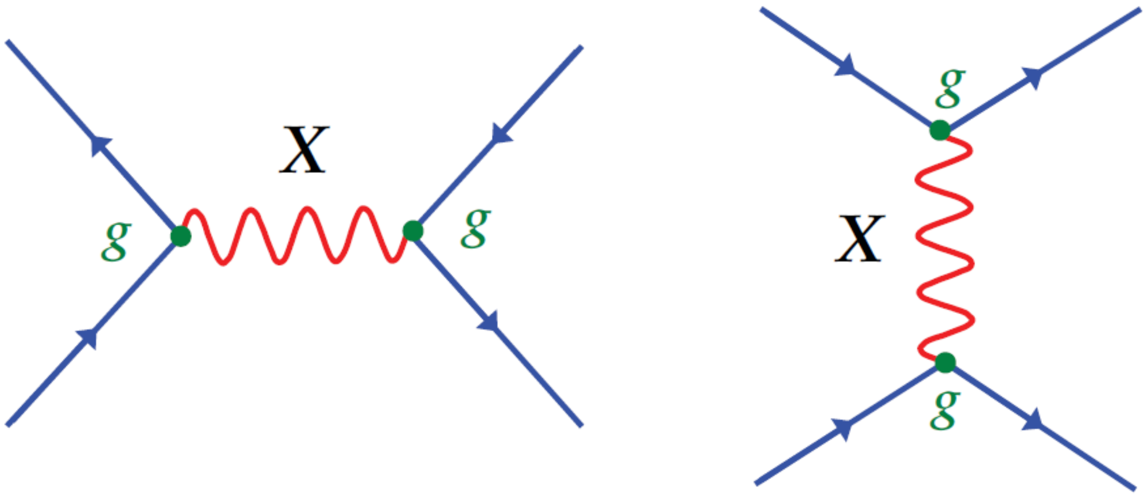
Elementary Particle Physics: theory and experiments

Interaction by Particle Exchange and QED



Follow the course/slides from M. A. Thomson lectures at Cambridge University

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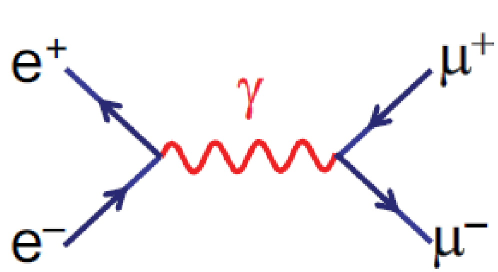
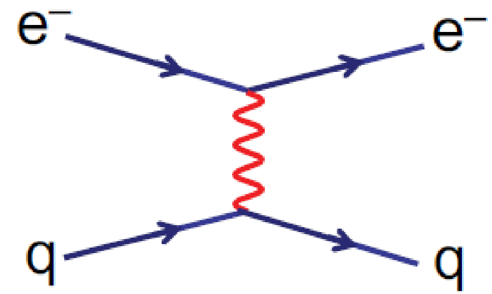
Recap

★ **Working towards a proper calculation of decay and scattering processes**

Initially concentrate on:

- $e^+e^- \rightarrow \mu^+\mu^-$
- $e^-q \rightarrow e^-q$





⅄ **In Handout 1 covered the <u>relativistic</u> calculation of particle decay rates and cross sections**

$$\sigma \propto \frac{|M|^2}{\text{flux}} \times (\text{phase space})$$

⅄ **In Handout 2 covered <u>relativistic</u> treatment of spin-half particles**

Dirac Equation

⅄ **This handout concentrate on the Lorentz Invariant Matrix Element**

- **Interaction by particle exchange**
- **Introduction to Feynman diagrams**
- **The Feynman rules for QED**

Interaction by Particle Exchange

- Calculate transition rates from **Fermi's Golden Rule**

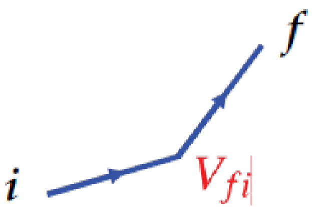
$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

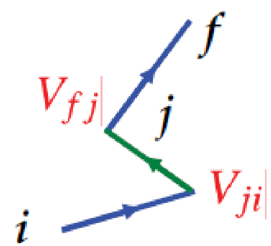
where T_{fi} is perturbation expansion for the Transition Matrix Element

$$T_{fi} = \langle f|V|i\rangle + \sum_{j\neq i} \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j} + \ldots$$

- For particle scattering, the first two terms in the perturbation series can be viewed as:

"scattering in a potential"

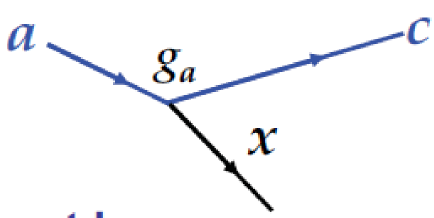




"scattering via an intermediate state"

- **"Classical picture"** – particles act as sources for fields which give rise a potential in which other particles scatter – **"action at a distance"**
- **"Quantum Field Theory picture"** – forces arise due to the exchange of virtual particles. No action at a distance + forces between particles now due to particles

- Need an expression for $\langle c+x|V|a\rangle$ in non-invariant matrix element T_{fi}



- Ultimately aiming to obtain Lorentz Invariant ME
- Recall T_{fi} is related to the invariant matrix element by

$$T_{fi} = \prod_k (2E_k)^{-1/2} M_{fi}$$

where k runs over all particles in the matrix element

- Here we have

$$\langle c+x|V|a\rangle = \frac{M_{(a\to c+x)}}{(2E_a 2E_c 2E_x)^{1/2}}$$

$M_{(a\to c+x)}$ is the "Lorentz Invariant" matrix element for $a \to c + x$

★ The simplest Lorentz Invariant quantity is a scalar, in this case

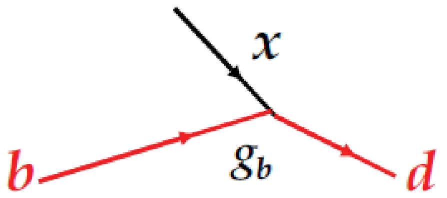
$$\langle c+x|V|a\rangle = \frac{g_a}{(2E_a 2E_c 2E_x)^{1/2}}$$

g_a is a measure of the strength of the interaction $a \to c + x$

Note : the matrix element is only LI in the sense that it is defined in terms of LI wave-function normalisations and that the form of the coupling is LI

Note : in this "illustrative" example g is not dimensionless.

Similarly $\langle d|V|x+b\rangle = \dfrac{g_b}{(2E_b 2E_d 2E_x)^{1/2}}$



Giving
$$T_{fi}^{ab} = \frac{\langle d|V|x+b\rangle\langle c+x|V|a\rangle}{(E_a+E_b)-(E_c+E_x+E_b)}$$
$$= \frac{1}{2E_x}\cdot\frac{1}{(2E_a 2E_b 2E_c 2E_d)^{1/2}}\cdot\frac{g_a g_b}{(E_a-E_c-E_x)}$$

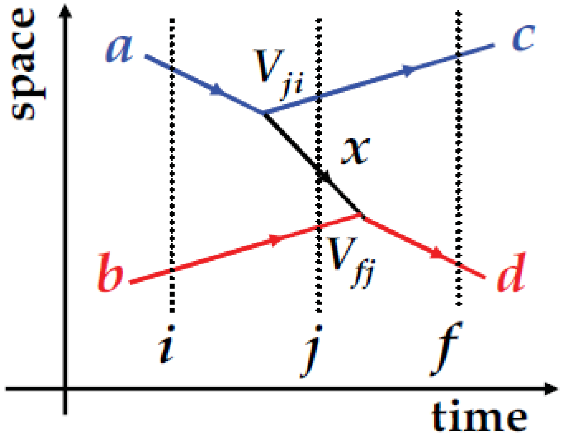
★ The "Lorentz Invariant" matrix element for the entire process is

$$M_{fi}^{ab} = (2E_a 2E_b 2E_c 2E_d)^{1/2} T_{fi}^{ab}$$
$$= \frac{1}{2E_x}\cdot\frac{g_a g_b}{(E_a-E_c-E_x)}$$

Note:

- M_{fi}^{ab} refers to the time-ordering where a emits x before b absorbs it
  It is not Lorentz invariant, order of events in time depends on frame
- Momentum is conserved at each interaction vertex but not energy
  $E_j \neq E_i$
- Particle x is "on-mass shell" i.e. $E_x^2 = \vec{p}_x^{\,2} + m^2$

- Consider the particle interaction $a+b \to c+d$ which occurs via an intermediate state corresponding to the exchange of particle x
- One possible space-time picture of this process is:



Initial state i : $a+b$
Final state f : $c+d$
Intermediate state j : $c+b+x$

- This **time-ordered** diagram corresponds to a "emitting" x and then b absorbing x

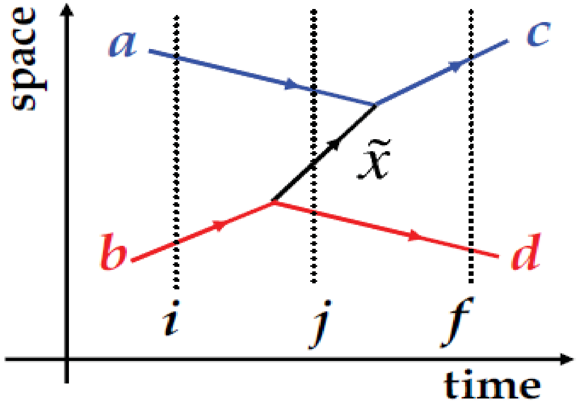
- The corresponding term in the perturbation expansion is:

$$T_{fi} = \frac{\langle f|V|j\rangle\langle j|V|i\rangle}{E_i - E_j}$$

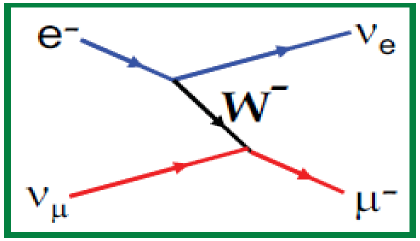
$$T_{fi}^{ab} = \frac{\langle d|V|x+b\rangle\langle c+x|V|a\rangle}{(E_a+E_b)-(E_c+E_x+E_b)}$$

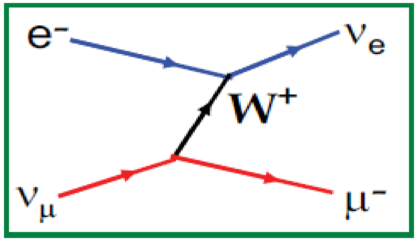
- T_{fi}^{ab} refers to the time-ordering where a emits x before b absorbs it

★ But need to consider also the other time ordering for the process



- This **time-ordered** diagram corresponds to *b* "emitting" $\tilde{x}$ and then *a* absorbing $\tilde{x}$
- $\tilde{x}$ is the anti-particle of x e.g.





