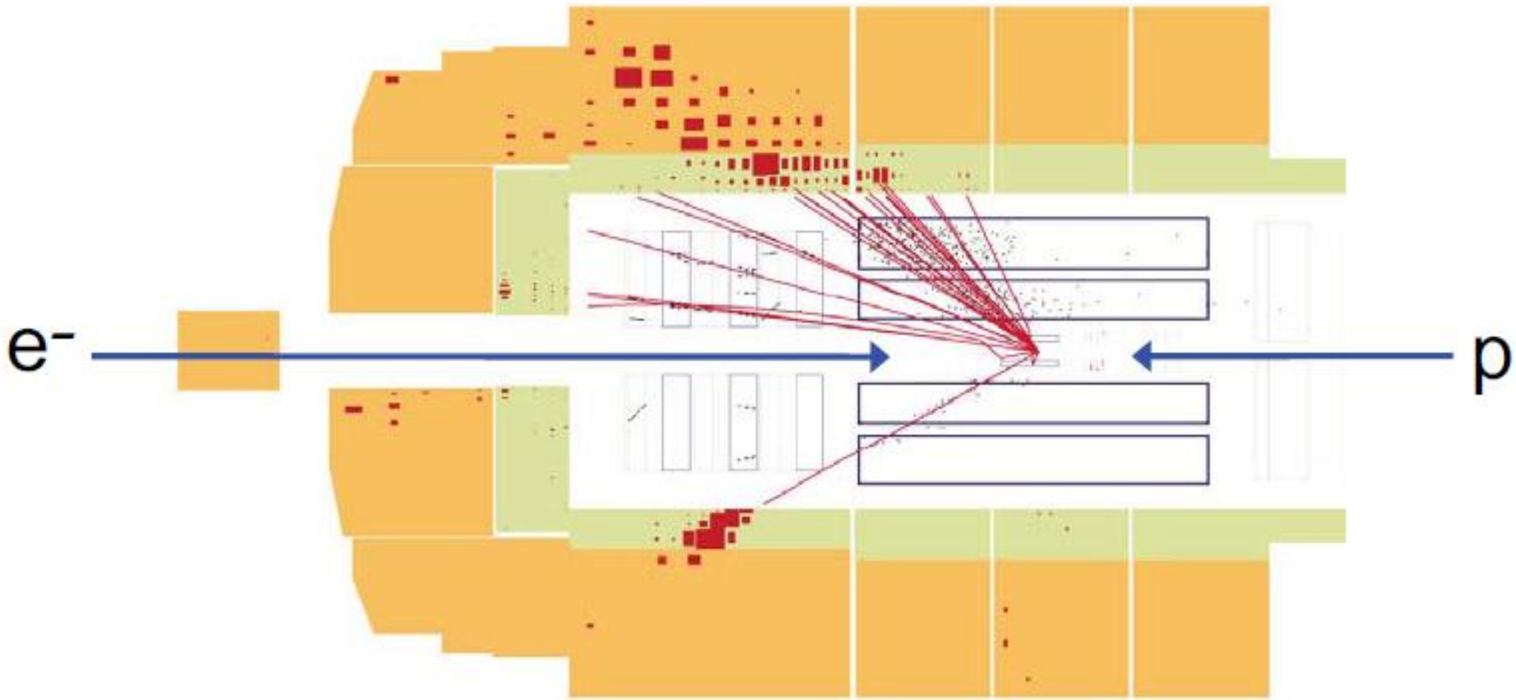
★ For elastic scattering of relativistic electrons from an extended proton:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \theta/2} \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} \cos^2 \frac{\theta}{2} + 2\tau G_M^2 \sin^2 \frac{\theta}{2} \right)$$

Rosenbluth Formula

★ Electron elastic scattering from protons demonstrates that the proton is an extended object with rms charge radius of ~0.8 fm

Deep Inelastic Scattering



e^- p Elastic Scattering at Very High q^2

★ At high q^2 the Rosenbluth expression for elastic scattering becomes

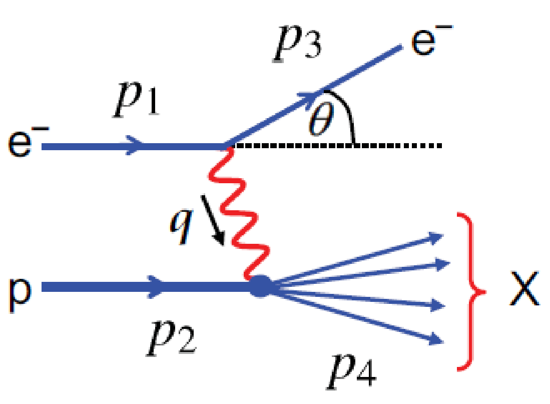
$$\left(\frac{d\sigma}{d\Omega}\right)_{elastic} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\frac{E_3}{E_1}\left(\frac{q^2}{2M^2}G_M^2\sin^2\frac{\theta}{2}\right) \qquad \tau = -\frac{q^2}{4M^2} \gg 1$$

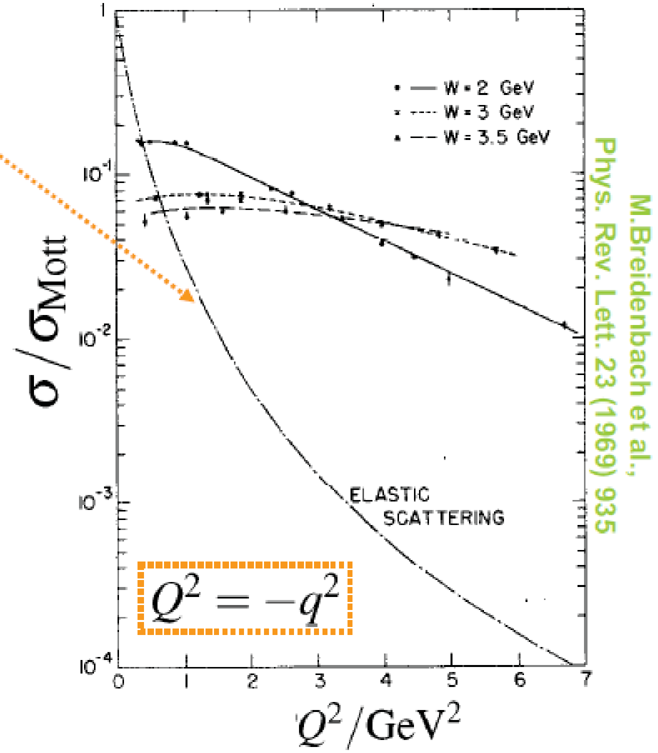
• From e^- p elastic scattering, the proton magnetic form factor is

$$G_M(q^2) \approx \frac{1}{(1+q^2/0.71\mathrm{GeV}^2)^2} \Rightarrow G_M(q^2) \propto q^{-4} \quad \text{at high } q^2$$

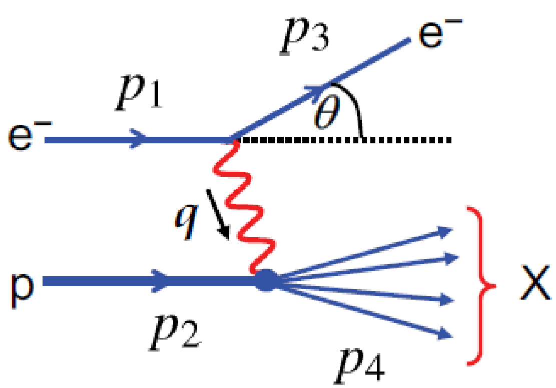
$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{elastic} \propto q^{-6}$$

• Due to the finite proton size, elastic scattering at high q^2 is unlikely and inelastic reactions where the proton breaks up dominate.





