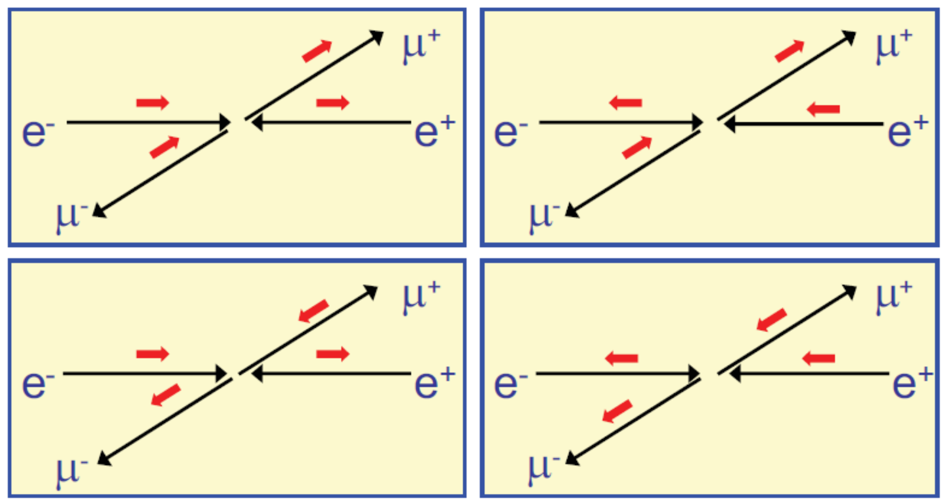
Elementary Particle Physics: theory and experiments

The Dirac equation



Follow the course/slides from M. A. Thomson lectures at Cambridge University

Prof. dr hab. Elżbieta Richter-Wąs

Non-relativistic QM (revision)

- For particle physics need a relativistic formulation of quantum mechanics. But first take a few moments to review the non-relativistic formulation QM
- Take as the starting point non-relativistic energy:

$$E = T + V = \frac{\vec{p}^2}{2m} + V$$

- In QM we identify the energy and momentum operators:

$$\vec{p} \to -i\vec{\nabla}, \quad E \to i\frac{\partial}{\partial t}$$

which gives the time dependent Schrödinger equation (take V=0 for simplicity)

$$-\frac{1}{2m}\vec{\nabla}^2\psi = i\frac{\partial\psi}{\partial t} \qquad \text{(S1)}$$

with plane wave solutions: $\psi = Ne^{i(\vec{p}.\vec{r}-Et)}$ where $\begin{cases} -i\nabla\psi = \vec{p}\psi \\ i\frac{\partial\psi}{\partial t} = E\psi \end{cases}$

- The SE is **first order** in the time derivatives and **second order** in spatial derivatives – and is manifestly not Lorentz invariant.
- In what follows we will use probability density/current extensively. For the non-relativistic case these are derived as follows

(S1)* ➡ $$-\frac{1}{2m}\vec{\nabla}^2\psi^* = -i\frac{\partial\psi^*}{\partial t} \qquad \text{(S2)}$$

$$\psi^* \times (\mathbf{S1}) - \psi \times (\mathbf{S2}): \quad -\frac{1}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) = i\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right)$$

$$-\frac{1}{2m}\vec{\nabla}.\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) = i\frac{\partial}{\partial t}(\psi^*\psi)$$

- Which by comparison with the continuity equation

$$\vec{\nabla}.\vec{j} + \frac{\partial\rho}{\partial t} = 0$$

leads to the following expressions for probability density and current:

$$\rho = \psi^*\psi = |\psi|^2 \qquad \vec{j} = \frac{1}{2mi}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$$

- For a plane wave $\psi = Ne^{i(\vec{p}.\vec{r} - Et)}$

$$\rho = |N|^2 \quad \text{and} \quad \vec{j} = |N|^2\frac{\vec{p}}{m} = |N|^2\vec{v}$$

★ The number of particles per unit volume is $|N|^2$

★ For $|N|^2$ particles per unit volume moving at velocity $\vec{v}$, have $|N|^2|\vec{v}|$ passing through a unit area per unit time (particle flux). Therefore $\vec{j}$ is a vector in the particle's direction with magnitude equal to the flux.

The Klein-Gordon equation

- Applying $\vec{p} \to -i\vec{\nabla}, \quad E \to i\partial/\partial t$ to the relativistic equation for energy:

$$E^2 = |\vec{p}|^2 + m^2 \qquad \text{(KG1)}$$

gives the **Klein-Gordon equation:**

$$\frac{\partial^2 \psi}{\partial t^2} = \vec{\nabla}^2 \psi - m^2 \psi \qquad \text{(KG2)}$$

- Using $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \Rightarrow \partial^\mu \partial_\mu \equiv \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$

KG can be expressed compactly as

$$\boxed{(\partial^\mu \partial_\mu + m^2)\psi = 0} \qquad \text{(KG3)}$$

- For plane wave solutions, $\psi = Ne^{i(\vec{p}.\vec{r} - Et)}$, the KG equation gives:

$$-E^2 \psi = -|\vec{p}|^2 \psi - m^2 \psi$$

$$\Rightarrow \quad E = \pm\sqrt{|\vec{p}|^2 + m^2}$$

★ Not surprisingly, the KG equation has negative energy solutions – this is just what we started with in eq. KG1