- The **Lorentz invariant matrix element** for this time ordering is:

$$M_{fi}^{ba} = \frac{1}{2E_x} \cdot \frac{g_a g_b}{(E_b - E_d - E_x)}$$

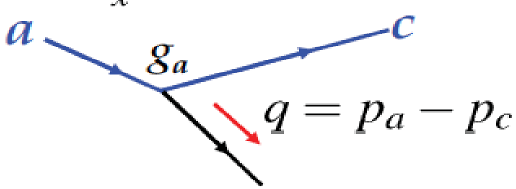
★ In QM need to sum over matrix elements corresponding to same final state:

$$\begin{aligned} M_{fi} &= M_{fi}^{ab} + M_{fi}^{ba} \\ &= \frac{g_a g_b}{2E_x} \cdot \left(\frac{1}{E_a - E_c - E_x} + \frac{1}{E_b - E_d - E_x} \right) \\ &= \frac{g_a g_b}{2E_x} \cdot \left(\frac{1}{E_a - E_c - E_x} - \frac{1}{E_a - E_c + E_x} \right) \end{aligned}$$

Energy conservation:
$(E_a + E_b = E_c + E_d)$

- Which gives
$$M_{fi} = \frac{g_a g_b}{2E_x} \cdot \frac{2E_x}{(E_a - E_c)^2 - E_x^2}$$
$$= \frac{g_a g_b}{(E_a - E_c)^2 - E_x^2}$$

- From 1st time ordering $E_x^2 = \vec{p}_x^{\,2} + m_x^2 = (\vec{p}_a - \vec{p}_c)^2 + m_x^2$



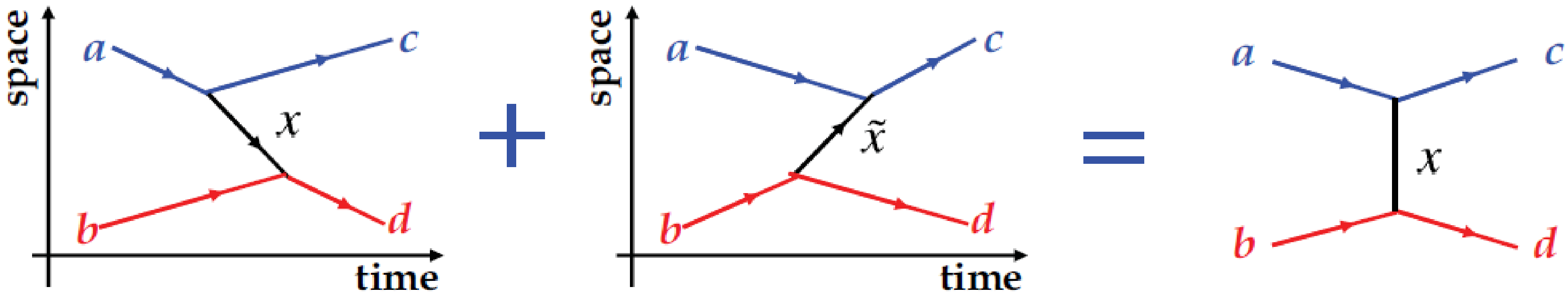
giving
$$M_{fi} = \frac{g_a g_b}{(E_a - E_c)^2 - (\vec{p}_a - \vec{p}_c)^2 - m_x^2}$$
$$= \frac{g_a g_b}{(p_a - p_c)^2 - m_x^2}$$

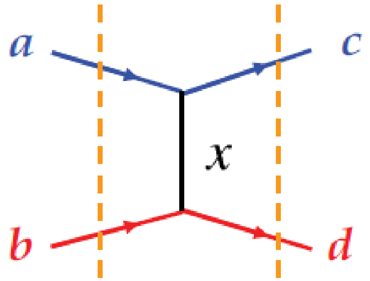
$$\Rightarrow \quad \boxed{M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}}$$

- After summing over all possible time orderings, M_{fi} is (as anticipated) **Lorentz invariant**. This is a remarkable result – the sum over all time orderings gives a frame independent matrix element.
- Exactly the same result would have been obtained by considering the annihilation process

Feynman diagrams

- The sum over all possible time-orderings is represented by a **FEYNMAN diagram**





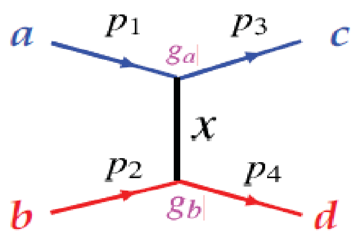
In a Feynman diagram:

- the LHS represents the initial state
- the RHS is the final state
- everything in between is "how the interaction happened"

- It is important to remember that energy and momentum are conserved at each interaction vertex in the diagram.
- The factor $1/(q^2 - m_x^2)$ is the propagator; it arises naturally from the above discussion of interaction by particle exchange

★ **The matrix element:** $M_{fi} = \dfrac{g_a g_b}{q^2 - m_x^2}$ **depends on:**

- **The fundamental strength of the interaction at the two vertices** g_a, g_b
- **The four-momentum, q, carried by the (virtual) particle which is determined from energy/momentum conservation at the vertices. Note q^2 can be either positive or negative.**

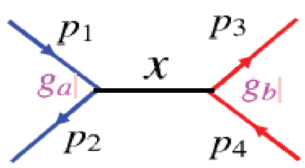


Here $q = p_1 - p_3 = p_4 - p_2 = t$ **"t-channel"**

For elastic scattering: $p_1 = (E, \vec{p}_1); \quad p_3 = (E, \vec{p}_3)$

$$q^2 = (E - E)^2 - (\vec{p}_1 - \vec{p}_3)^2$$

$q^2 < 0$ **termed "space-like"**



Here $q = p_1 + p_2 = p_3 + p_4 = s$ **"s-channel"**

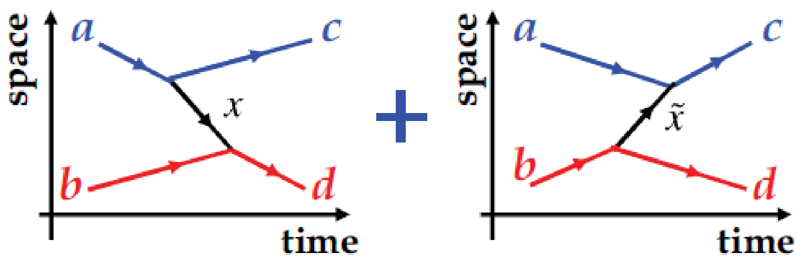
In CoM: $p_1 = (E, \vec{p}); \quad p_2 = (E, -\vec{p})$

$$q^2 = (E + E)^2 - (\vec{p} - \vec{p})^2 = 4E^2$$

$q^2 > 0$ **termed "time-like"**

Virtual Particles

"Time-ordered QM"



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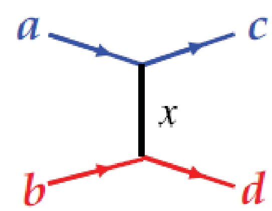
Feynman diagram



$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- Momentum conserved at vertices
- Energy **not** conserved at vertices
- Exchanged particle **"on mass shell"**

$$E_x^2 - |\vec{p}_x|^2 = m_x^2$$

- Momentum **AND** energy conserved at interaction vertices
- Exchanged particle **"off mass shell"**

$$E_x^2 - |\vec{p}_x|^2 = q^2 \neq m_x^2$$

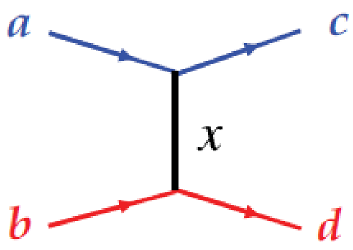
VIRTUAL PARTICLE

- Can think of observable "on mass shell" particles as propagating waves and unobservable virtual particles as normal modes between the source particles:

Aside: V(r) from Particle Exchange

★ Can view the scattering of an electron by a proton at rest in two ways:

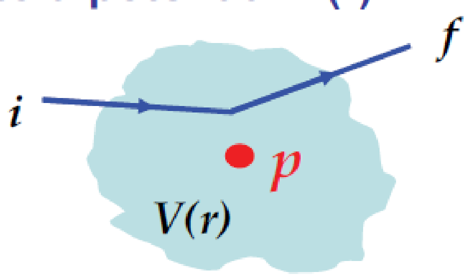
- Interaction by particle exchange in 2nd order perturbation theory.



$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

- Could also evaluate the same process in first order perturbation theory treating proton as a fixed source of a field which gives rise to a potential V(r)

$$M = \langle \psi_f | V(r) | \psi_i \rangle$$



Obtain same expression for M_{fi} using

$$V(r) = g_a g_b \frac{e^{-mr}}{r}$$

YUKAWA potential

★ In this way can relate potential and forces to the particle exchange picture

★ However, scattering from a fixed potential $V(r)$ is not a relativistic invariant view

Quantum Electrodynamics (QED)

★ Now consider the interaction of an electron and tau lepton by the exchange of a photon. Although the general ideas we applied previously still hold, we now have to account for the spin of the electron/tau-lepton and also the spin (polarization) of the virtual photon.

- The basic interaction between a photon and a charged particle can be introduced by making the minimal substitution

$$\vec{p} \rightarrow \vec{p} - q\vec{A}; \quad E \rightarrow E - q\phi$$

(here $q =$ charge)

In QM:

$$\vec{p} = -i\vec{\nabla}; \qquad E = i\partial/\partial t$$

Therefore make substitution: $i\partial_\mu \rightarrow i\partial_\mu - qA_\mu$

where $A_\mu = (\phi, -\vec{A}); \qquad \partial_\mu = (\partial/\partial t, +\vec{\nabla})$

- The Dirac equation:

$$\gamma^\mu \partial_\mu \psi + im\psi = 0 \quad \Rightarrow \quad \gamma^\mu \partial_\mu \psi + iq\gamma^\mu A_\mu \psi + im\psi = 0$$

$$(\times i) \quad \Rightarrow \quad i\gamma^0 \frac{\partial \psi}{\partial t} + i\vec{\gamma}.\vec{\nabla}\psi - q\gamma^\mu A_\mu \psi - m\psi = 0$$

$$i\gamma^0 \frac{\partial \psi}{\partial t} = \gamma^0 \hat{H}\psi \quad = \quad m\psi - i\vec{\gamma}.\vec{\nabla}\psi + q\gamma^\mu A_\mu \psi$$

$$\times\gamma^0 : \qquad \hat{H}\psi \quad = \quad \underbrace{(\gamma^0 m - i\gamma^0\vec{\gamma}.\vec{\nabla})}_{\text{Combined rest mass + K.E.}}\psi + \underbrace{q\gamma^0\gamma^\mu A_\mu}_{\text{Potential energy}}\psi$$

- We can identify the potential energy of a charged spin-half particle in an electromagnetic field as:

$$\boxed{\hat{V}_D = q\gamma^0\gamma^\mu A_\mu}$$

(note the A_0 term is just: $q\gamma^0\gamma^0 A_0 = q\phi$)

- The final complication is that we have to account for the photon polarization states.

$$A_\mu = \varepsilon_\mu^{(\lambda)} e^{i(\vec{p}.\vec{r} - Et)}$$

e.g. for a real photon propagating in the z direction we have two orthogonal transverse polarization states

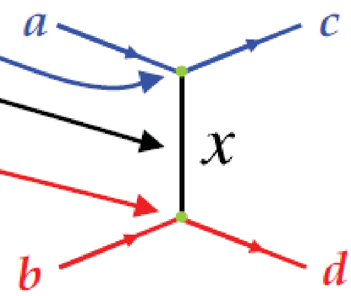
$$\varepsilon^{(1)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \varepsilon^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Could equally have chosen circularly polarized states

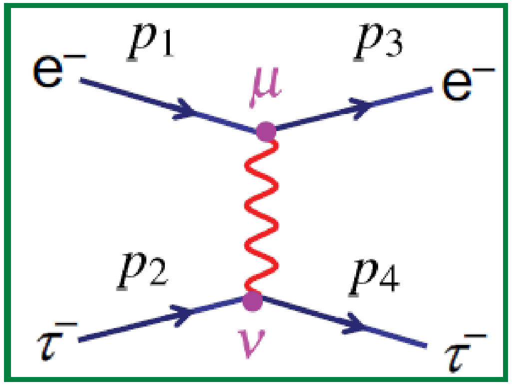
- Previously with the example of a simple spin-less interaction we had:

$$M = \langle\psi_c|V|\psi_a\rangle \frac{1}{q^2 - m_x^2} \langle\psi_d|V|\psi_b\rangle$$

$\langle\psi_c|V|\psi_a\rangle = g_a$, $\langle\psi_d|V|\psi_b\rangle = g_b$



★ In QED we could again go through the procedure of summing the time-orderings using Dirac spinors and the expression for $\hat{V}_D$. If we were to do this, remembering to sum over all photon polarizations, we would obtain:



$$M = \left[u_e^\dagger(p_3) q_e \gamma^0 \gamma^\mu u_e(p_1)\right] \sum_\lambda \frac{\varepsilon_\mu^\lambda (\varepsilon_\nu^\lambda)^*}{q^2} \left[u_\tau^\dagger(p_4) q_\tau \gamma^0 \gamma^\nu u_\tau(p_2)\right]$$

- Interaction of e^- with photon
- Massless photon propagator summing over polarizations
- Interaction of τ^- with photon

- All the physics of **QED** is in the above expression !