Kinematics of Inelastic Scattering



- For inelastic scattering the mass of the final state hadronic system is no longer the proton mass, M
- The final state hadronic system must contain at least one baryon which implies the final state invariant mass $M_X > M$

$$M_X^2 = p_4^2 = (E_4^2 - |\vec{p}_4|^2)$$

★ For inelastic scattering introduce four new kinematic variables: x, y, ν, Q^2

★ **Define:** $x \equiv \frac{Q^2}{2p_2.q}$ **Bjorken x** (Lorentz Invariant)

where $Q^2 \equiv -q^2$ $Q^2 > 0$

- Here $M_X^2 = p_4^2 = (q + p_2)^2 = -Q^2 + 2p_2.q + M^2$

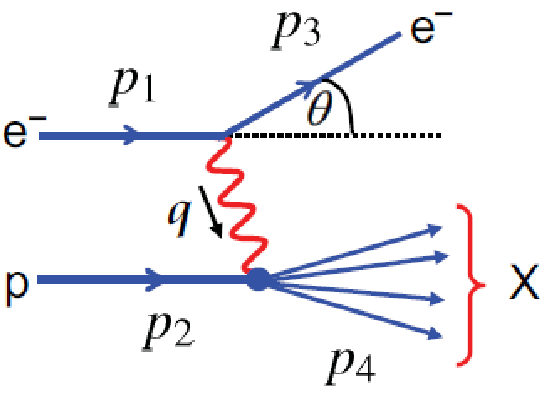
⇨ $Q^2 = 2p_2.q + M^2 - M_X^2$ ➡ $Q^2 \leq 2p_2.q$

Note: in many text books W is often used in place of M_X

hence $0 < x < 1$ inelastic $x = 1$ elastic

Proton intact $M_X = M$

★ **Define:** $y \equiv \dfrac{p_2 \cdot q}{p_2 \cdot p_1}$ (Lorentz Invariant)



- In the Lab. Frame:

$$p_1 = (E_1, 0, 0, E_1) \quad p_2 = (M, 0, 0, 0)$$
$$q = (E_1 - E_3, \vec{p}_1 - \vec{p}_3)$$

$$\rightarrow \quad y = \frac{M(E_1 - E_3)}{ME_1} = 1 - \frac{E_3}{E_1}$$

So y is the fractional energy loss of the incoming particle

$$0 < y < 1$$

- In the C.o.M. Frame (neglecting the electron and proton masses):

$$p_1 = (E, 0, 0, E); \quad p_2 = (E, 0, 0, -E); \quad p_3 = (E, E\sin\theta^*, 0, E\cos\theta^*)$$

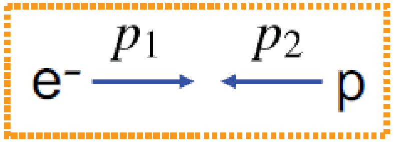
$$\rightarrow \quad y = \tfrac{1}{2}(1 - \cos\theta^*) \qquad \text{for } E \gg M$$

★ **Finally Define:** $\nu \equiv \dfrac{p_2 \cdot q}{M}$ (Lorentz Invariant)

- In the Lab. Frame: $\nu = E_1 - E_3$

  ν is the energy lost by the incoming particle

Relationships between Kinematic Variables



- Can rewrite the new kinematic variables in terms of the squared centre-of-mass energy, s, for the electron-proton collision

$$s = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1.p_2 = 2p_1.p_2 + M^2 + \cancel{m_e^2}$$

$$2p_1.p_2 = s - M^2$$

Neglect mass of electron

- For a fixed centre-of-mass energy, it can then be shown that the four kinematic variables

$$Q^2 \equiv -q^2 \qquad x \equiv \frac{Q^2}{2p_2.q} \qquad y \equiv \frac{p_2.q}{p_2.p_1} \qquad \nu \equiv \frac{p_2.q}{M}$$

are not independent.

- i.e. the scaling variables x and y can be expressed as

$$x = \frac{Q^2}{2M\nu} \qquad y = \frac{2M}{s - M^2}\nu$$

Note the simple relationship between y and ν

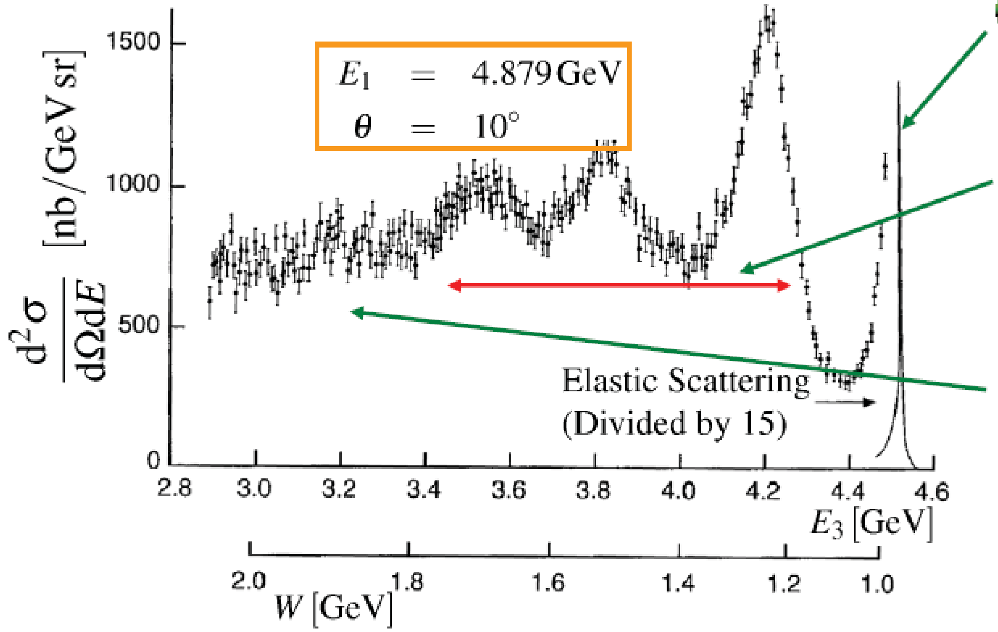
and $$xy = \frac{Q^2}{s - M^2} \quad \Rightarrow \quad Q^2 = (s - M^2)xy$$

- For a fixed centre of mass energy, the interaction kinematics are completely defined by any <u>two</u> of the above kinematic variables (except y and ν)
- For elastic scattering $(x = 1)$ there is only one independent variable. As we saw previously if you measure electron scattering angle know everything else.

Inelastic Scattering

Example: Scattering of 4.879 GeV electrons from protons at rest

- Place detector at 10° to beam and measure the energies of scattered e⁻
- Kinematics fully determined from the electron energy and angle !
- e.g. for this energy and angle : the invariant mass of the final state hadronic system $W^2 = M_X^2 = 10.06 - 2.03E_3$ (try and show this)



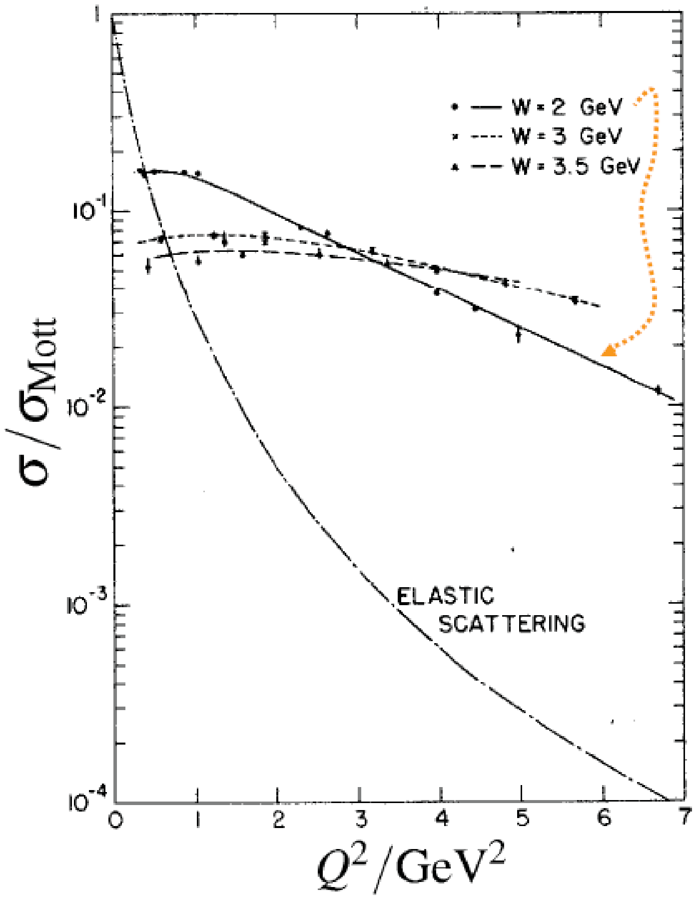
- **Elastic Scattering**
  proton remains intact
  $W = M$
- **Inelastic Scattering**
  produce "excited states" of proton e.g. $\Delta^+(1232)$
  $W = M_\Delta$
- **Deep Inelastic Scattering**
  proton breaks up resulting in a many particle final state

DIS = large W

Inelastic Cross Section

- Repeat experiments at different angles/beam energies and determine q^2 dependence of elastic and inelastic cross-sections

M.Breidenbach et al., Phys. Rev. Lett. 23 (1969) 935



- Elastic scattering falls of rapidly with q^2 due to the proton not being point-like (i.e. form factors)
- Inelastic scattering cross sections only weakly dependent on q^2
- Deep Inelastic scattering cross sections almost independent of q^2 !

i.e. "Form factor" → 1

→ **Scattering from point-like objects within the proton !**

Elastic → Inelastic Scattering

★ **Recall: Elastic scattering**

- Only one independent variable. In Lab. frame express differential cross section in terms of the electron scattering angle (Rosenbluth formula)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\left(\frac{G_E^2+\tau G_M^2}{(1+\tau)}\cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2}\right) \qquad \tau = \frac{Q^2}{4M^2}$$

Note: here the energy of the scattered electron is determined by the angle.