♦ Historically the **–ve** energy solutions were viewed as problematic. **But** for the KG there is also a problem with the **probability density…**

- Proceeding as before to calculate the probability and current densities:

(KG2)* $$\frac{\partial^2 \psi^*}{\partial t^2} = \vec{\nabla}^2 \psi^* - m^2 \psi^*$$ (KG4)

$\psi^* \times (\mathbf{KG2}) - \psi \times (\mathbf{KG4}):$

$$\psi^* \frac{\partial^2 \psi}{\partial t^2} - \psi \frac{\partial^2 \psi^*}{\partial t^2} = \psi^*(\nabla^2 \psi - m^2 \psi) - \psi(\nabla^2 \psi^* - m^2 \psi^*)$$

$$\frac{\partial}{\partial t}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) = \vec{\nabla}.(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*)$$

- Which, again, by comparison with the continuity equation allows us to identify

$$\rho = i\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) \quad \text{and} \quad \vec{j} = i(\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*)$$

- For a plane wave $\psi = N e^{i(\vec{p}.\vec{r} - Et)}$

$$\rho = 2E|N|^2 \quad \text{and} \quad \vec{j} = |N|^2 \vec{p}$$

★ Particle densities are proportional to E. We might have anticipated this from the previous discussion of Lorentz invariant phase space (i.e. density of $1/V$ in the particles rest frame will appear as E/V in a frame where the particle has energy E due to length contraction).

The Dirac equation

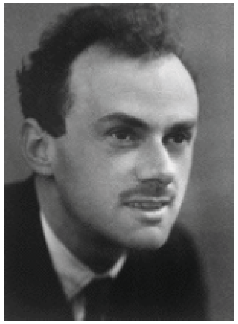
★ Historically, it was thought that there were **two** main problems with the Klein-Gordon equation:

- Negative energy solutions
- The negative particle densities associated with these solutions

$$\rho = 2E|N|^2$$

★ We now know that in Quantum Field Theory these problems are overcome and the KG equation is used to describe **spin-0** particles (inherently single particle description → multi-particle quantum excitations of a scalar field).

Nevertheless:



★ These problems motivated **Dirac (1928)** to search for a different formulation of relativistic quantum mechanics in which all particle densities are positive.

★ The resulting wave equation had solutions which not only solved this problem but also fully describe the intrinsic spin and magnetic moment of the electron!

The Dirac equation

- Schrödinger eqn: $-\frac{1}{2m}\vec{\nabla}^2\psi = i\frac{\partial\psi}{\partial t}$ 1st order in $\partial/\partial t$; 2nd order in $\partial/\partial x, \partial/\partial y, \partial/\partial z$
- Klein-Gordon eqn: $(\partial^\mu\partial_\mu + m^2)\psi = 0$ 2nd order throughout
- Dirac looked for an alternative which was 1st order throughout:

$$\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\frac{\partial\psi}{\partial t} \qquad \textbf{(D1)}$$

where $\hat{H}$ is the Hamiltonian operator and, as usual, $\vec{p} = -i\vec{\nabla}$

- Writing (D1) in full:

$$\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi = \left(i\frac{\partial}{\partial t}\right)\psi$$

"squaring" this equation gives

$$\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right)\psi = -\frac{\partial^2\psi}{\partial t^2}$$

- Which can be expanded in gory details as…

$$
\begin{aligned}
-\frac{\partial^2\psi}{\partial t^2} \quad = \quad & -\alpha_x^2\frac{\partial^2\psi}{\partial x^2}-\alpha_y^2\frac{\partial^2\psi}{\partial y^2}-\alpha_z^2\frac{\partial^2\psi}{\partial z^2}+\beta^2 m^2\psi \\
& -(\alpha_x\alpha_y+\alpha_y\alpha_x)\frac{\partial^2\psi}{\partial x\partial y}-(\alpha_y\alpha_z+\alpha_z\alpha_y)\frac{\partial^2\psi}{\partial y\partial z}-(\alpha_z\alpha_x+\alpha_x\alpha_z)\frac{\partial^2\psi}{\partial z\partial x} \\
& -(\alpha_x\beta+\beta\alpha_x)m\frac{\partial\psi}{\partial x}-(\alpha_y\beta+\beta\alpha_y)m\frac{\partial\psi}{\partial y}-(\alpha_z\beta+\beta\alpha_z)m\frac{\partial\psi}{\partial z}
\end{aligned}
$$

- **For this to be a reasonable formulation of relativistic QM, a free particle must also obey $E^2=\vec{p}^2+m^2$, i.e. it must satisfy the Klein-Gordon equation:**

$$
-\frac{\partial^2\psi}{\partial t^2}=-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial z^2}+m^2\psi
$$

- **Hence for the Dirac Equation to be consistent with the KG equation require:**

$$
\alpha_x^2=\alpha_y^2=\alpha_z^2=\beta^2=1 \qquad \text{(D2)}
$$

$$
\alpha_j\beta+\beta\alpha_j=0 \qquad \text{(D3)}
$$

$$
\alpha_j\alpha_k+\alpha_k\alpha_j=0 \quad (j\neq k) \qquad \text{(D4)}
$$

★ **Immediately we see that the α_j and β cannot be numbers. Require 4 mutually anti-commuting matrices**

★ **Must be (at least) 4x4 matrices (see Appendix I)**

- Consequently the wave-function must be a <u>four-component</u> Dirac Spinor

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

> A consequence of introducing an equation that is 1st order in time/space derivatives is that the wave-function has new degrees of freedom !