- The sum over the polarizations of the VIRTUAL photon has to include longitudinal and scalar contributions, i.e. 4 polarisation states

$$\varepsilon^{(0)} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad \varepsilon^{(1)} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \qquad \varepsilon^{(2)} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad \varepsilon^{(3)} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

and gives:

$$\sum_{\lambda} \varepsilon_{\mu}^{\lambda} (\varepsilon_{\nu}^{\lambda})^{*} = -g_{\mu\nu}$$

This is not obvious – for the moment just take it on trust

and the invariant matrix element becomes:

$$M = \left[u_e^{\dagger}(p_3) q_e \gamma^0 \gamma^{\mu} u_e(p_1)\right] \frac{-g_{\mu\nu}}{q^2} \left[u_{\tau}^{\dagger}(p_4) q_{\tau} \gamma^0 \gamma^{\nu} u_{\tau}(p_2)\right]$$

- Using the definition of the adjoint spinor $\overline{\psi} = \psi^{\dagger}\gamma^0$

$$M = [\overline{u}_e(p_3) q_e \gamma^{\mu} u_e(p_1)] \frac{-g_{\mu\nu}}{q^2} [\overline{u}_{\tau}(p_4) q_{\tau} \gamma^{\nu} u_{\tau}(p_2)]$$

★ This is a remarkably simple expression ! It is shown in Appendix V of Handout 2 that $\overline{u}_1 \gamma^{\mu} u_2$ transforms as a four vector. Writing

$$j_e^{\mu} = \overline{u}_e(p_3) \gamma^{\mu} u_e(p_1) \qquad j_{\tau}^{\nu} = \overline{u}_{\tau}(p_4) \gamma^{\nu} u_{\tau}(p_2)$$

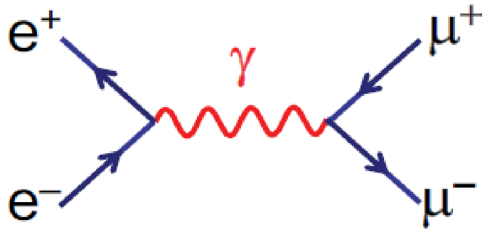
$$M = -q_e q_{\tau} \frac{j_e \cdot j_{\tau}}{q^2}$$ showing that M is Lorentz Invariant

Feynman rules for QED

- It should be remembered that the expression

$$M = [\overline{u}_e(p_3) q_e \gamma^\mu u_e(p_1)] \frac{-g_{\mu\nu}}{q^2} [\overline{u}_\tau(p_4) q_\tau \gamma^\nu u_\tau(p_2)]$$

hides a lot of complexity. We have summed over all possible **time-orderings** and summed over all **polarization states** of the virtual photon. If we are then presented with a new Feynman diagram we don't want to go through the full calculation again. Fortunately this isn't necessary – can just write down matrix element using a set of simple rules



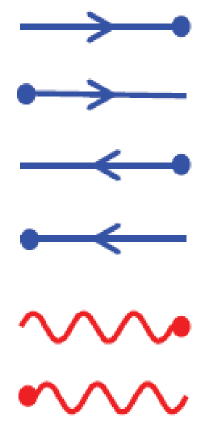
Basic Feynman Rules:

- Propagator factor for each internal line
  (i.e. each internal virtual particle)
- Dirac Spinor for each external line
  (i.e. each real incoming or outgoing particle)
- Vertex factor for each vertex

Basic rules for QED

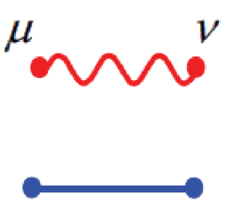
- **External Lines**

spin 1/2	incoming particle	$u(p)$	
	outgoing particle	$\bar{u}(p)$	
	incoming antiparticle	$\bar{v}(p)$	
	outgoing antiparticle	$v(p)$	
spin 1	incoming photon	$\varepsilon^{\mu}(p)$	
	outgoing photon	$\varepsilon^{\mu}(p)^{*}$	



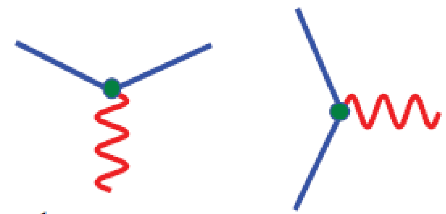
- **Internal Lines (propagators)**

spin 1	photon	$-\dfrac{ig_{\mu\nu}}{q^2}$	
spin 1/2	fermion	$\dfrac{i(\gamma^{\mu} q_{\mu} + m)}{q^2 - m^2}$	



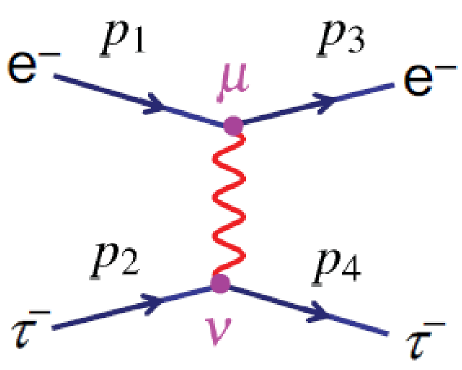
- **Vertex Factors**

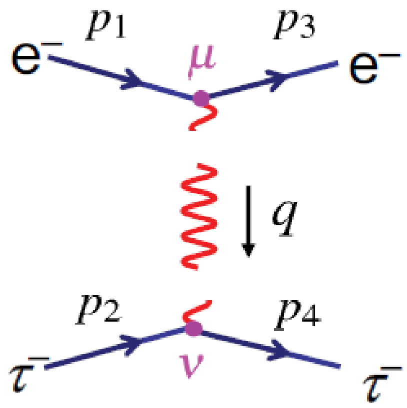
spin 1/2	fermion (charge -|e|)	$ie\gamma^{\mu}$	



- **Matrix Element** $-iM$ = product of all factors

e.g.





$$\overline{u}_e(p_3)[ie\gamma^\mu]u_e(p_1)$$

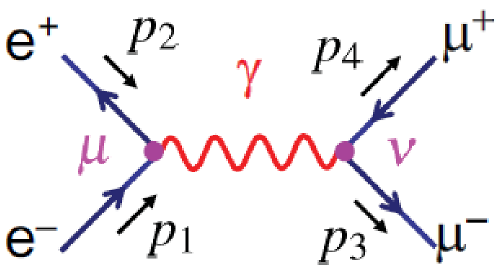
$$\frac{-ig_{\mu\nu}}{q^2}$$

$$\overline{u}_\tau(p_4)[ie\gamma^\nu]u_\tau(p_2)$$

$$-iM = [\overline{u}_e(p_3)ie\gamma^\mu u_e(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}_\tau(p_4)ie\gamma^\nu u_\tau(p_2)]$$

- Which is the same expression as we obtained previously

e.g.



$$-iM = [\overline{v}(p_2)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_3)ie\gamma^\nu v(p_4)]$$

Note:
- At each vertex the adjoint spinor is written first
- Each vertex has a different index
- The $g_{\mu\nu}$ of the propagator connects the indices at the vertices

Summary

★ Interaction by particle exchange naturally gives rise to **Lorentz Invariant Matrix Element** of the form

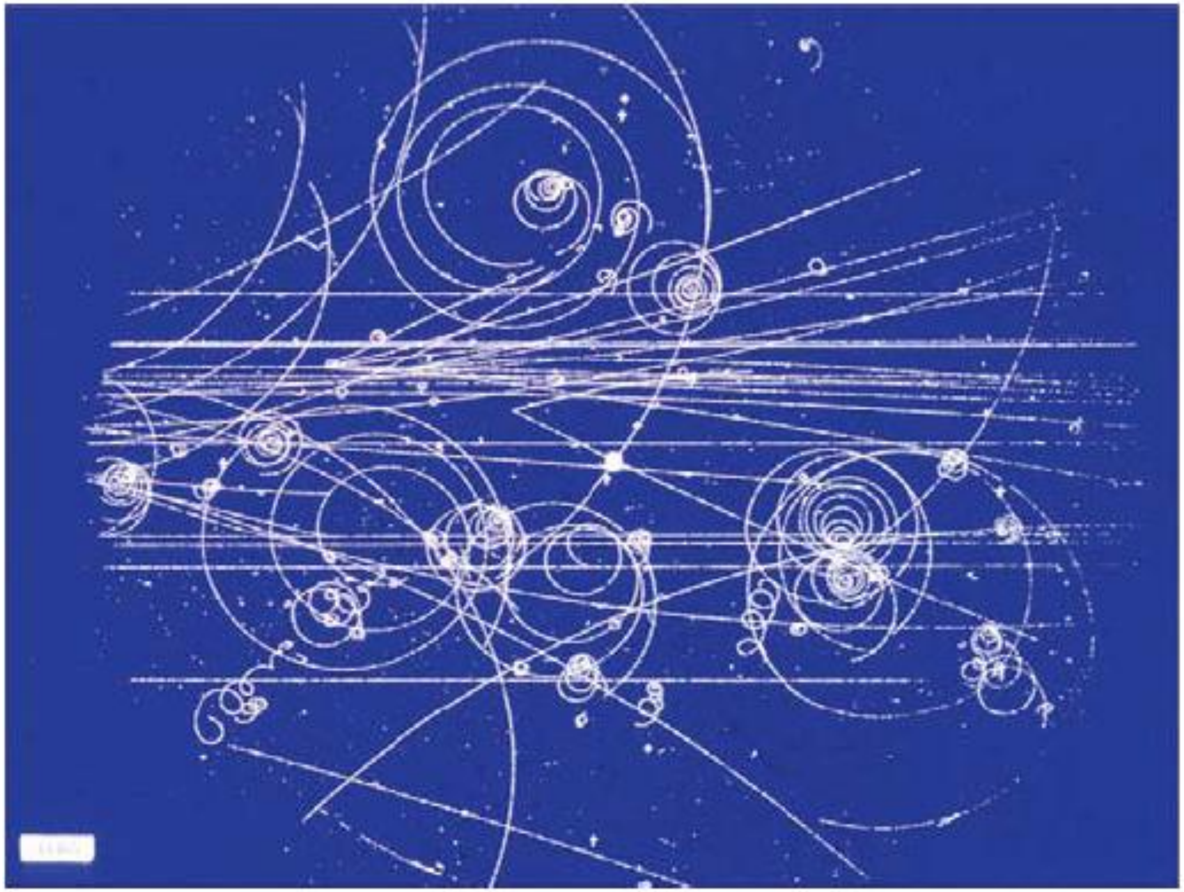
$$M_{fi} = \frac{g_a g_b}{q^2 - m_x^2}$$

★ Derived the basic interaction in **QED** taking into account the spins of the fermions and polarization of the virtual photons:

$$-iM = [\overline{u}(p_3) ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\overline{u}(p_4) ie\gamma^\nu u(p_2)]$$

★ We now have all the elements to perform proper calculations in QED !