- In terms of the Lorentz invariant kinematic variables can express this differential cross section in terms of Q^2

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\frac{G_E^2+\tau G_M^2}{(1+\tau)}\left(1-y-\frac{M^2y^2}{Q^2}\right) + \frac{1}{2}y^2 G_M^2\right]$$

which can be written as:

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[f_2(Q^2)\left(1-y-\frac{M^2y^2}{Q^2}\right) + \frac{1}{2}y^2 f_1(Q^2)\right]$$

★ **Inelastic scattering**

- For Deep Inelastic Scattering have two independent variables. Therefore need a double differential cross section

Deep Inelastic Scattering

★ It can be shown that **the most general Lorentz Invariant expression** for $e^-p \rightarrow e^-X$ inelastic scattering (via a single exchanged photon is):

$$\frac{d^2\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{M^2y^2}{Q^2}\right)\frac{F_2(x,Q^2)}{x} + y^2F_1(x,Q^2)\right] \quad (1)$$

INELASTIC SCATTERING

c.f.

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{M^2y^2}{Q^2}\right)f_2(Q^2) + \frac{1}{2}y^2f_1(Q^2)\right]$$

ELASTIC SCATTERING

We will soon see how this connects to the quark model of the proton

- **NOTE:** The form factors have been replaced by the **STRUCTURE FUNCTIONS**

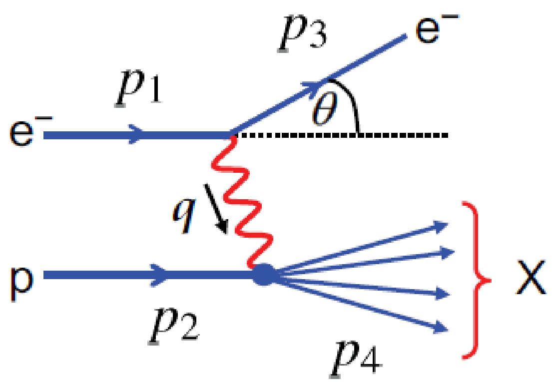
$$F_1(x,Q^2) \quad \text{and} \quad F_2(x,Q^2)$$

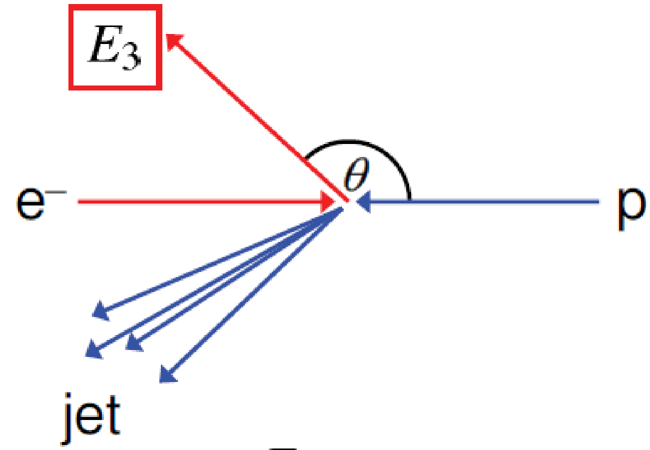
which are a function of x and Q^2: can not be interpreted as the Fourier transforms of the charge and magnetic moment distributions. We shall soon see that they describe the momentum distribution of the quarks within the proton

★ In the limit of high energy (or more correctly $Q^2 \gg M^2y^2$) eqn. (1) becomes:

$$\frac{d^2\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2F_1(x,Q^2)\right] \quad (2)$$

- In the Lab. frame it is convenient to express the cross section in terms of the angle, θ, and energy, E_3, of the scattered electron – **experimentally well measured.**





$$Q^2 = 4E_1E_3\sin^2\theta/2; \quad x = \frac{Q^2}{2M(E_1 - E_3)}; \quad y = 1 - \frac{E_3}{E_1}; \quad \nu = E_1 - E_3$$

- In the Lab. frame, Equation (2) becomes:

$$\frac{d^2\sigma}{dE_3 d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\left[\frac{1}{\nu}F_2(x,Q^2)\cos^2\frac{\theta}{2} + \frac{2}{M}F_1(x,Q^2)\sin^2\frac{\theta}{2}\right] \quad (3)$$

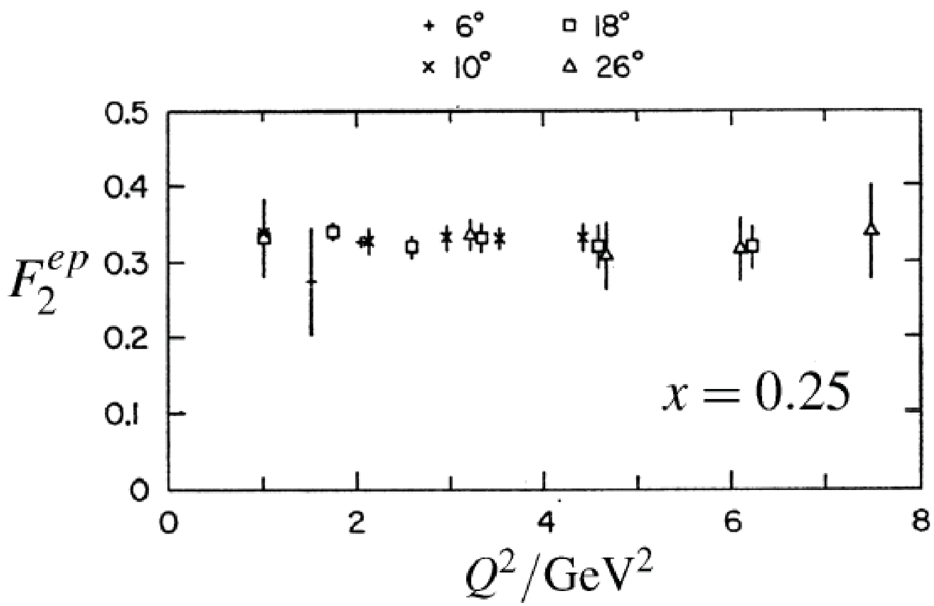
Electromagnetic Structure Function

Pure Magnetic Structure Function

Measuring the Structure Functions

★ To determine $F_1(x,Q^2)$ and $F_2(x,Q^2)$ for a given x and Q^2 need measurements of the differential cross section at several different scattering angles and incoming electron beam energies

Example: electron-proton scattering F_2 vs. Q^2 at fixed x



J.T.Friedman + H.W.Kendall, Ann. Rev. Nucl. Sci. 22 (1972) 203

♦ Experimentally it is observed that both F_1 and F_2 are (almost) independent of Q^2

Bjorken Scaling and the Calla-Gross Relation

★ The near (see later) independence of the structure functions on Q^2 is known as Bjorken Scaling, i.e.

$$F_1(x,Q^2) \rightarrow F_1(x) \qquad F_2(x,Q^2) \rightarrow F_2(x)$$

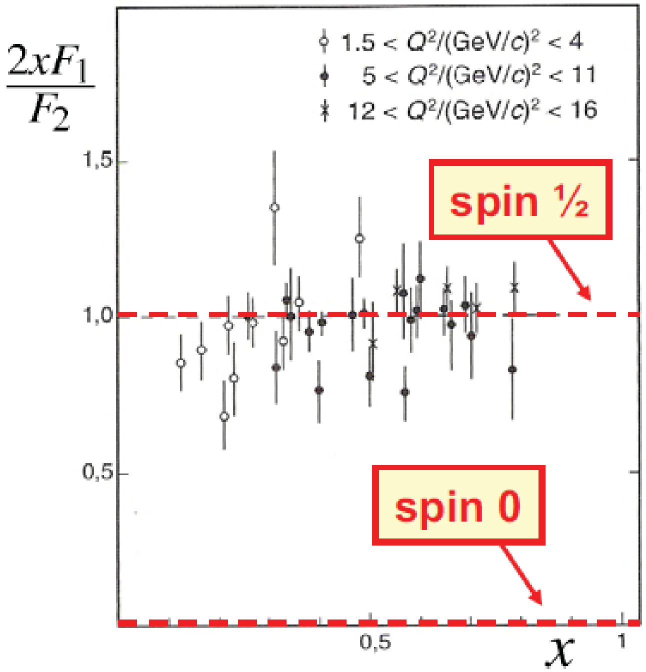
- It is strongly suggestive of scattering from point-like constituents within the proton

★ It is also observed that $F_1(x)$ and $F_2(x)$ are not independent but satisfy the Callan-Gross relation

$$F_2(x) = 2xF_1(x)$$

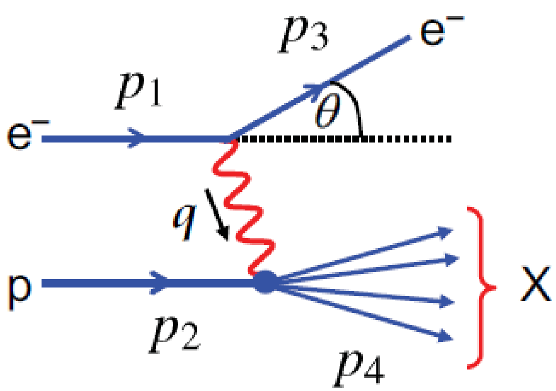
- As we shall soon see this is exactly what is expected for scattering from spin-half quarks.