- For the Hamiltonian $\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\partial\psi/\partial t$ to be Hermitian requires

$$\alpha_x = \alpha_x^\dagger; \quad \alpha_y = \alpha_y^\dagger; \quad \alpha_z = \alpha_z^\dagger; \quad \beta = \beta^\dagger; \qquad \text{(D5)}$$

i.e. the require four anti-commuting Hermitian 4x4 matrices.

- At this point it is convenient to introduce an explicit representation for $\vec{\alpha}, \beta$. It should be noted that physical results do not depend on the particular representation – everything is in the commutation relations.
- A convenient choice is based on the Pauli spin matrices:

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

with $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

- The matrices are Hermitian and anti-commute with each other

Dirac Equation: Probability Density and Current

- Now consider probability density/current – this is where the perceived problems with the Klein-Gordon equation arose.
- Start with the Dirac equation

$$-i\alpha_x\frac{\partial\psi}{\partial x}-i\alpha_y\frac{\partial\psi}{\partial y}-i\alpha_z\frac{\partial\psi}{\partial z}+m\beta\psi=i\frac{\partial\psi}{\partial t} \qquad \textbf{(D6)}$$

and its **Hermitian** conjugate

$$+i\frac{\partial\psi^\dagger}{\partial x}\alpha_x^\dagger+i\frac{\partial\psi^\dagger}{\partial y}\alpha_y^\dagger+i\frac{\partial\psi^\dagger}{\partial z}\alpha_z^\dagger+m\psi^\dagger\beta^\dagger=-i\frac{\partial\psi^\dagger}{\partial t} \qquad \textbf{(D7)}$$

- Consider $\psi^\dagger\times(\mathbf{D6})-(\mathbf{D7})\times\psi$ remembering α,β are Hermitian ➡

$$\psi^\dagger\left(-i\alpha_x\frac{\partial\psi}{\partial x}-i\alpha_y\frac{\partial\psi}{\partial y}-i\alpha_z\frac{\partial\psi}{\partial z}+\beta m\psi\right)-\left(i\frac{\partial\psi^\dagger}{\partial x}\alpha_x+i\frac{\partial\psi^\dagger}{\partial y}\alpha_y+i\frac{\partial\psi^\dagger}{\partial z}\alpha_z+m\psi^\dagger\beta\right)\psi=i\psi^\dagger\frac{\partial\psi}{\partial t}+i\frac{\partial\psi^\dagger}{\partial t}\psi$$

➡ $$\underbrace{\psi^\dagger\left(\alpha_x\frac{\partial\psi}{\partial x}+\alpha_y\frac{\partial\psi}{\partial y}+\alpha_z\frac{\partial\psi}{\partial z}\right)}+\underbrace{\left(\frac{\partial\psi^\dagger}{\partial x}\alpha_x+\frac{\partial\psi^\dagger}{\partial y}\alpha_y+\frac{\partial\psi^\dagger}{\partial z}\alpha_z\right)\psi}+\frac{\partial(\psi^\dagger\psi)}{\partial t}=0$$

- Now using the identitv:

$$\psi^\dagger\alpha_x\frac{\partial\psi}{\partial x}+\frac{\partial\psi^\dagger}{\partial x}\alpha_x\psi\equiv\frac{\partial(\psi^\dagger\alpha_x\psi)}{\partial x}$$

gives the continuity equation $$\vec{\nabla}.(\psi^\dagger \vec{\alpha} \psi) + \frac{\partial(\psi^\dagger \psi)}{\partial t} = 0$$ (D8)

where $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$

- The probability density and current can be identified as:

$$\rho = \psi^\dagger \psi \quad \text{and} \quad \vec{j} = \psi^\dagger \vec{\alpha} \psi$$

where $\rho = \psi^\dagger \psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 > 0$

- Unlike the KG equation, the Dirac equation has probability densities which are always positive.
- In addition, the solutions to the Dirac equation are the four component Dirac Spinors. A great success of the Dirac equation is that these components naturally give rise to the property of intrinsic spin.
- It can be shown that Dirac spinors represent spin-half particles (appendix II) with an intrinsic magnetic moment of

$$\vec{\mu} = \frac{q}{m} \vec{S}$$ (appendix III)

Covariant Notation: the Dirac γ Matrices

- The Dirac equation can be written more elegantly by introducing the four Dirac gamma matrices:

$$\gamma^0 \equiv \beta; \quad \gamma^1 \equiv \beta\alpha_x; \quad \gamma^2 \equiv \beta\alpha_y; \quad \gamma^3 \equiv \beta\alpha_z$$