Electron-positron annihilation

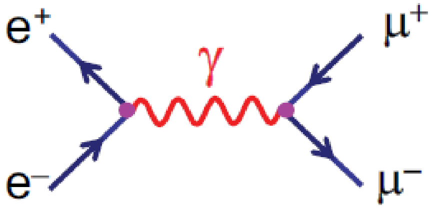


QED calculations

- How to calculate a cross section using QED (e.g. $e^+e^- \rightarrow \mu^+\mu^-$):

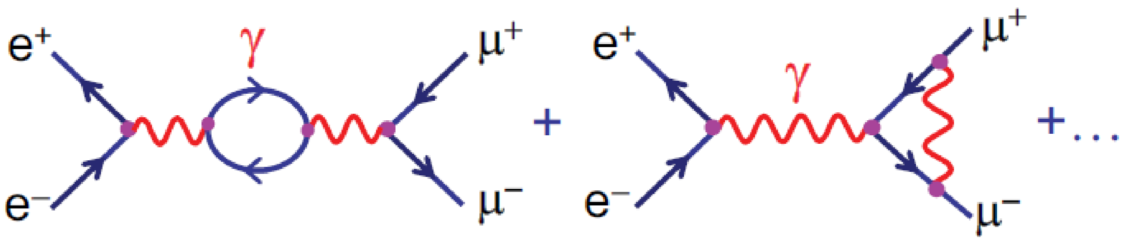
❶ Draw all possible Feynman Diagrams

- For $e^+e^- \rightarrow \mu^+\mu^-$ there is just one lowest order diagram



$$M \propto e^2 \propto \alpha_{em}$$

+ many second order diagrams + ...



$$M \propto e^4 \propto \alpha_{em}^2$$

❷ For each diagram calculate the matrix element using Feynman rules derived in handout 4.

❸ Sum the individual matrix elements (i.e. sum the amplitudes)

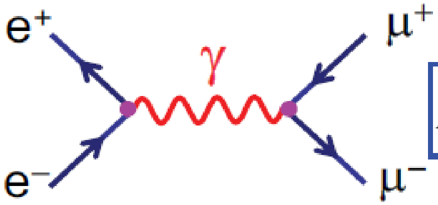
$$M_{fi} = M_1 + M_2 + M_3 + \ldots.$$

- Note: summing amplitudes therefore different diagrams for the same final state can interfere either positively or negatively!

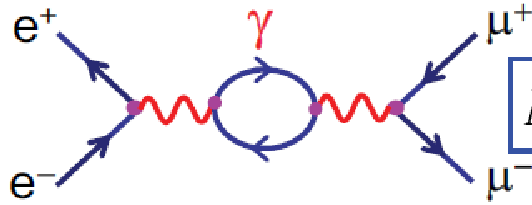
and then square $|M_{fi}|^2 = (M_1 + M_2 + M_3 + ...)(M_1^* + M_2^* + M_3^* + ...)$

➡ this gives the full perturbation expansion in α_{em}

- For QED $\alpha_{em} \sim 1/137$ the lowest order diagram dominates and for most purposes it is sufficient to neglect higher order diagrams.



$M^2 \propto \alpha_{em}^2$



$M^2 \propto \alpha_{em}^4$

❹ Calculate decay rate/cross section using formulae from handout 1.

- e.g. for a decay

$$\Gamma = \frac{p^*}{32\pi^2 m_a^2} \int |M_{fi}|^2 \mathrm{d}\Omega$$

- For scattering in the centre-of-mass frame

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} |M_{fi}|^2 \qquad (1)$$

- For scattering in lab. frame (neglecting mass of scattered particle)

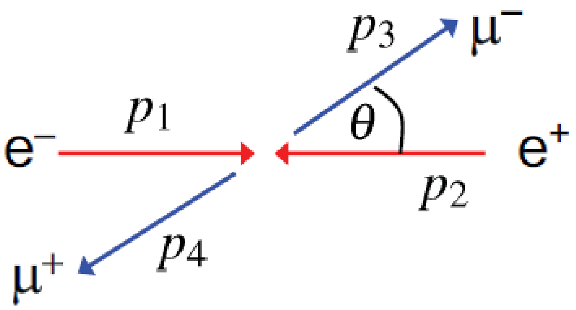
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{ME_1} \right)^2 |M_{fi}|^2$$

Electron Positron Annihilation

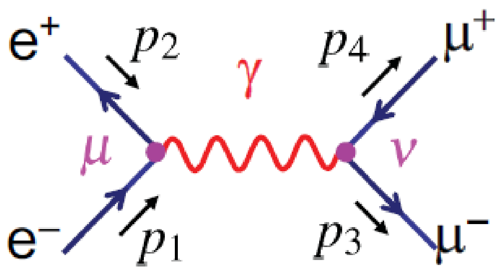
★ **Consider the process:** $e^+e^- \rightarrow \mu^+\mu^-$

- **Work in C.o.M. frame (this is appropriate for most e^+e^- colliders).**

$$p_1 = (E, 0, 0, p) \qquad p_2 = (E, 0, 0, -p)$$
$$p_3 = (E, \vec{p}_f) \qquad p_4 = (E, -\vec{p}_f)$$



- **Only consider the lowest order Feynman diagram:**



- **Feynman rules give:**

$$-iM = [\overline{v}(p_2) ie\gamma^\mu u(p_1)] \frac{-ig_{\mu\nu}}{q^2} [\overline{u}(p_3) ie\gamma^\nu v(p_4)]$$

NOTE:
- **Incoming anti-particle** $\overline{v}$
- **Incoming particle** u
- **Adjoint spinor written first**

- **In the C.o.M. frame have**

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f|}{|\vec{p}_i|} |M_{fi}|^2 \qquad \text{with} \qquad s = (p_1 + p_2)^2 = (E+E)^2 = 4E^2$$

Electron and Muon Currents

- Here $q^2 = (p_1 + p_2)^2 = s$ and matrix element

$$-iM = [\overline{v}(p_2)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_3)ie\gamma^\nu v(p_4)]$$

$$\Rightarrow \quad M = -\frac{e^2}{s}g_{\mu\nu}[\overline{v}(p_2)\gamma^\mu u(p_1)][\overline{u}(p_3)\gamma^\nu v(p_4)]$$

- In handout 2 introduced the four-vector current

$$j^\mu = \overline{\psi}\gamma^\mu\psi$$

which has same form as the two terms in [ ] in the matrix element

- The matrix element can be written in terms of the electron and muon currents

$$(j_e)^\mu = \overline{v}(p_2)\gamma^\mu u(p_1) \quad \text{and} \quad (j_\mu)^\nu = \overline{u}(p_3)\gamma^\nu v(p_4)$$

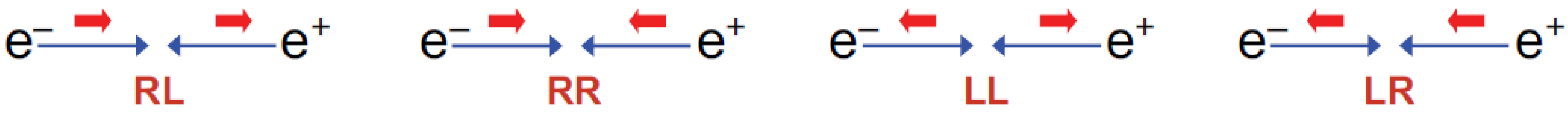
$$\Rightarrow \quad M = -\frac{e^2}{s}g_{\mu\nu}(j_e)^\mu(j_\mu)^\nu$$

$$\boxed{M = -\frac{e^2}{s}j_e.j_\mu}$$

- Matrix element is a four-vector scalar product – confirming it is Lorentz Invariant

Spin in e^+e^- Annihilation

- In general the electron and positron will not be polarized, i.e. there will be equal numbers of positive and negative helicity states
- There are four possible combinations of spins in the initial state !



- Similarly there are four possible helicity combinations in the final state
- In total there are 16 combinations e.g. RL→RR, RL→RL, ...
- To account for these states we need to sum over all 16 possible helicity combinations and then average over the number of <u>initial</u> helicity states:

$$\langle |M|^2 \rangle = \frac{1}{4} \sum_{\text{spins}} |M_i|^2 = \frac{1}{4}\left(|M_{LL\to LL}|^2 + |M_{LL\to LR}|^2 + \ldots\right)$$

★ i.e. need to evaluate:

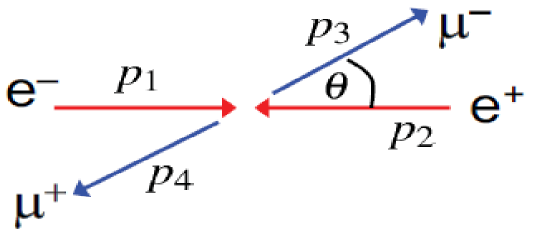
$$M = -\frac{e^2}{s} j_e \cdot j_\mu$$

for all 16 helicity combinations !