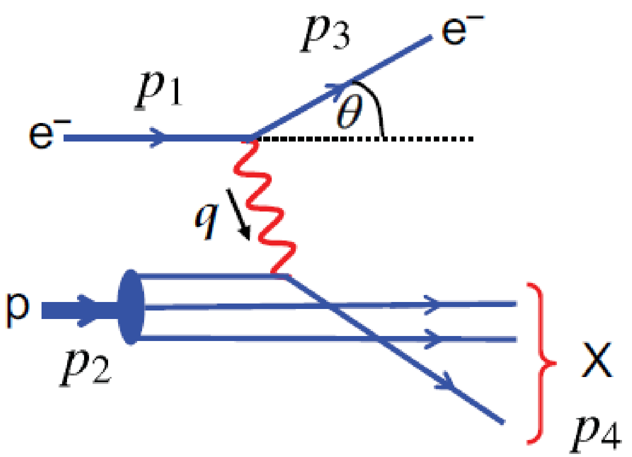
- Note if quarks were spin zero particles we would expect the purely magnetic structure function to be zero, i.e. $F_1(x) = 0$



The Quark-Parton Model

- Before quarks and gluons were generally accepted Feynman proposed that the proton was made up of point-like constituents "**partons**"
- Both Bjorken Scaling and the Callan-Gross relationship can be explained by assuming that Deep Inelastic Scattering is dominated by the scattering of a single virtual photon from point-like spin-half constituents of the proton.

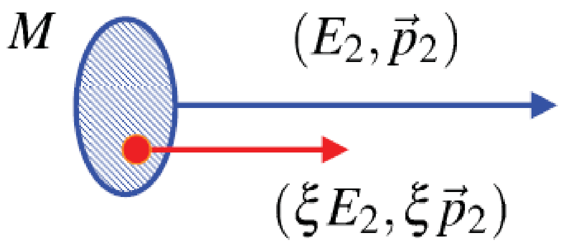


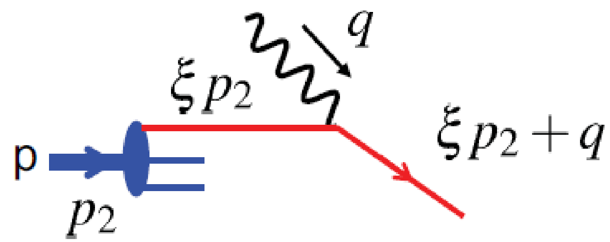


Scattering from a proton with structure functions → Scattering from a point-like quark within the proton

★ How do these two pictures of the interaction relate to each other?

- In the parton model the basic interaction is **ELASTIC** scattering from a **"quasi-free"** spin-½ quark in the proton, i.e. treat the quark as a free particle!
- The parton model is most easily formulated in a frame where the proton has very high energy, often referred to as the **"infinite momentum frame"**, where we can neglect the proton mass and $p_2 = (E_2, 0, 0, E_2)$
- In this frame can also neglect the mass of the quark and any momentum transverse to the direction of the proton.
- Let the quark carry a fraction ξ of the proton's four-momentum.





- After the interaction the struck quark's four-momentum is $\xi p_2 + q$

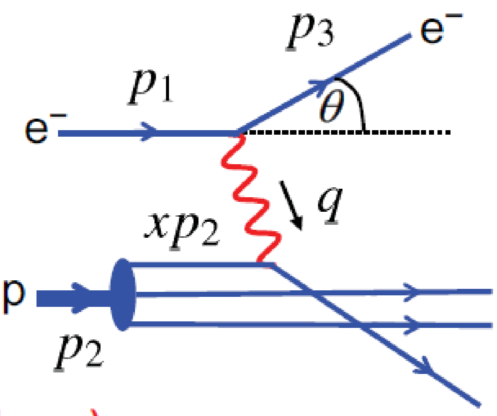
$$(\xi p_2 + q)^2 = m_q^2 \approx 0 \quad \Rightarrow \quad \cancel{\xi^2 p_2^2} + q^2 + 2\xi p_2.q = 0 \qquad (\xi^2 p_2^2 = m_q^2 \approx 0)$$

$$\Rightarrow \quad \xi = \frac{Q^2}{2p_2.q} = x$$

Bjorken x can be identified as the fraction of the proton momentum carried by the struck quark (in a frame where the proton has very high energy)

- In terms of the proton momentum

$$s = (p_1 + p_2)^2 \simeq 2p_1.p_2 \qquad y = \frac{p_2.q}{p_2.p_1} \qquad x = \frac{Q^2}{2p_2.q}$$



- But for the underlying quark interaction

$$s^q = (p_1 + xp_2)^2 = 2xp_1.p_2 = xs$$

$$y_q = \frac{p_q.q}{p_q.p_1} = \frac{xp_2.q}{xp_2.p_1} = y$$

$x_q = 1$ (elastic, i.e. assume quark does not break up)

- Previously derived the Lorentz Invariant cross section for $e^-\mu^- \to e^-\mu^-$ elastic scattering in the ultra-relativistic limit .

Now apply this to $e^-q \to e^-q$

$$\frac{d\sigma}{dq^2} = \frac{2\pi\alpha^2 e_q^2}{q^4}\left[1 + \left(1 + \frac{q^2}{s_q}\right)^2\right]$$

e_q is quark charge, i.e. $e_u = +2/3; \quad e_d = -1/3$

- Using $-q^2 = Q^2 = (s_q - m^2)x_q y_q$ ➡ $\frac{q^2}{s_q} = -y_q = -y$

$$\frac{d\sigma}{dQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}\left[1 + (1-y)^2\right]$$

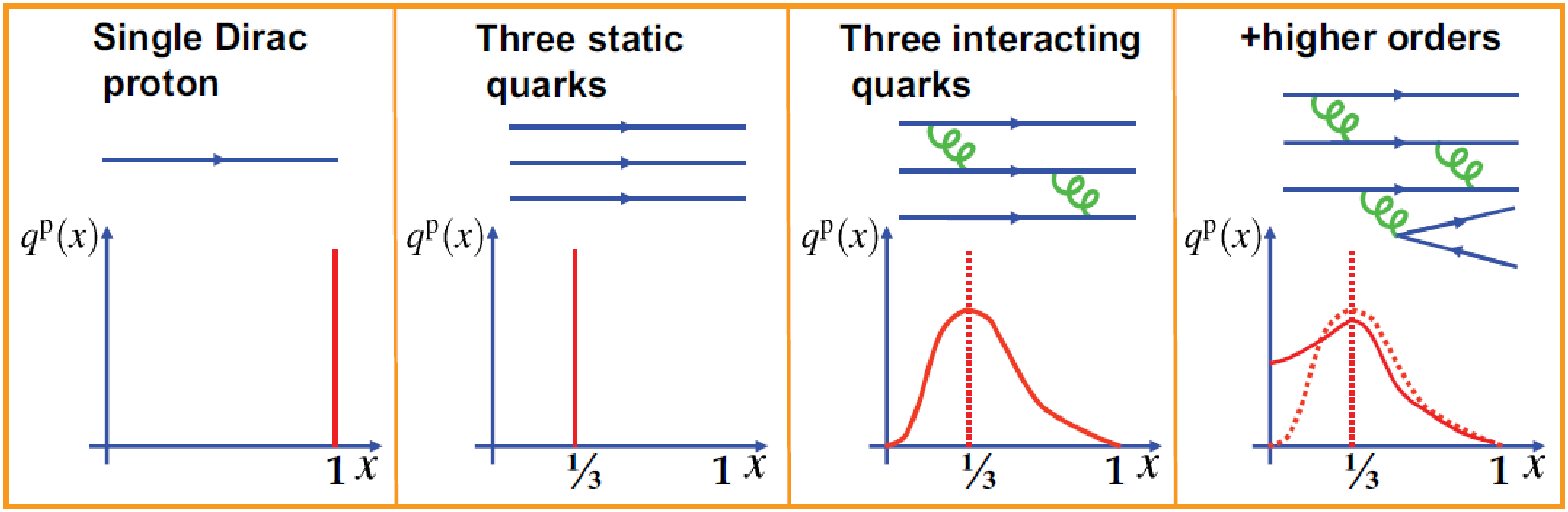
$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2 e_q^2}{Q^4}\left[(1-y) + \frac{y^2}{2}\right] \quad (3)$$

★ This is the expression for the differential cross-section for **elastic** e^-q scattering from a quark carrying a fraction x of the proton momentum.