Premultiply the Dirac equation (D6) by β

$$i\beta\alpha_x \frac{\partial\psi}{\partial x} + i\beta\alpha_y \frac{\partial\psi}{\partial y} + i\beta\alpha_z \frac{\partial\psi}{\partial z} - \beta^2 m\psi = -i\beta \frac{\partial\psi}{\partial t}$$

$$\Rightarrow \quad i\gamma^1 \frac{\partial\psi}{\partial x} + i\gamma^2 \frac{\partial\psi}{\partial y} + i\gamma^3 \frac{\partial\psi}{\partial z} - m\psi = -i\gamma^0 \frac{\partial\psi}{\partial t}$$

using $\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ this can be written compactly as:

$$\boxed{(i\gamma^\mu \partial_\mu - m)\psi = 0} \qquad \text{(D9)}$$

★ **NOTE:** it is important to realise that the Dirac gamma matrices are <u>not</u> four-vectors - they are constant matrices which remain invariant under a Lorentz transformation. However it can be shown that the Dirac equation is itself Lorentz covariant (see Appendix IV)

Properties of the γ matrices

- From the properties of the α and β matrices (D2)-(D4) immediately obtain:

$$(\gamma^0)^2 = \beta^2 = 1 \quad \text{and} \quad (\gamma^1)^2 = \beta\alpha_x\beta\alpha_x = -\alpha_x\beta\beta\alpha_x = -\alpha_x^2 = -1$$

- The full set of relations is

$$\begin{aligned}(\gamma^0)^2 &= 1\\ (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 &= -1\\ \gamma^0\gamma^j + \gamma^j\gamma^0 &= 0\\ \gamma^j\gamma^k + \gamma^k\gamma^j &= 0 \quad (j \neq k)\end{aligned}$$

which can be expressed as:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$ (defines the algebra)

- Are the gamma matrices Hermitian?
  - β is Hermitian so γ^0 is Hermitian.
  - The α matrices are also Hermitian, giving

$$\gamma^{1\dagger} = (\beta\alpha_x)^\dagger = \alpha_x^\dagger\beta^\dagger = \alpha_x\beta = -\beta\alpha_x = -\gamma^1$$

  - Hence γ^1, γ^2, γ^3 are anti-Hermitian

$$\boxed{\gamma^{0\dagger} = \gamma^0, \quad \gamma^{1\dagger} = -\gamma^1, \quad \gamma^{2\dagger} = -\gamma^2, \quad \gamma^{3\dagger} = -\gamma^3}$$

Pauli-Dirac representation

- From now on we will use the Pauli-Dirac representation of the gamma matrices:

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$ which when written in full are

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \; \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}; \; \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}; \; \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

- Using the gamma matrices $\rho = \psi^\dagger \psi$ and $\vec{j} = \psi^\dagger \vec{\alpha} \psi$ can be written as:

$$j^\mu = (\rho, \vec{j}) = \psi^\dagger \gamma^0 \gamma^\mu \psi$$

where j^μ is the <u>four-vector</u> current.

(The proof that j^μ is indeed a four vector is given in Appendix V.)

- In terms of the four-vector current the continuity equation becomes

$$\partial_\mu j^\mu = 0$$

- Finally the expression for the four-vector current

$$j^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi$$

can be simplified by introducing the adjoint spinor

The Adjoint Spinor

- The adjoint spinor is defined as

$$\overline{\psi} = \psi^\dagger \gamma^0$$

i.e. $\overline{\psi} = \psi^\dagger \gamma^0 \quad = \quad (\psi^*)^T \gamma^0 = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$$\overline{\psi} \quad = \quad (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*)$$

- In terms the adjoint spinor the four vector current can be written:

$$j^\mu = \overline{\psi} \gamma^\mu \psi$$

★ We will use this expression in deriving the Feynman rules for the Lorentz invariant matrix element for the fundamental interactions.

★ That's enough notation, start to investigate the free particle solutions of the Dirac equation...

Dirac Equation: Free Particle at Rest

- Look for free particle solutions to the Dirac equation of form:

$$\psi = u(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}$$

where $u(\vec{p},E)$, which is a constant four-component spinor which must satisfy the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

- Consider the derivatives of the free particle solution

$$\partial_0 \psi = \frac{\partial \psi}{\partial t} = -iE\psi; \quad \partial_1 \psi = \frac{\partial \psi}{\partial x} = ip_x \psi, \quad \ldots$$

substituting these into the Dirac equation gives:

$$(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)u = 0$$

which can be written: $$(\gamma^\mu p_\mu - m)u = 0 \qquad \text{(D10)}$$

- This is the Dirac equation in "momentum" – note it contains no derivatives.
- For a particle at rest $\vec{p} = 0$

and $\psi = u(E,0)e^{-iEt}$

eq. (D10) → $$E\gamma^0 u - mu = 0$$

$$\Rightarrow \quad E\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix} = m\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\\\phi_4\end{pmatrix} \qquad \text{(D11)}$$