★ Fortunately, in the limit $E \gg m_\mu$ only 4 helicity combinations give non-zero matrix elements – we will see that this is an important feature of QED/QCD

- In the C.o.M. frame in the limit $E \gg m$

$$p_1 = (E,0,0,E); \quad p_2 = (E,0,0,-E)$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta);$$
$$p_4 = (E, -\sin\theta, 0, -E\cos\theta)$$



- Left- and right-handed helicity spinors (handout 3) for particles/anti-particles are:

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix} \quad v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \\ -s \\ e^{i\phi}c \end{pmatrix} \quad v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \\ c \\ e^{i\phi}s \end{pmatrix}$$

where $s = \sin\frac{\theta}{2}$; $c = \cos\frac{\theta}{2}$ and $N = \sqrt{E+m}$

- In the limit $E \gg m$ these become:

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}; \; u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix}; \; v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix}; \; v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

- The initial-state electron can either be in a left- or right-handed helicity state

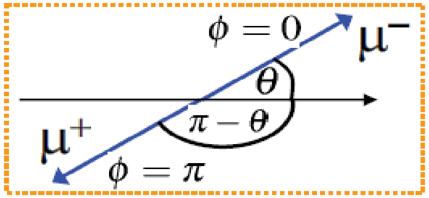
$$u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}; \; u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix};$$

- For the initial state positron $(\theta = \pi)$ can have either:

$$v_\uparrow(p_2) = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix};\; v_\downarrow(p_2) = \sqrt{E}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

- Similarly for the final state μ⁻ which has polar angle θ and choosing $\phi = 0$

$$u_\uparrow(p_3) = \sqrt{E}\begin{pmatrix}c\\s\\c\\s\end{pmatrix};\; u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix}-s\\c\\s\\-c\end{pmatrix};$$



- And for the final state μ⁺ replacing $\theta \to \pi - \theta;\quad \phi \to \pi$ obtain

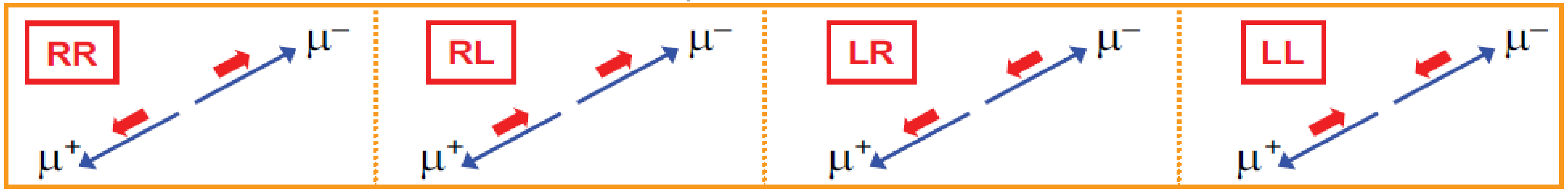
$$v_\uparrow(p_4) = \sqrt{E}\begin{pmatrix}c\\s\\-c\\-s\end{pmatrix};\; v_\downarrow(p_4) = \sqrt{E}\begin{pmatrix}s\\-c\\s\\-c\end{pmatrix};$$

using

$$\sin\left(\tfrac{\pi-\theta}{2}\right) = \cos\tfrac{\theta}{2}$$
$$\cos\left(\tfrac{\pi-\theta}{2}\right) = \sin\tfrac{\theta}{2}$$
$$e^{i\pi} = -1$$

- Wish to calculate the matrix element $M = -\frac{e^2}{s} j_e . j_\mu$

★ first consider the muon current j_μ for 4 possible helicity combinations



The muon current

- Want to evaluate $(j_\mu)^\nu = \overline{u}(p_3)\gamma^\nu v(p_4)$ for all four helicity combinations
- For arbitrary spinors $\psi,\ \phi$ with it is straightforward to show that the components of $\overline{\psi}\gamma^\mu\phi$ are

$$\overline{\psi}\gamma^0\phi = \psi^\dagger\gamma^0\gamma^0\phi = \psi_1^*\phi_1 + \psi_2^*\phi_2 + \psi_3^*\phi_3 + \psi_4^*\phi_4 \quad (3)$$

$$\overline{\psi}\gamma^1\phi = \psi^\dagger\gamma^0\gamma^1\phi = \psi_1^*\phi_4 + \psi_2^*\phi_3 + \psi_3^*\phi_2 + \psi_4^*\phi_1 \quad (4)$$

$$\overline{\psi}\gamma^2\phi = \psi^\dagger\gamma^0\gamma^2\phi = -i(\psi_1^*\phi_4 - \psi_2^*\phi_3 + \psi_3^*\phi_2 - \psi_4^*\phi_1) \quad (5)$$

$$\overline{\psi}\gamma^3\phi = \psi^\dagger\gamma^0\gamma^3\phi = \psi_1^*\phi_3 - \psi_2^*\phi_4 + \psi_3^*\phi_1 - \psi_4^*\phi_2 \quad (6)$$

- Consider the $\mu_R^-\mu_L^+$ combination using $\psi = u_\uparrow \quad \phi = v_\downarrow$

with $v_\downarrow = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}; \quad u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix};$

$$\overline{u}_\uparrow(p_3)\gamma^0 v_\downarrow(p_4) = E(cs - sc + cs - sc) = 0$$

➡ $$\overline{u}_\uparrow(p_3)\gamma^1 v_\downarrow(p_4) = E(-c^2 + s^2 - c^2 + s^2) = 2E(s^2 - c^2) = -2E\cos\theta$$

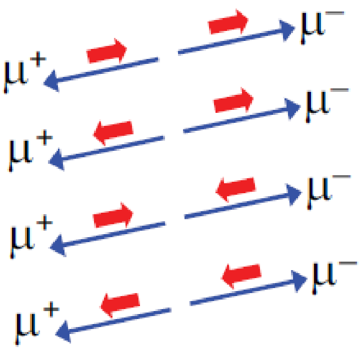
$$\overline{u}_\uparrow(p_3)\gamma^2 v_\downarrow(p_4) = -iE(-c^2 - s^2 - c^2 - s^2) = 2iE$$

$$\overline{u}_\uparrow(p_3)\gamma^3 v_\downarrow(p_4) = E(cs + sc + cs + sc) = 4Esc = 2E\sin\theta$$

- Hence the four-vector muon current for the RL combination is

$$\bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) \quad = \quad 2E(0, -\cos\theta, i, \sin\theta)$$

- The results for the 4 helicity combinations (obtained in the same manner) are:



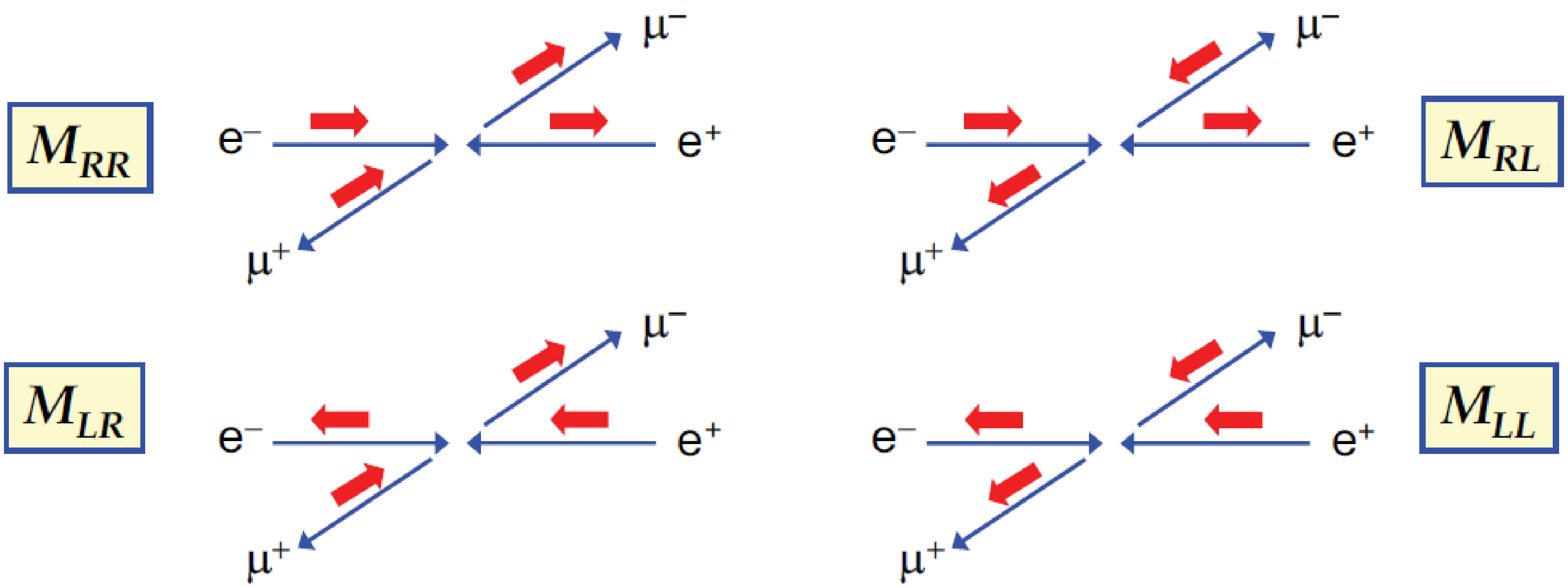
$$\begin{aligned}
\bar{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) &= 2E(0, -\cos\theta, i, \sin\theta) && \text{RL} \\
\bar{u}_\uparrow(p_3)\gamma^\nu v_\uparrow(p_4) &= (0,0,0,0) && \text{RR} \\
\bar{u}_\downarrow(p_3)\gamma^\nu v_\downarrow(p_4) &= (0,0,0,0) && \text{LL} \\
\bar{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) &= 2E(0, -\cos\theta, -i, \sin\theta) && \text{LR}
\end{aligned}$$

★ IN THE LIMIT $E \gg m$ only two helicity combinations are non-zero !

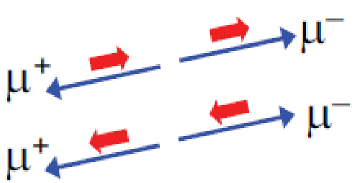
- This is an important feature of QED. It applies equally to QCD.
- In the Weak interaction only one helicity combination contributes.
- The origin of this will be discussed in the last part of this lecture
- But as a consequence of the 16 possible helicity combinations only four given non-zero matrix elements

Electron Positron Annihilation cont.

★ For $e^+e^- \rightarrow \mu^+\mu^-$ now only have to consider the 4 matrix elements:



• Previously we derived the muon currents for the allowed helicities:



$$
\begin{aligned}
\mu_R^-\mu_L^+ &: \quad \overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0,-\cos\theta,i,\sin\theta) \\
\mu_L^-\mu_R^+ &: \quad \overline{u}_\downarrow(p_3)\gamma^\nu v_\uparrow(p_4) = 2E(0,-\cos\theta,-i,\sin\theta)
\end{aligned}
$$

• Now need to consider the electron current

The electron current

- The incoming electron and positron spinors (L and R helicities) are:

$$u_\uparrow = \sqrt{E}\begin{pmatrix}1\\0\\1\\0\end{pmatrix};\; u_\downarrow = \sqrt{E}\begin{pmatrix}0\\1\\0\\-1\end{pmatrix};\quad v_\uparrow = \sqrt{E}\begin{pmatrix}1\\0\\-1\\0\end{pmatrix};\; v_\downarrow = \sqrt{E}\begin{pmatrix}0\\1\\0\\1\end{pmatrix}$$

- The electron current can either be obtained from equations (3)-(6) as before or it can be obtained directly from the expressions for the muon current.

$$(j_e)^\mu = \overline{v}(p_2)\gamma^\mu u(p_1) \qquad\qquad (j_\mu)^\mu = \overline{u}(p_3)\gamma^\mu v(p_4)$$

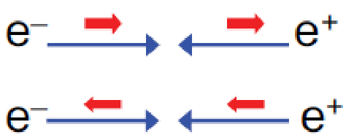
- Taking the Hermitian conjugate of the muon current gives

$$\begin{aligned}
[\overline{u}(p_3)\gamma^\mu v(p_4)]^\dagger &= \left[u(p_3)^\dagger\gamma^0\gamma^\mu v(p_4)\right]^\dagger \\
&= v(p_4)^\dagger\gamma^{\mu\dagger}\gamma^{0\dagger}u(p_3) && (AB)^\dagger = B^\dagger A^\dagger \\
&= v(p_4)^\dagger\gamma^{\mu\dagger}\gamma^0 u(p_3) && \gamma^{0\dagger} = \gamma^0 \\
&= v(p_4)^\dagger\gamma^0\gamma^\mu u(p_3) && \gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu \\
&= \overline{v}(p_4)\gamma^\mu u(p_3)
\end{aligned}$$