- Now need to account for distribution of quark momenta within proton

★ Introduce parton distribution functions such that $q^p(x)dx$ is the number of quarks of type q within a proton with momenta between $x \rightarrow x + dx$

- Expected **form of the parton distribution function ?**



★ The cross section for scattering from a particular quark type within the proton which in the range $x \rightarrow x + \mathrm{d}x$ is

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y) + \frac{y^2}{2}\right] \times e_q^2 q^{\mathrm{p}}(x)\mathrm{d}x$$

★ Summing over all types of quark within the proton gives the expression for the electron-proton scattering cross section

$$\frac{\mathrm{d}^2\sigma^{\mathrm{ep}}}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y) + \frac{y^2}{2}\right]\sum_q e_q^2 q^{\mathrm{p}}(x) \qquad (5)$$

★ Compare with the electron-proton scattering cross section in terms of structure functions (equation (2)):

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right] \qquad (6)$$

★ By comparing (5) and (6) obtain the parton model prediction for the structure functions in the general L.I. form for the differential cross section

$$F_2^{\mathrm{p}}(x,Q^2) = 2xF_1^{\mathrm{p}}(x,Q^2) = x\sum_q e_q^2 q^{\mathrm{p}}(x)$$

⇒ Can relate measured structure functions to the underlying quark distributions

The parton model predicts:

- **Bjorken Scaling** $F_1(x,Q^2) \to F_1(x) \quad F_2(x,Q^2) \to F_2(x)$
  - ✶ Due to scattering from point-like particles within the proton
- **Callan-Gross Relation** $F_2(x) = 2xF_1(x)$
  - ✶ Due to scattering from spin half Dirac particles where the magnetic moment is directly related to the charge; hence the "electro-magnetic" and "pure magnetic" terms are fixed with respect to each other.

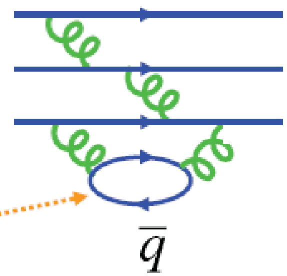
★ At present parton distributions cannot be calculated from QCD

- Can't use perturbation theory due to large coupling constant

★ Measurements of the structure functions enable us to determine the parton distribution functions !

★ For electron-proton scattering we have:

$$F_2^{\mathrm{p}}(x) = x\sum_q e_q^2 q^{\mathrm{p}}(x)$$



- Due to higher orders, the proton contains not only up and down quarks but also anti-up and anti-down quarks (will neglect the small contributions from heavier quarks)

- For electron-proton scattering have:

$$F_2^{\text{ep}}(x) = x\sum_q e_q^2 q^{\text{p}}(x) = x\left(\frac{4}{9}u^{\text{p}}(x) + \frac{1}{9}d^{\text{p}}(x) + \frac{4}{9}\overline{u}^{\text{p}}(x) + \frac{1}{9}\overline{d}^{\text{p}}(x)\right)$$

- For electron-neutron scattering have:

$$F_2^{\text{en}}(x) = x\sum_q e_q^2 q^{\text{n}}(x) = x\left(\frac{4}{9}u^{\text{n}}(x) + \frac{1}{9}d^{\text{n}}(x) + \frac{4}{9}\overline{u}^{\text{n}}(x) + \frac{1}{9}\overline{d}^{\text{n}}(x)\right)$$

★ Now assume "isospin symmetry", i.e. that the neutron (ddu) is the same as a proton (uud) with up and down quarks interchanged, i.e.

$$d^{\text{n}}(x) = u^{\text{p}}(x); \qquad u^{\text{n}}(x) = d^{\text{p}}(x)$$

and define the neutron distributions functions in terms of those of the proton

$$u(x) \equiv u^{\text{p}}(x) = d^{\text{n}}(x); \qquad d(x) \equiv d^{\text{p}}(x) = u^{\text{n}}(x)$$

$$\overline{u}(x) \equiv \overline{u}^{\text{p}}(x) = \overline{d}^{\text{n}}(x); \qquad \overline{d}(x) \equiv \overline{d}^{\text{p}}(x) = \overline{u}^{\text{n}}(x)$$

giving:

$$F_2^{\text{ep}}(x) = 2xF_1^{\text{ep}}(x) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x) + \frac{4}{9}\overline{u}(x) + \frac{1}{9}\overline{d}(x)\right) \qquad (7)$$

$$F_2^{\text{en}}(x) = 2xF_1^{\text{en}}(x) = x\left(\frac{4}{9}d(x) + \frac{1}{9}u(x) + \frac{4}{9}\overline{d}(x) + \frac{1}{9}\overline{u}(x)\right) \qquad (8)$$

- Integrating (7) and (8) :

$$\int_0^1 F_2^{ep}(x)\mathrm{d}x = \int_0^1 x\left(\frac{4}{9}[u(x)+\overline{u}(x)] + \frac{1}{9}[d(x)+\overline{d}(x)]\right)\mathrm{d}x = \frac{4}{9}f_u + \frac{1}{9}f_d$$

$$\int_0^1 F_2^{en}(x)\mathrm{d}x = \int_0^1 x\left(\frac{4}{9}[d(x)+\overline{d}(x)] + \frac{1}{9}[u(x)+\overline{u}(x)]\right)\mathrm{d}x = \frac{4}{9}f_d + \frac{1}{9}f_u$$

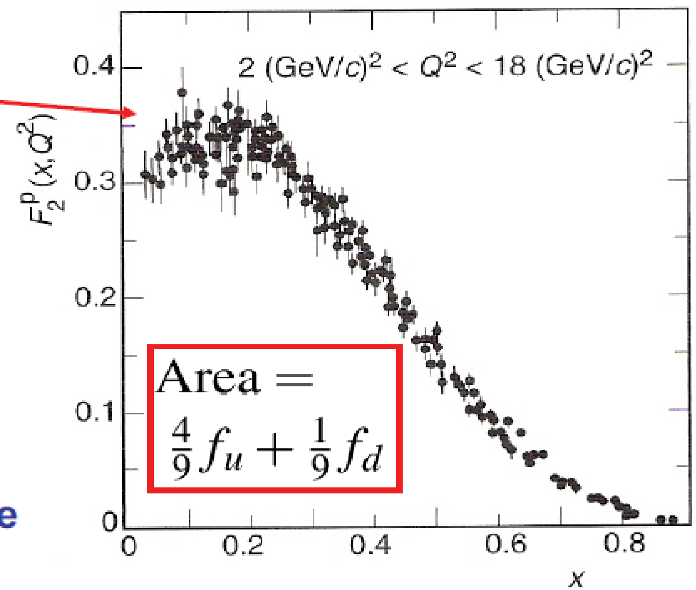
★ $f_u = \int_0^1 [xu(x) + x\overline{u}(x)]\mathrm{d}x$ is the fraction of the proton momentum carried by the up and anti-up quarks

Experimentally

$$\int F_2^{ep}(x)\mathrm{d}x \approx 0.18$$

$$\int F_2^{en}(x)\mathrm{d}x \approx 0.12$$

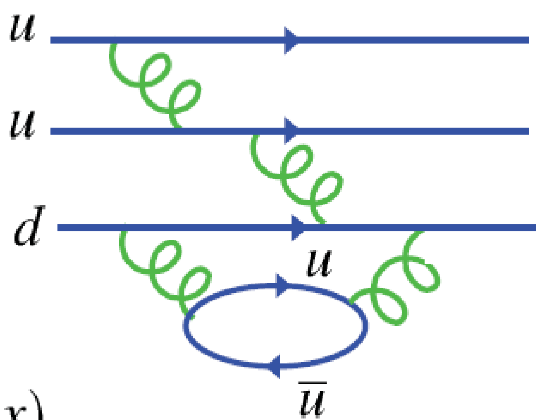
➡ $f_u \approx 0.36 \qquad f_d \approx 0.18$



✶ In the proton, as expected, the up quarks carry twice the momentum of the down quarks

✶ The quarks carry just over 50 % of the total proton momentum. The rest is carried by gluons (which being neutral doesn't contribute to electron-nucleon scattering).