- This equation has four orthogonal solutions:

$$u_1(m,0)=\begin{pmatrix}1\\0\\0\\0\end{pmatrix};\ u_2(m,0)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix};\ u_3(m,0)=\begin{pmatrix}0\\0\\1\\0\end{pmatrix};\ u_4(m,0)=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

(D11) ⇒ $E = m$ (for u_1, u_2)   (D11) ⇒ $E = -m$ (for u_3, u_4)

still have **NEGATIVE ENERGY SOLUTIONS**

- Including the time dependence from $\psi = u(E,0)e^{-iEt}$ gives

$$\psi_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-imt};\quad \psi_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{-imt};\quad \psi_3=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{+imt};\ \text{and}\ \ \psi_4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{+imt}$$

Two spin states with **E>0** (ψ_1, ψ_2)   Two spin states with **E<0** (ψ_3, ψ_4)

★ In QM mechanics can't just discard the **E<0** solutions as unphysical as we require a complete set of states - i.e. 4 SOLUTIONS

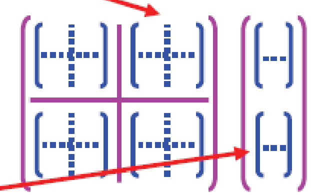
Dirac Equation: Plane Wave Solutions

- Now aim to find general plane wave solutions: $\psi = u(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}$
- Start from Dirac equation (D10): $(\gamma^\mu p_\mu - m)u = 0$

and use
$$\begin{aligned}\gamma^\mu p_\mu - m &= E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3 - m \\ &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} .\vec{p} - m\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} (E-m)I & -\vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & -(E+m)I \end{pmatrix}\end{aligned}$$

Note
$\vec{\sigma}.\vec{p} = p_x\sigma_x + p_y\sigma_y + p_z\sigma_z$

Note in the above equation the **4x4 matrix** is written in terms of four **2x2 sub-matrices**



- Writing the four component spinor as $u = \begin{pmatrix} u_A \\ u_B \end{pmatrix}$

$$(\gamma^\mu p_\mu - m)u = 0 \quad \Rightarrow \quad \begin{pmatrix} (E-m)I & -\vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & (-E-m)I \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Giving two coupled simultaneous equations for $u_A,\ u_B$

$$\left.\begin{aligned}(\vec{\sigma}.\vec{p})u_B &= (E-m)u_A \\ (\vec{\sigma}.\vec{p})u_A &= (E+m)u_B\end{aligned}\right\} \qquad \text{(D12)}$$

Expanding

$$\vec{\sigma}.\vec{p} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_z$$

$$\vec{\sigma}.\vec{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix}$$

- Therefore (D12)

$$\left.\begin{aligned} (\vec{\sigma}.\vec{p})u_B &= (E-m)u_A \\ (\vec{\sigma}.\vec{p})u_A &= (E+m)u_B \end{aligned}\right\}$$

gives

$$u_B = \frac{\vec{\sigma}.\vec{p}}{E+m} u_A = \frac{1}{E+m} \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} u_A$$

- Solutions can be obtained by making the arbitrary (but simplest) choices for u_A

i.e. $u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

giving

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}; \quad \text{and} \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

where N is the wave-function normalisation

NOTE: For $\vec{p} = 0$ these correspond to the E>0 particle at rest solutions

★ The choice of u_A is arbitrary, but this isn't an issue since we can express any other choice as a linear combination. It is analogous to choosing a basis for spin which could be eigenfunctions of S_x, S_y or S_z

Repeating for $u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives the solutions u_3 and u_4

★ The four solutions are: $\psi_i = u_i(E, \vec{p}) e^{i(\vec{p}.\vec{r} - Et)}$

$$u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}; \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}; \quad u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x + ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}; \quad u_4 = N_4 \begin{pmatrix} \frac{p_x - ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

- If any of these solutions is put back into the Dirac equation, as expected, we obtain

$$E^2 = \vec{p}^2 + m^2$$

which doesn't in itself identify the negative energy solutions.

- One rather subtle point: One could ask the question whether we can interpret all four solutions as positive energy solutions. The answer is no. If we take all solutions to have the same value of E, i.e. E = +|E|, only two of the solutions are found to be independent.
- There are only four independent solutions when the two are taken to have E<0.

★ To identify which solutions have E<0 energy refer back to particle at rest (eq. D11).