- Taking the complex conjugate of the muon currents for the two non-zero helicity configurations:

$$\begin{aligned}\overline{v}_{\downarrow}(p_4)\gamma^{\mu}u_{\uparrow}(p_3) &= \left[\overline{u}_{\uparrow}(p_3)\gamma^{\nu}v_{\downarrow}(p_4)\right]^{*} = 2E(0,-\cos\theta,-i,\sin\theta)\\ \overline{v}_{\uparrow}(p_4)\gamma^{\mu}u_{\downarrow}(p_3) &= \left[\overline{u}_{\downarrow}(p_3)\gamma^{\nu}v_{\uparrow}(p_4)\right]^{*} = 2E(0,-\cos\theta,i,\sin\theta)\end{aligned}$$

To obtain the electron currents we simply need to set $\theta = 0$

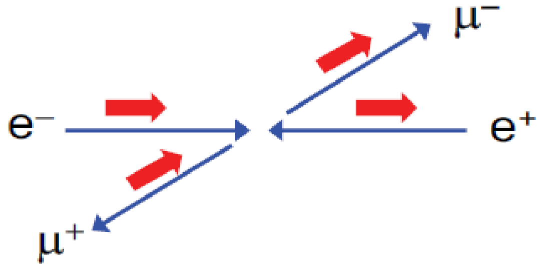


$$\boxed{\begin{aligned} e^-_R e^+_L &: \quad \overline{v}_{\downarrow}(p_2)\gamma^{\nu}u_{\uparrow}(p_1) = 2E(0,-1,-i,0)\\ e^-_L e^+_R &: \quad \overline{v}_{\uparrow}(p_2)\gamma^{\nu}u_{\downarrow}(p_1) = 2E(0,-1,i,0)\end{aligned}}$$

Matrix element calculation

- We can now calculate $M = -\frac{e^2}{s} j_e . j_\mu$ for the four possible helicity combinations.

e.g. the matrix element for $e^-_R e^+_L \rightarrow \mu^-_R \mu^+_L$ which will denote M_{RR}



Here the first subscript refers to the helicity of the e⁻ and the second to the helicity of the μ⁻. Don't need to specify other helicities due to "helicity conservation", only certain chiral combinations are non-zero.

★ Using:

$$e^-_R e^+_L : \quad (j_e)^\mu = \overline{v}_\downarrow(p_2)\gamma^\mu u_\uparrow(p_1) = 2E(0,-1,-i,0)$$

$$\mu^-_R \mu^+_L : \quad (j_\mu)^\nu = \overline{u}_\uparrow(p_3)\gamma^\nu v_\downarrow(p_4) = 2E(0,-\cos\theta,i,\sin\theta)$$

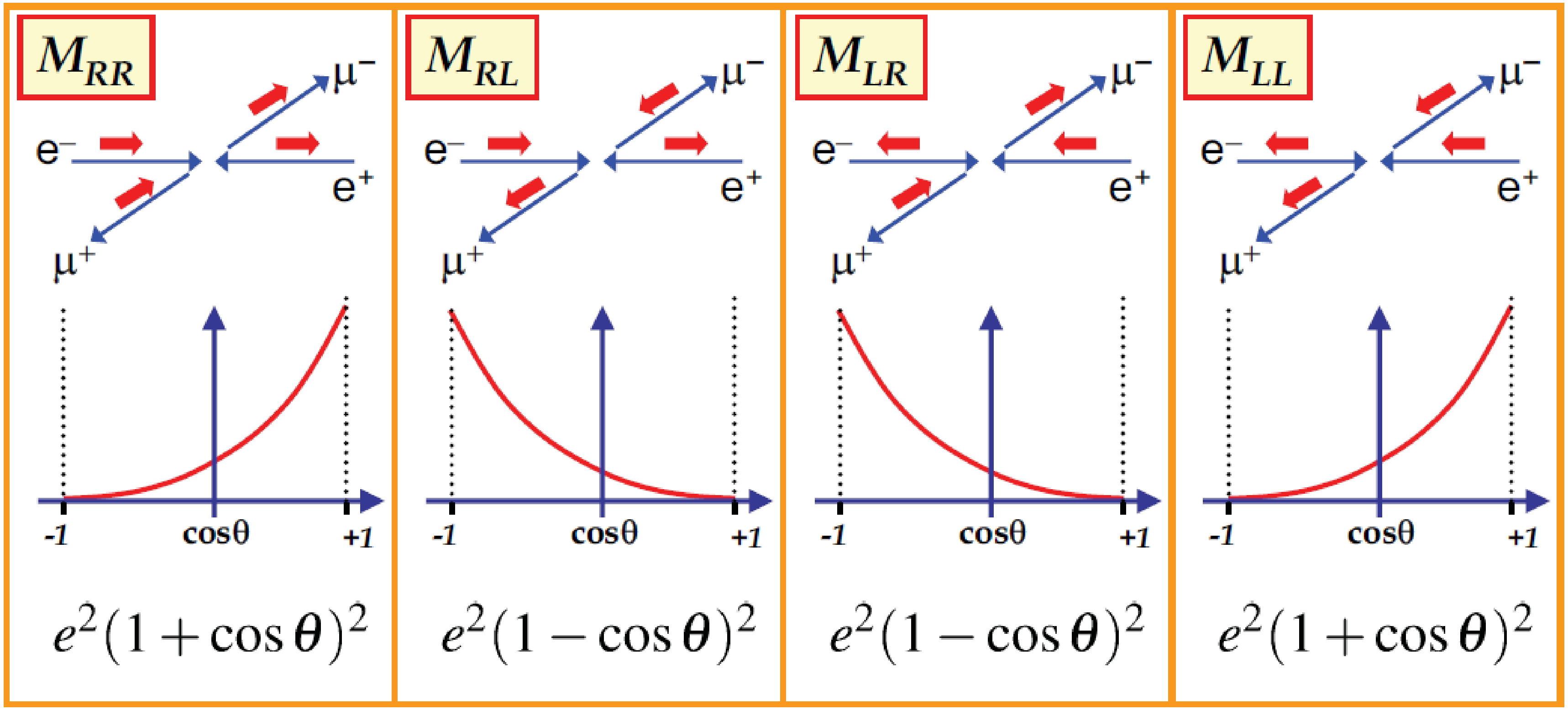
gives

$$\begin{aligned} M_{RR} &= -\frac{e^2}{s}[2E(0,-1,-i,0)].[2E(0,-\cos\theta,i,\sin\theta)] \\ &= -e^2(1+\cos\theta) \\ &= -4\pi\alpha(1+\cos\theta) \end{aligned}$$

where $\alpha = e^2/4\pi \approx 1/137$

Similarly

$$|M_{RR}|^2 = |M_{LL}|^2 = (4\pi\alpha)^2(1+\cos\theta)^2$$
$$|M_{RL}|^2 = |M_{LR}|^2 = (4\pi\alpha)^2(1-\cos\theta)^2$$



- Assuming that the incoming electrons and positrons are unpolarized, all 4 possible initial helicity states are equally likely.

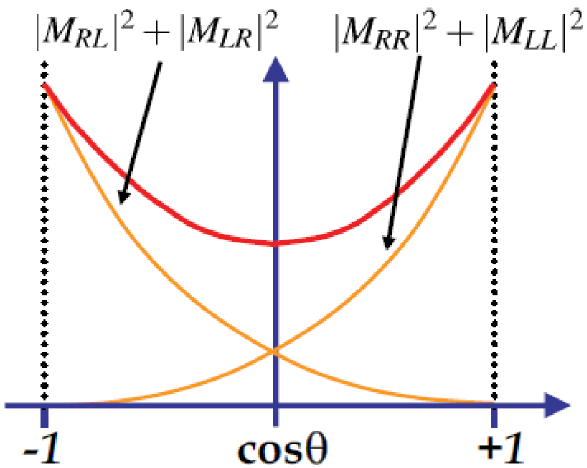
Differential cross-section

- The cross section is obtained by averaging over the initial spin states and summing over the final spin states:

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \times \frac{1}{64\pi^2 s}\left(|M_{RR}|^2 + |M_{RL}|^2 + |M_{LR}|^2 + |M_{LL}^2|\right)$$

$$= \frac{(4\pi\alpha)^2}{256\pi^2 s}\left(2(1+\cos\theta)^2 + 2(1-\cos\theta)^2\right)$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)}$$

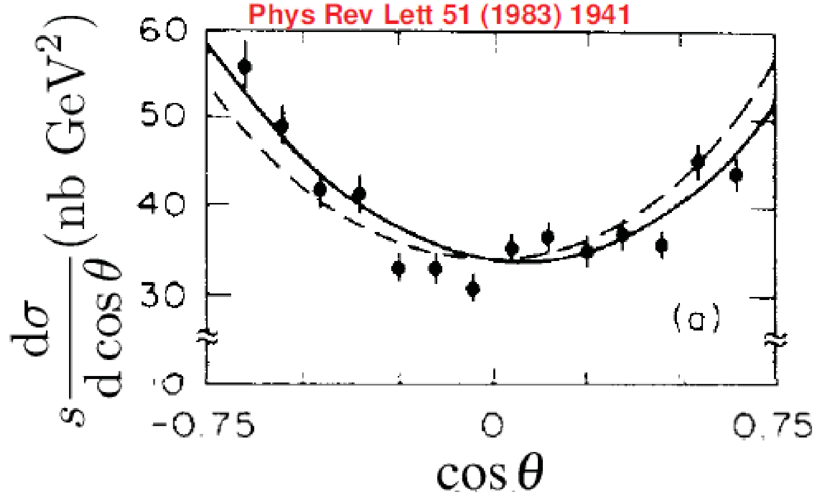


Example:

$e^+e^- \rightarrow \mu^+\mu^-$

$\sqrt{s} = 29$ GeV

Mark II Expt., M.E.Levi et al., Phys Rev Lett 51 (1983) 1941



- - - - - pure QED, $O(\alpha^3)$

——— QED plus Z contribution

Angular distribution becomes slightly asymmetric in higher order QED or when Z contribution is included

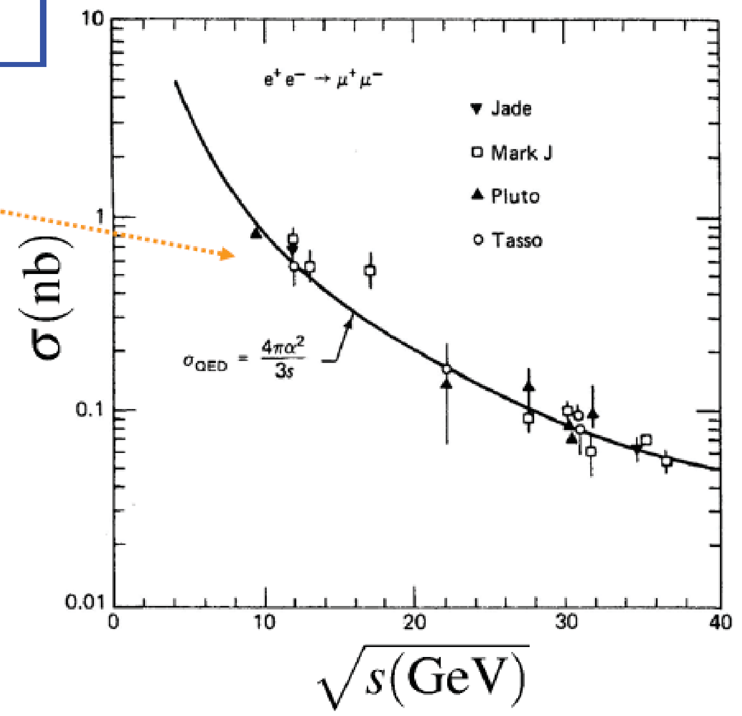
- The total cross section is obtained by integrating over $\theta,\ \phi$ using

$$\int (1+\cos^2\theta)\mathrm{d}\Omega = 2\pi\int_{-1}^{+1}(1+\cos^2\theta)\mathrm{d}\cos\theta = \frac{16\pi}{3}$$

giving the **QED** total cross-section for the process $e^+e^- \to \mu^+\mu^-$

$$\sigma = \frac{4\pi\alpha^2}{3s}$$

★ Lowest order cross section calculation provides a good description of the data !