Valence and Sea Quarks



- As we are beginning to see the proton is complex…
- The parton distribution function $u^{\mathrm{p}}(x) = u(x)$ includes contributions from the "valence" quarks and the virtual quarks produced by gluons: the "sea"
- Resolving into valence and sea contributions:

$$u(x) = u_{\mathrm{V}}(x) + u_{\mathrm{S}}(x) \qquad d(x) = d_{\mathrm{V}}(x) + d_{\mathrm{S}}(x)$$
$$\overline{u}(x) = \overline{u}_{\mathrm{S}}(x) \qquad \overline{d}(x) = \overline{d}_{\mathrm{S}}(x)$$

- The proton contains two valence up quarks and one valence down quark and would expect: $\int_0^1 u_{\mathrm{V}}(x)\mathrm{d}x = 2 \qquad \int_0^1 d_{\mathrm{V}}(x)\mathrm{d}x = 1$
- But no *a priori* expectation for the total number of sea quarks !
- But sea quarks arise from gluon quark/anti-quark pair production and with $m_u = m_d$ it is reasonable to expect

$$u_{\mathrm{S}}(x) = d_{\mathrm{S}}(x) = \overline{u}_{\mathrm{S}}(x) = \overline{d}_{\mathrm{S}}(x) = S(x)$$

- With these relations (7) and (8) become

$$F_2^{\mathrm{ep}}(x) = x\left(\frac{4}{9}u_{\mathrm{V}}(x) + \frac{1}{9}d_{\mathrm{V}}(x) + \frac{10}{9}S(x)\right) \qquad F_2^{\mathrm{en}}(x) = x\left(\frac{4}{9}d_{\mathrm{V}}(x) + \frac{1}{9}u_{\mathrm{V}}(x) + \frac{10}{9}S(x)\right)$$

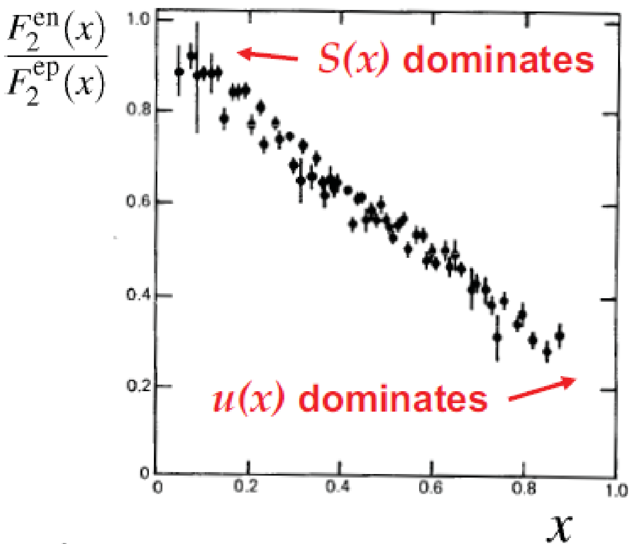
Giving the ratio

$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} = \frac{4d_V(x) + u_V(x) + 10S(x)}{4u_V(x) + d_V(x) + 10S(x)}$$

- The sea component arises from processes such as $g \rightarrow \overline{u}u$. Due to the $1/q^2$ dependence of the gluon propagator, much more likely to produce low energy gluons. Expect the sea to comprise of low energy $q/\overline{q}$
- Therefore at low x expect the sea to dominate:

$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} \rightarrow 1 \quad \text{as} \quad x \rightarrow 0$$

Observed experimentally



- At high x expect the sea contribution to be small

$$\frac{F_2^{en}(x)}{F_2^{ep}(x)} \rightarrow \frac{4d_V(x) + u_V(x)}{4u_V(x) + d_V(x)} \quad \text{as} \quad x \rightarrow 1$$

Note: $u_V = 2d_V$ would give ratio 2/3 as $x \rightarrow 1$

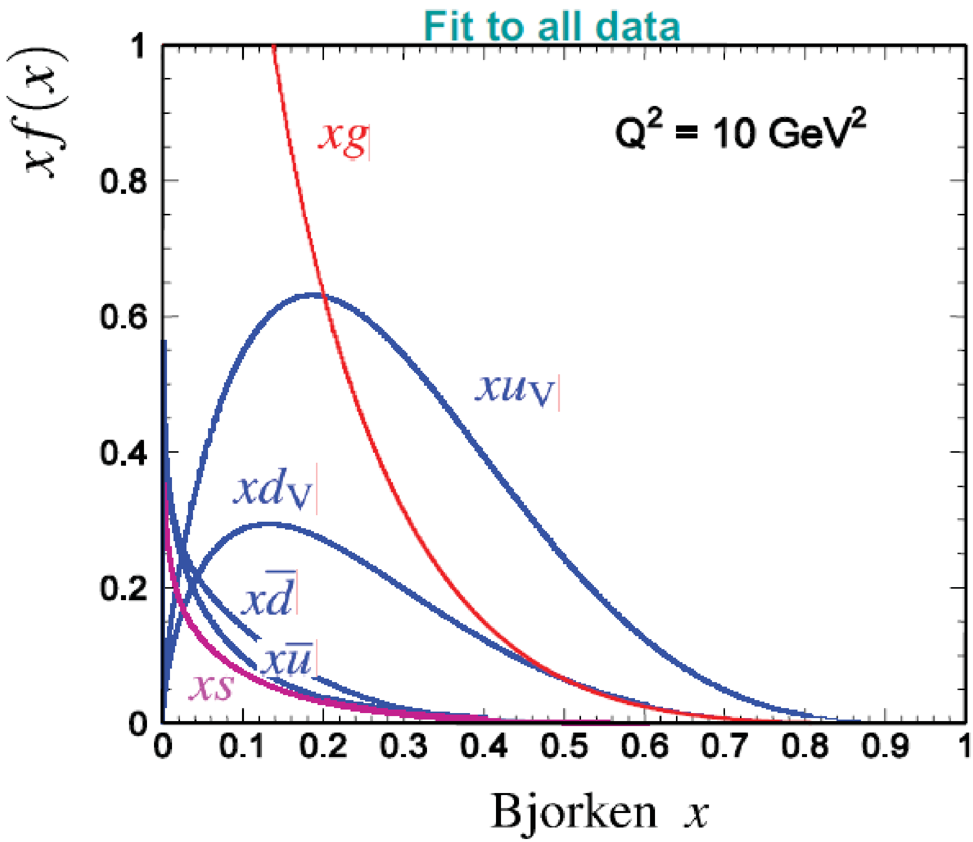
Experimentally $F_2^{en}(x)/F_2^{ep}(x) \rightarrow 1/4 \quad \text{as} \quad x \rightarrow 1$

➡ $d(x)/u(x) \rightarrow 0 \quad \text{as} \quad x \rightarrow 1$

This behaviour is not understood.

Parton Distribution Functions

- Ultimately the parton distribution functions are obtained from a fit to all experimental data including neutrino scattering
  - Hadron-hadron collisions give information on gluon pdf $g(x)$



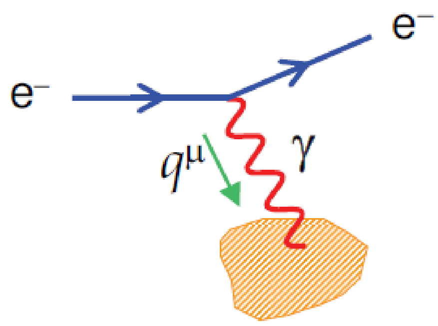
Note:

- Apart from at large x
  $u_V(x) \approx 2d_V(x)$
- For $x < 0.2$ gluons dominate
- In fits to data assume
  $u_s(x) = \overline{u}(x)$
- $\overline{d}(x) > \overline{u}(x)$
  not understood – exclusion principle?
- Small strange quark component $s(x)$