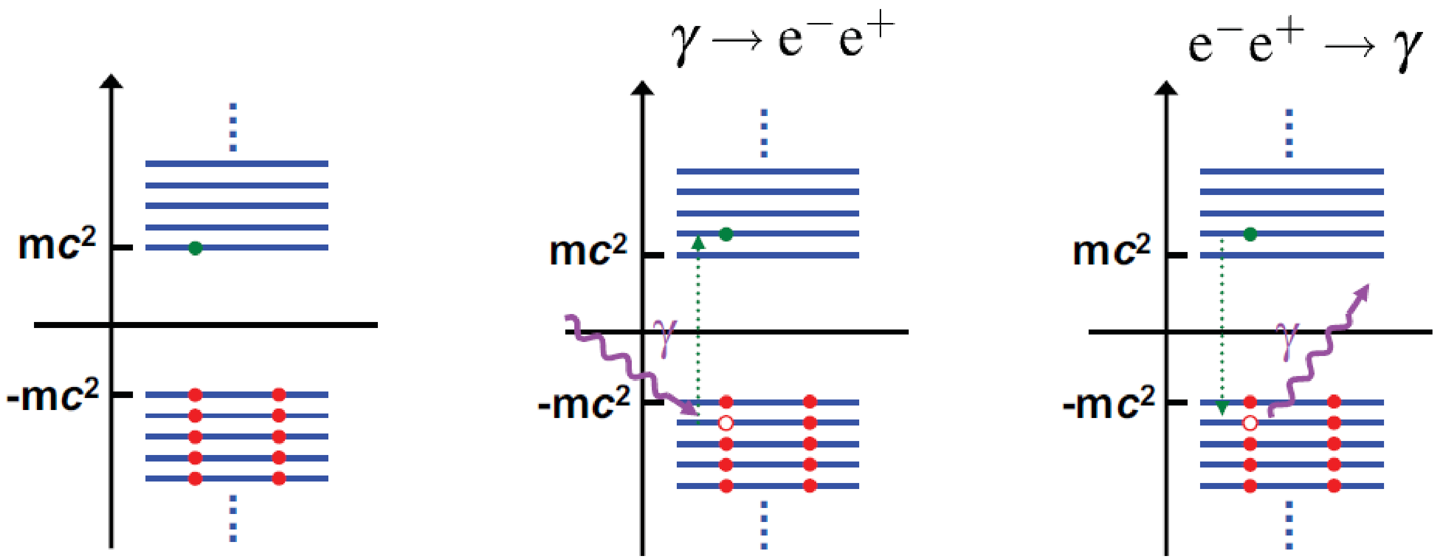
- For $\vec{p} = 0$: u_1, u_2 correspond to the E>0 particle at rest solutions
  u_3, u_4 correspond to the E<0 particle at rest solutions

★ So u_1, u_2 are the +ve energy solutions and u_3, u_4 are the -ve energy solutions

Interpretation of –ve Energy Solutions

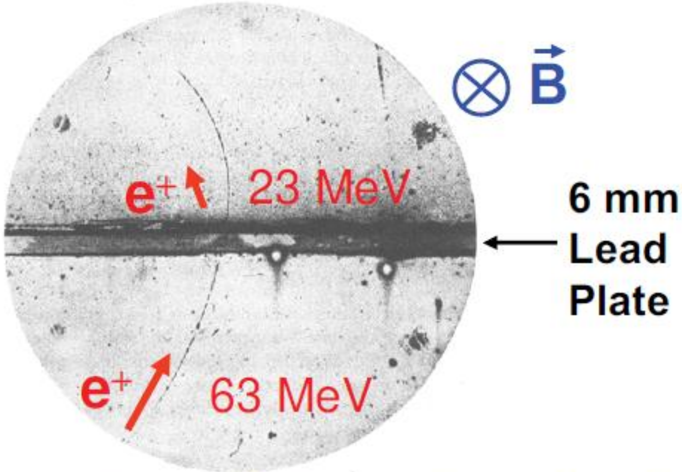
★ The Dirac equation has negative energy solutions. Unlike the KG equation these have positive probability densities. But how should –ve energy solutions be interpreted? Why don't all +ve energy electrons fall into to the lower energy –ve energy states?

<u>**Dirac Interpretation:**</u> the vacuum corresponds to all –ve energy states being full with the Pauli exclusion principle preventing electrons falling into -ve energy states. Holes in the –ve energy states correspond to +ve energy anti-particles with opposite charge. Provides a picture for pair-production and annihilation.

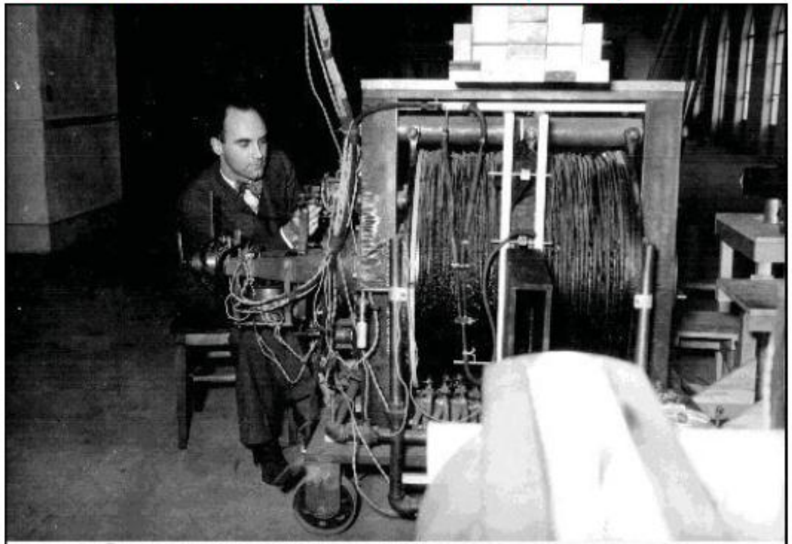


Discovery of the Positron

★ Cosmic ray track in cloud chamber:



C.D.Anderson, Phys Rev 43 (1933) 491



- e^+ enters at bottom, slows down in the lead plate – know direction
- Curvature in B-field shows that it is a positive particle
- Can't be a proton as would have stopped in the lead

➡ **Provided Verification of Predictions of Dirac Equation**

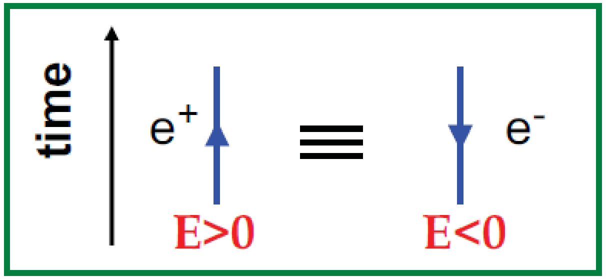
★ Anti-particle solutions exist ! But the picture of the vacuum corresponding to the state where all –ve energy states are occupied is rather unsatisfactory, what about bosons (no exclusion principle),....

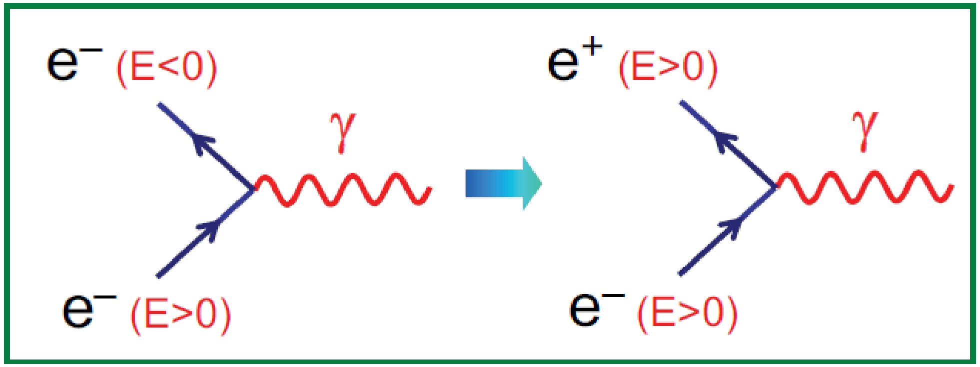
Feynman-Stuckelberg Interpretation

★ There are many problems with the Dirac interpretation of anti-particles and it is best viewed as of historical interest – don't take it too seriously.

Feynman-Stückelberg Interpretation:

★ Interpret a negative energy solution as a negative energy particle which propagates backwards in time or equivalently a positive energy anti-particle which propagates forwards in time





$$e^{-i(-E)(-t)} \rightarrow e^{-iEt}$$

NOTE: in the Feynman diagram the arrow on the anti-particle remains in the backwards in time direction to label it an anti-particle solution.

★ At this point it become more convenient to work with anti-particle wave-functions with $E = \sqrt{|\vec{p}|^2 + m^2}$ motivated by this interpretation

Anti-Particle Spinors

- Want to redefine our –ve energy solutions such that: $E = |\sqrt{|\vec{p}|^2 + m^2}|$
  i.e. the energy of the physical anti-particle.

We can look at this in two ways:

❶ Start from the negative energy solutions

$$u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}; \quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

Where E is understood to be negative

- Can simply "define" anti-particle wave-function by flipping the sign of E and $\vec{p}$ following the Feynman-Stückelburg interpretation:

$$v_1(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)} = u_4(-E,-\vec{p})e^{i(\vec{p}.\vec{r}-Et)}$$

$$v_2(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)} = u_3(-E,-\vec{p})e^{i(\vec{p}.\vec{r}-Et)}$$

where E is now understood to be positive, $E = |\sqrt{|\vec{p}|^2 + m^2}|$

Anti-Particle Spinors

❷ Find negative energy plane wave solutions to the Dirac equation of the form: $\psi = v(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)}$ where $E = |\sqrt{|\vec{p}|^2+m^2}|$

- Note that although $E>0$ these are still negative energy solutions in the sense that

$$\hat{H}v_1 = i\frac{\partial}{\partial t}v_1 = -Ev_1$$

- Solving the Dirac equation $(i\gamma^\mu\partial_\mu - m)\psi = 0$

$$\Rightarrow \quad (-\gamma^0 E + \gamma^1 p_x + \gamma^2 p_y + \gamma^3 p_z - m)v = 0$$

$$\boxed{(\gamma^\mu p_\mu + m)v = 0} \qquad \textbf{(D13)}$$

✶ The Dirac equation in terms of momentum for ANTI-PARTICLES (c.f. D10)

- Proceeding as before:

$$\left.\begin{aligned}(\vec{\sigma}.\vec{p})v_A &= (E-m)v_B \\ (\vec{\sigma}.\vec{p})v_B &= (E+m)v_A\end{aligned}\right\} \quad \text{etc., ...}$$

$$\Rightarrow \quad v_1 = N_1'\begin{pmatrix}\frac{p_x-ip_y}{E+m}\\ \frac{-p_z}{E+m}\\ 0\\ 1\end{pmatrix}; \qquad v_2 = N_2'\begin{pmatrix}\frac{p_z}{E+m}\\ \frac{p_x+ip_y}{E+m}\\ 1\\ 0\end{pmatrix}$$

- The same wave-functions that were written down on the previous page.