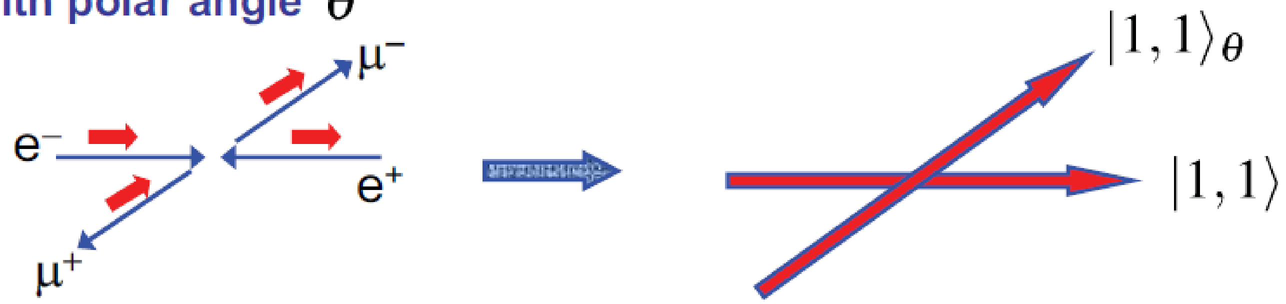
This is an impressive result. From first principles we have arrived at an expression for the electron-positron annihilation cross section which is good to **1%**

Spin considerations (E >> m)

★ The angular dependence of the QED electron-positron matrix elements can be understood in terms of angular momentum

- Because of the allowed helicity states, the electron and positron interact in a spin state with $S_z = \pm 1$, i.e. in a total spin 1 state aligned along the z axis: $|1,+1\rangle$ or $|1,-1\rangle$
- Similarly the muon and anti-muon are produced in a total spin 1 state aligned along an axis with polar angle θ

e.g. M_{RR}



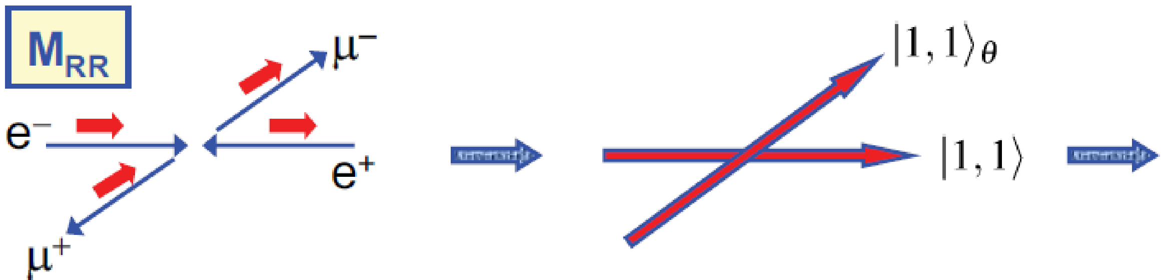
- Hence $M_{RR} \propto \langle \psi | 1,1 \rangle$ where ψ corresponds to the spin state, $|1,1\rangle_\theta$, of the muon pair.
- To evaluate this need to express $|1,1\rangle_\theta$ in terms of eigenstates of S_z
- In the appendix it is shown that:

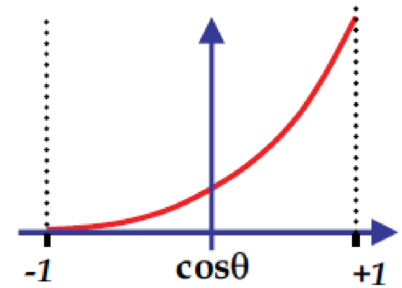
$$|1,1\rangle_\theta = \tfrac{1}{2}(1-\cos\theta)|1,-1\rangle + \tfrac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \tfrac{1}{2}(1+\cos\theta)|1,+1\rangle$$

- Using the wave-function for a spin 1 state along an axis at angle θ

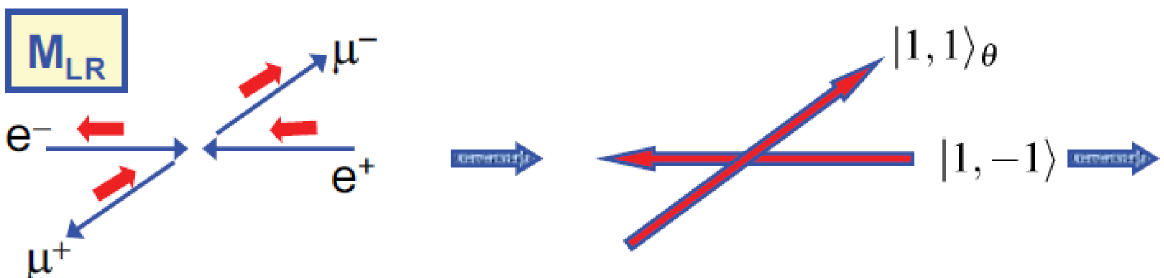
$$\psi = |1,1\rangle_\theta = \tfrac{1}{2}(1-\cos\theta)|1,-1\rangle + \tfrac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \tfrac{1}{2}(1+\cos\theta)|1,+1\rangle$$

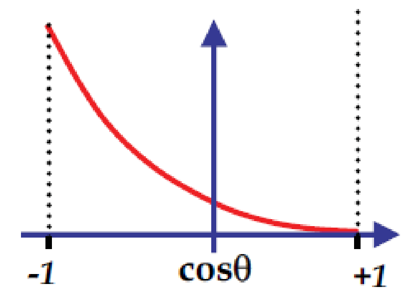
can immediately understand the angular dependence





$$|M_{RR}|^2 \propto |\langle\psi|1,+1\rangle|^2 = \tfrac{1}{4}(1+\cos\theta)^2$$



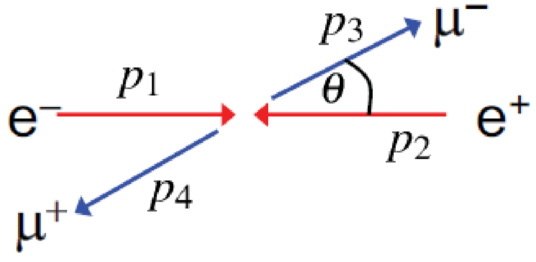


$$|M_{LR}|^2 \propto |\langle\psi|1,-1\rangle|^2 = \tfrac{1}{4}(1-\cos\theta)^2$$

Lorentz Invariant form of Matrix Element

- Before concluding this discussion, note that the spin-averaged Matrix Element derived above is written in terms of the muon angle in the C.o.M. frame.

$$\begin{aligned}\langle|M_{fi}|^2\rangle &= \frac{1}{4}\times(|M_{RR}|^2+|M_{RL}|^2+|M_{LR}|^2+|M_{LL}^2|) \\ &= \frac{1}{4}e^4(2(1+\cos\theta)^2+2(1-\cos\theta)^2) \\ &= e^4(1+\cos^2\theta)\end{aligned}$$



- The matrix element is Lorentz Invariant (scalar product of 4-vector currents) and it is desirable to write it in a frame-independent form, i.e. express in terms of Lorentz Invariant 4-vector scalar products
- In the C.o.M. $p_1 = (E,0,0,E) \qquad p_2 = (E,0,0,-E)$

$$p_3 = (E, E\sin\theta, 0, E\cos\theta) \qquad p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$

giving: $p_1.p_2 = 2E^2; \quad p_1.p_3 = E^2(1-\cos\theta); \quad p_1.p_4 = E^2(1+\cos\theta)$

- Hence we can write

$$\langle|M_{fi}|^2\rangle = 2e^4\frac{(p_1.p_3)^2+(p_1.p_4)^2}{(p_1.p_2)^2}$$

$$\equiv 2e^4\left(\frac{t^2+u^2}{s^2}\right)$$

★Valid in any frame !

Chirality

- The helicity eigenstates for a particle/anti-particle for $E \gg m$ are:

$$u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix};\ u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi} \end{pmatrix};\ v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi} \end{pmatrix};\ v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi} \end{pmatrix}$$

where $s = \sin\frac{\theta}{2};\quad c = \cos\frac{\theta}{2}$

- Define the matrix

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

- <u>In the limit</u> $E \gg m$ the helicity states are also eigenstates of γ^5

$$\gamma^5 u_\uparrow = +u_\uparrow;\ \gamma^5 u_\downarrow = -u_\downarrow;\ \gamma^5 v_\uparrow = -v_\uparrow;\ \gamma^5 v_\downarrow = +v_\downarrow$$

★ <u>In general</u>, define the eigenstates of γ^5 as LEFT and RIGHT HANDED <u>CHIRAL</u> states $u_R;\quad u_L;\quad v_R;\quad v_L$

i.e. $\gamma^5 u_R = +u_R;\ \gamma^5 u_L = -u_L;\ \gamma^5 v_R = -v_R;\ \gamma^5 v_L = +v_L$

- In the LIMIT $E \gg m$ (and ONLY IN THIS LIMIT):

$$u_R \equiv u_\uparrow;\quad u_L \equiv u_\downarrow;\quad v_R \equiv v_\uparrow;\quad v_L \equiv v_\downarrow$$

★ This is a subtle but important point: in general the HELICITY and CHIRAL eigenstates are not the same. It is only in the ultra-relativistic limit that the chiral eigenstates correspond to the helicity eigenstates.