Scaling Violations

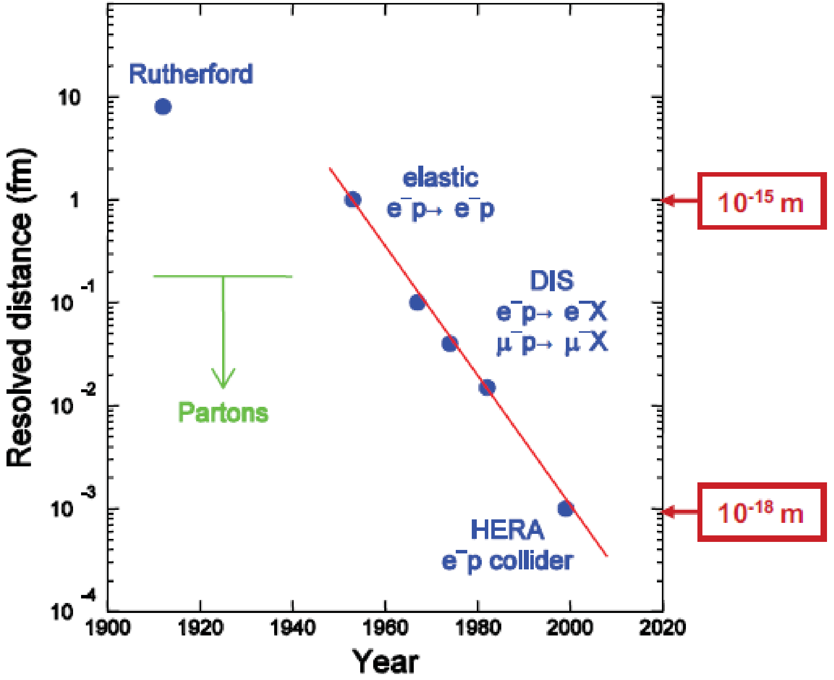
- In last 40 years, experiments have probed the proton with virtual photons of ever increasing energy
- Non-point like nature of the scattering becomes apparent when λ_γ ~ size of scattering centre

$$\lambda_\gamma = \frac{h}{|\vec{q}|} \sim \frac{1\,\mathrm{GeV\,fm}}{|\vec{q}|(\mathrm{GeV})}$$



- Scattering from point-like quarks gives rise to Bjorken scaling: no q^2 cross section dependence
- IF quarks were not point-like, at high q^2 (when the wavelength of the virtual photon ~ size of quark) would observe rapid decrease in cross section with increasing q^2.
- To search for quark sub-structure want to go to highest q^2

HERA

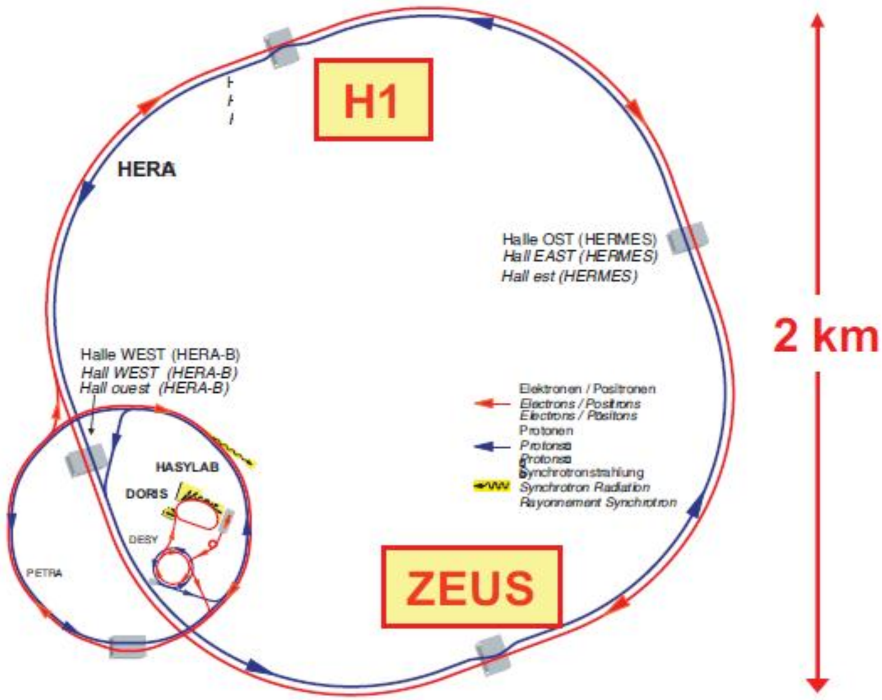


HERA $e^{\pm}p$ Collider: 1991 - 2007

★ **DESY (Deutsches Elektronen-Synchroton) Laboratory, Hamburg, Germany**

$e^{\pm}$ → 27.5 GeV ← 820 GeV p

$\sqrt{s}$ = 300 GeV

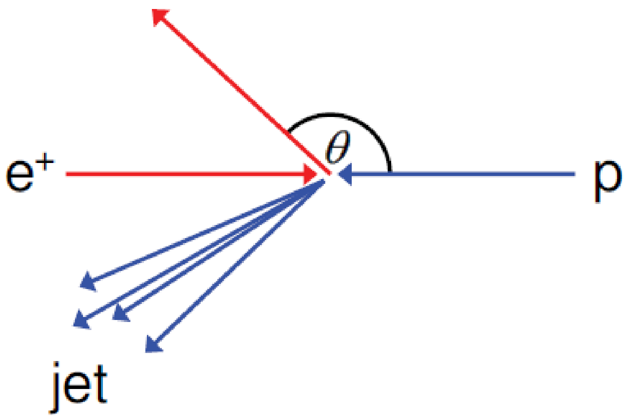


★ **Two large experiments : H1 and ZEUS**

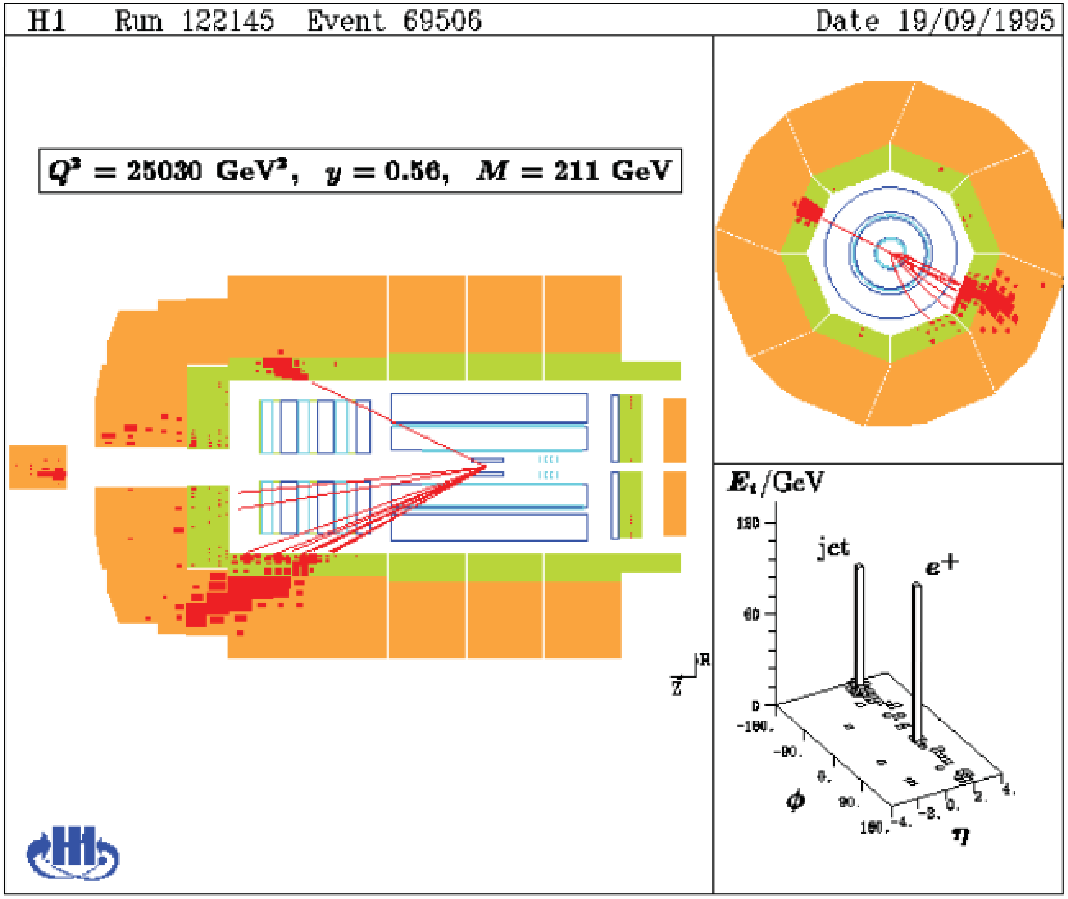
★ **Probe proton at very high Q^2 and very low x**

Example of a High Q^2 Event in H1

- ✶ **Event kinematics determined from electron angle and energy**



- ✶ **Also measure hadronic system (although not as precisely) – gives some redundancy**