★ Chirality is an import concept in the structure of QED, and any interaction of the form $\overline{u}\gamma^{\nu} u$

- In general, the eigenstates of the chirality operator are:

$$\gamma^5 u_R = +u_R; \quad \gamma^5 u_L = -u_L; \quad \gamma^5 v_R = -v_R; \quad \gamma^5 v_L = +v_L$$

- Define the projection operators:

$$P_R = \tfrac{1}{2}(1+\gamma^5); \qquad P_L = \tfrac{1}{2}(1-\gamma^5)$$

- The projection operators, project out the chiral eigenstates

$$P_R u_R = u_R; \quad P_R u_L = 0; \quad P_L u_R = 0; \quad P_L u_L = u_L$$
$$P_R v_R = 0; \quad P_R v_L = v_L; \quad P_L v_R = v_R; \quad P_L v_L = 0$$

- Note P_R projects out right-handed particle states and left-handed anti-particle states
- We can then write any spinor in terms of it left and right-handed chiral components:

$$\psi = \psi_R + \psi_L = \tfrac{1}{2}(1+\gamma^5)\psi + \tfrac{1}{2}(1-\gamma^5)\psi$$

Chirality in QED

- In QED the basic interaction between a fermion and photon is:

$$ie\overline{\psi}\gamma^{\mu}\phi$$

- Can decompose the spinors in terms of Left and Right-handed chiral components:

$$\begin{aligned} ie\overline{\psi}\gamma^{\mu}\phi &= ie(\overline{\psi}_L + \overline{\psi}_R)\gamma^{\mu}(\phi_R + \phi_L) \\ &= ie(\overline{\psi}_R\gamma^{\mu}\phi_R + \overline{\psi}_R\gamma^{\mu}\phi_L + \overline{\psi}_L\gamma^{\mu}\phi_R + \overline{\psi}_L\gamma^{\mu}\phi_L) \end{aligned}$$

- Using the properties of γ^5

$$(\gamma^5)^2 = 1; \quad \gamma^{5\dagger} = \gamma^5; \quad \gamma^5\gamma^{\mu} = -\gamma^{\mu}\gamma^5$$

it is straightforward to show

$$\overline{\psi}_R\gamma^{\mu}\phi_L = 0; \quad \overline{\psi}_L\gamma^{\mu}\phi_R = 0$$

★ Hence only certain combinations of chiral eigenstates contribute to the interaction. This statement is **ALWAYS** true.

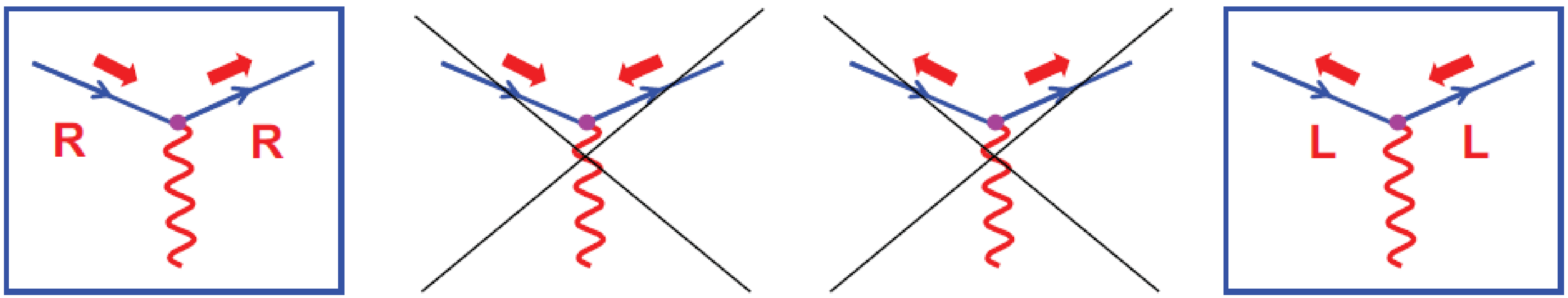
- For $E \gg m$, the chiral and helicity eigenstates are equivalent. This implies that for $E \gg m$ only certain helicity combinations contribute to the QED vertex ! This is why previously we found that for two of the four helicity combinations for the muon current were zero

Allowed QED Helicity Combinations

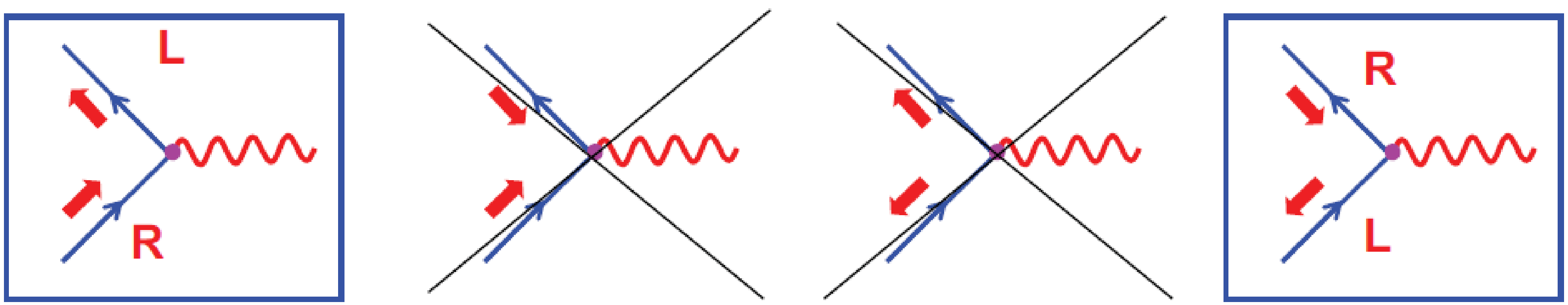
- In the ultra-relativistic limit the helicity eigenstates ≡ chiral eigenstates
- In this limit, the only non-zero helicity combinations in QED are:

Scattering:

"Helicity conservation"



Annihilation:



Summary

★ In the centre-of-mass frame the $e^+e^- \rightarrow \mu^+\mu^-$ differential cross-section is

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)$$

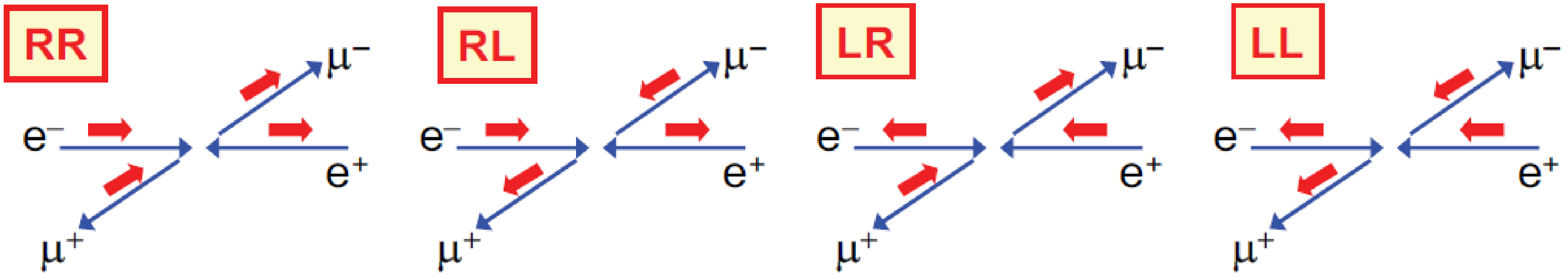
NOTE: neglected masses of the muons, i.e. assumed $E \gg m_\mu$

★ In QED only certain combinations of LEFT- and RIGHT-HANDED CHIRAL states give non-zero matrix elements

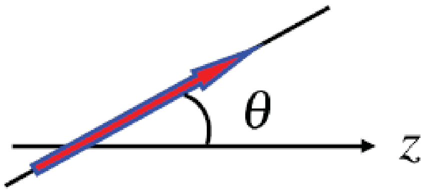
★ CHIRAL states defined by chiral projection operators

$$P_R = \tfrac{1}{2}(1+\gamma^5); \qquad P_L = \tfrac{1}{2}(1-\gamma^5)$$

★ In limit $E \gg m$ the chiral eigenstates correspond to the HELICITY eigenstates and only certain HELICITY combinations give non-zero matrix elements



Appendix: Spin 1 Rotation Matrices



- Consider the spin-1 state with spin +1 along the axis defined by unit vector

$$\vec{n} = (\sin\theta, 0, \cos\theta)$$

- Spin state is an eigenstate of $\vec{n}.\vec{S}$ with eigenvalue +1

$$(\vec{n}.\vec{S})|\psi\rangle = +1|\psi\rangle \quad \text{(A1)}$$

- Express in terms of linear combination of spin 1 states which are eigenstates of S_z

$$|\psi\rangle = \alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle$$

with $\alpha^2 + \beta^2 + \gamma^2 = 1$

- (A1) becomes

$$(\sin\theta S_x + \cos\theta S_z)(\alpha|1,1\rangle + \beta|1,0\rangle + \gamma|1,-1\rangle) = \alpha|1,1\rangle + \beta|1,0\rangle\gamma|1,-1\rangle \quad \text{(A2)}$$

- Write S_x in terms of ladder operators $S_x = \frac{1}{2}(S_+ + S_-)$

where $S_+|1,1\rangle = 0 \quad S_+|1,0\rangle = \sqrt{2}|1,1\rangle \quad S_+|1,-1\rangle = \sqrt{2}|1,0\rangle$

$S_-|1,1\rangle = \sqrt{2}|1,0\rangle \quad S_-|1,0\rangle = \sqrt{2}|1,-1\rangle \quad S_-|1,-1\rangle = 0$

- from which we find
$$S_x|1,1\rangle = \tfrac{1}{\sqrt{2}}|1,0\rangle$$
$$S_x|1,0\rangle = \tfrac{1}{\sqrt{2}}(|1,1\rangle + |1,-1\rangle)$$
$$S_x|1,-1\rangle = \tfrac{1}{\sqrt{2}}|1,0\rangle$$
- (A2) becomes
$$\sin\theta\left[\frac{\alpha}{\sqrt{2}}|1,0\rangle + \frac{\beta}{\sqrt{2}}|1,-1\rangle + \frac{\beta}{\sqrt{2}}|1,1\rangle + \frac{\gamma}{\sqrt{2}}|1,0\rangle\right] + \alpha\cos\theta|1,1\rangle - \gamma\cos\theta|1,-1\rangle = \alpha|1,1\rangle + \beta|1,0\rangle\gamma|1,-1\rangle$$
- which gives
$$\left.\begin{aligned} \beta\frac{\sin\theta}{\sqrt{2}} + \alpha\cos\theta &= \alpha \\ (\alpha+\gamma)\frac{\sin\theta}{\sqrt{2}} &= \beta \\ \beta\frac{\sin\theta}{\sqrt{2}} - \gamma\cos\theta &= \gamma \end{aligned}\right\}$$
- using $\alpha^2+\beta^2+\gamma^2 = 1$ the above equations yield
$$\alpha = \tfrac{1}{\sqrt{2}}(1+\cos\theta) \qquad \beta = \tfrac{1}{\sqrt{2}}\sin\theta \qquad \gamma = \tfrac{1}{\sqrt{2}}(1-\cos\theta)$$
- hence
$$\psi = \tfrac{1}{2}(1-\cos\theta)|1,-1\rangle + \tfrac{1}{\sqrt{2}}\sin\theta|1,0\rangle + \tfrac{1}{2}(1+\cos\theta)|1,+1\rangle$$

- The coefficients α, β, γ are examples of what are known as quantum mechanical rotation matrices. The express how angular momentum eigenstate in a particular direction is expressed in terms of the eigenstates defined in a different direction

$$d^{j}_{m',m}(\theta)$$

- For spin-1 $(j=1)$ we have just shown that

$$d^{1}_{1,1}(\theta) = \tfrac{1}{2}(1+\cos\theta) \qquad d^{1}_{0,1}(\theta) = \tfrac{1}{\sqrt{2}}\sin\theta \qquad d^{1}_{-1,1}(\theta) = \tfrac{1}{2}(1-\cos\theta)$$

- For spin-1/2 it is straightforward to show

$$d^{\frac{1}{2}}_{\frac{1}{2},\frac{1}{2}}(\theta) = \cos\frac{\theta}{2} \qquad d^{\frac{1}{2}}_{-\frac{1}{2},\frac{1}{2}}(\theta) = \sin\frac{\theta}{2}$$