$F_2(x, Q^2)$ Results

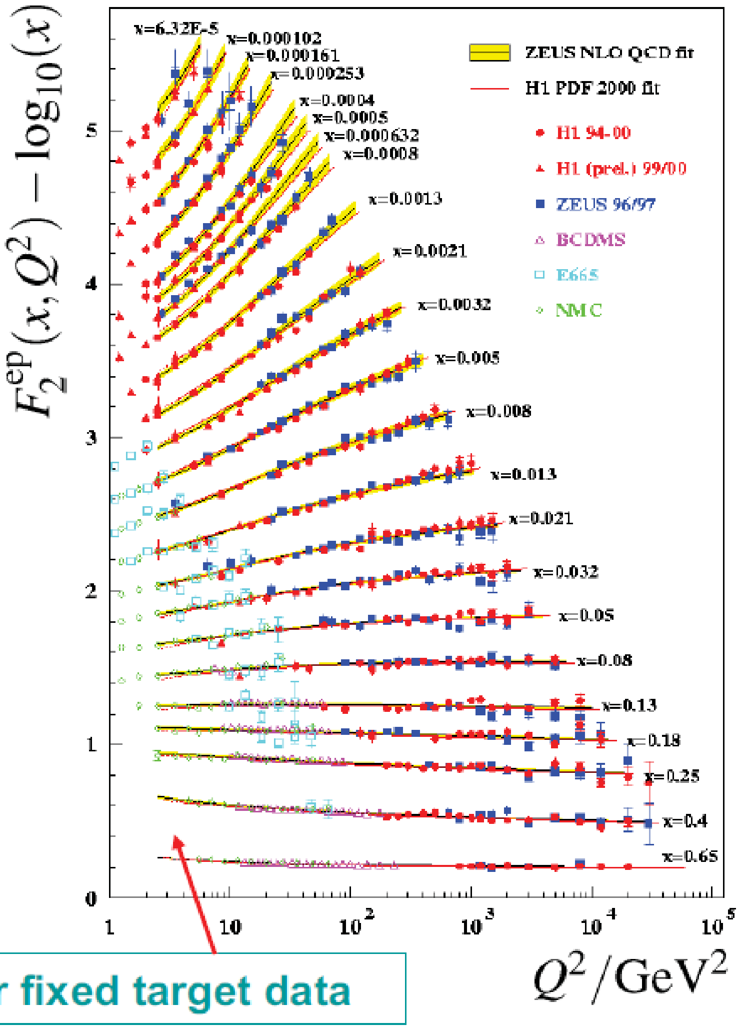
★ No evidence of rapid decrease of cross section at highest Q^2

➡ $R_{\text{quark}} < 10^{-18}\,\text{m}$

★ For $x > 0.05$, only weak dependence of F_2 on Q^2 : consistent with the expectation from the quark-parton model

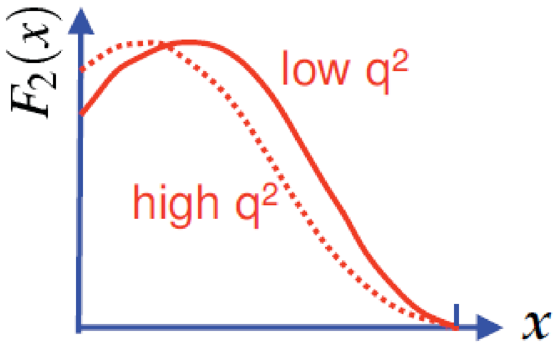
★ But observe clear scaling violations, particularly at low x

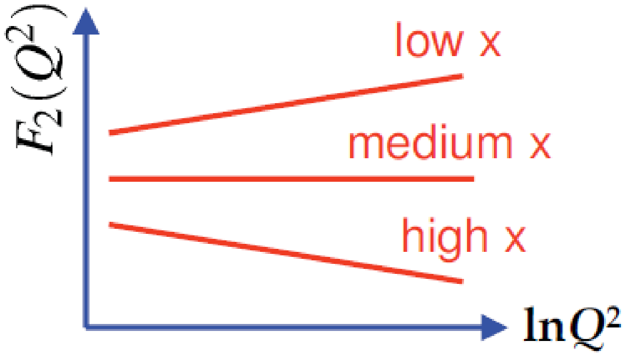
$$F_2(x, Q^2) \neq F_2(x)$$



Origin of Scaling Violations

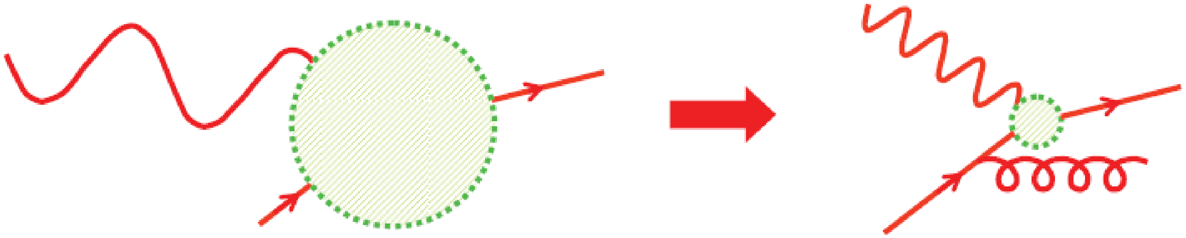
★ Observe "small" deviations from exact Bjorken scaling $F_2(x) \to F_2(x, Q^2)$





★ At high Q^2 observe more low x quarks

★ **"Explanation"**: at high Q^2 (shorter wave-length) resolve finer structure: i.e. reveal quark is sharing momentum with gluons. At higher Q^2 expect to "see" more low x quarks



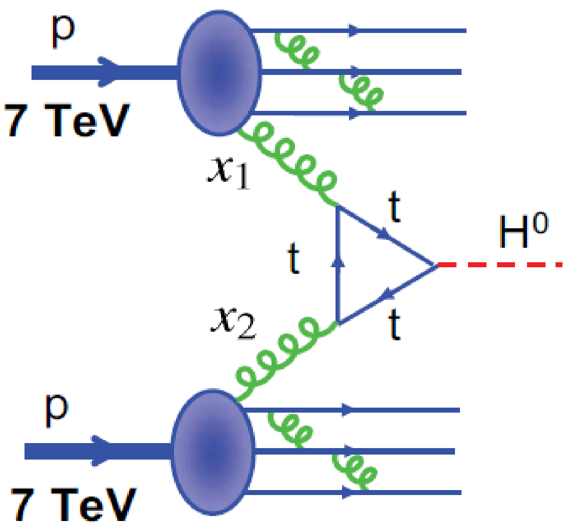
★ QCD cannot predict the x dependence of $F_2(x, Q^2)$

★ But QCD **can** predict the Q^2 dependence of $F_2(x, Q^2)$

Proton-proton collisions at the LHC

★ Measurements of structure functions not only provide a powerful test of QCD, the parton distribution functions are essential for the calculation of cross sections at pp and p$\bar{p}$ colliders.

- <u>Example:</u> Higgs production at the Large Hadron Collider LHC (2009-)
  - The LHC will collide 7 TeV protons on 7 TeV protons
  - However underlying collisions are between partons
  - Higgs production the LHC dominated by "gluon-gluon fusion"



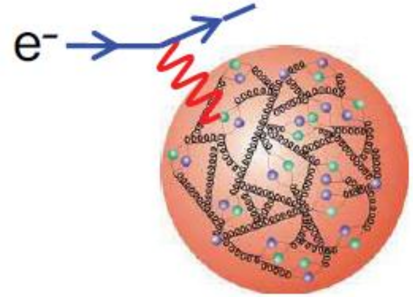
- Cross section depends on gluon PDFs

$$\sigma(pp \to HX) \sim \int_0^1 \int_0^1 g(x_1)g(x_2)\sigma(gg \to H)\mathrm{d}x_1\mathrm{d}x_2$$

- Uncertainty in gluon PDFs lead to a ±5 % uncertainty in Higgs production cross section

- Prior to HERA data uncertainty was ±25 %

Summary



- At very high electron energies $\lambda \ll r_p$: the proton appears to be a sea of quarks and gluons.
- Deep Inelastic Scattering = Elastic scattering from the quasi-free constituent quarks
  - ➡ Bjorken Scaling $F_1(x,Q^2) \to F_1(x)$ **point-like scattering**
  - ➡ Callan-Gross $F_2(x) = 2xF_1(x)$ **Scattering from spin-1/2**
- Describe scattering in terms of parton distribution functions $u(x), d(x), \ldots$ which describe momentum distribution inside a nucleon
- The proton is much more complex than just uud - sea of anti-quarks/gluons
- Quarks carry only 50 % of the protons momentum – the rest is due to low energy gluons
- We will come back to this topic when we discuss neutrino scattering...