Particle and anti-particle Spinors

★ Four solutions of form: $\psi_i = u_i(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}$

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix};\quad u_2 = N\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix};\quad u_3 = N\begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix};\quad u_4 = N\begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

$$E = +\left|\sqrt{|\vec{p}|^2+m^2}\right| \qquad E = -\left|\sqrt{|\vec{p}|^2+m^2}\right|$$

★ Four solutions of form $\psi_i = v_i(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)}$

$$v_1 = N\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix};\quad v_2 = N\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix};\quad v_3 = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \end{pmatrix};\quad v_4 = N\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \end{pmatrix}$$

$$E = +\left|\sqrt{|\vec{p}|^2+m^2}\right| \qquad E = -\left|\sqrt{|\vec{p}|^2+m^2}\right|$$

★ Since we have a four component spinor, **only four are linearly independent**

- Could choose to work with $\{u_1,u_2,u_3,u_4\}$ or $\{v_1,v_2,v_3,v_4\}$ or …
- **Natural** to use choose +ve energy solutions

$$\{u_1,u_2,v_1,v_2\}$$

Wave function normalisation

- From Lecture 1 want to normalise wave-functions to $2E$ particles per unit volume

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}$$

- Consider $\psi = u_1 e^{+i(\vec{p}.\vec{r}-Et)}$

  Probability density $\rho = \psi^\dagger \psi = (\psi^*)^T \psi = u_1^\dagger u_1$

$$\begin{aligned} u_1^\dagger u_1 &= |N|^2 \left(1 + \frac{p_z^2}{(E+m)^2} + \frac{p_x^2+p_y^2}{(E+m)^2}\right) \\ &= |N|^2 \left(\frac{(E+m)^2+|\vec{p}|^2}{(E+m)^2}\right) = |N|^2 \left(\frac{(E+m)^2+E^2-m^2}{(E+m)^2}\right) \\ &= |N|^2 \frac{2E^2+2Em}{(E+m)^2} = |N|^2 \frac{2E}{E+m} \end{aligned}$$

which for the desired $2E$ particles per unit volume, requires that

$$\boxed{N = \sqrt{E+m}}$$

- Obtain same value of ***N*** for $u_1,\ u_2,\ v_1,\ v_2$

Charge Conjugation

- In the Relativity and Electrodynamics course it was shown that the motion of a charged particle in an electromagnetic field $A^\mu = (\phi, \vec{A})$ can be obtained by making the *minimal substitution*

$$\vec{p} \rightarrow \vec{p} - e\vec{A}; \quad E \rightarrow E - e\phi$$

with

$$\vec{p} = -i\vec{\nabla}; \quad E = i\partial/\partial t$$

this can be written

$$\boxed{\partial_\mu \rightarrow \partial_\mu + ieA_\mu}$$

and the Dirac equation becomes:

$$\boxed{\gamma^\mu(\partial_\mu + ieA_\mu)\psi + im\psi = 0}$$

- Taking the complex conjugate and pre-multiplying by $-i\gamma^2$

$$\Rightarrow \quad -i\gamma^2\gamma^{\mu*}(\partial_\mu - ieA_\mu)\psi^* - m\gamma^2\psi^* = 0$$

But $\gamma^{0*} = \gamma^0;\ \gamma^{1*} = \gamma^1;\ \gamma^{2*} = -\gamma^2;\ \gamma^{3*} = \gamma^3$ and $\gamma^2\gamma^{\mu*} = -\gamma^\mu\gamma^2$

$$\Rightarrow \quad \gamma^\mu(\partial_\mu - ieA_\mu)\underbrace{i\gamma^2\psi^*} + im\underbrace{i\gamma^2\psi^*} = 0 \qquad \textbf{(D14)}$$

- Define the charge conjugation operator:

$$\boxed{\psi' = \hat{C}\psi = i\gamma^2\psi^*}$$

D14 becomes: $\gamma^\mu(\partial_\mu - ieA_\mu)\psi' + im\psi' = 0$

- Comparing to the original equation

$$\gamma^\mu(\partial_\mu + ieA_\mu)\psi + im\psi = 0$$

we see that the spinor ψ' describes a particle of the same mass but with opposite charge, i.e. an anti-particle !

$\hat{C}$ ➡ particle spinor ↔ anti-particle spinor

- Now consider the action of $\hat{C}$ on the free particle wave-function:

$$\psi = u_1 e^{i(\vec{p}.\vec{r}-Et)}$$

$$\psi' = \hat{C}\psi = i\gamma^2\psi^* = i\gamma^2 u_1^* e^{-i(\vec{p}.\vec{r}-Et)}$$

$$i\gamma^2 u_1^* = i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}\sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}^* = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} = v_1$$

hence $\psi = u_1 e^{i(\vec{p}.\vec{r}-Et)} \xrightarrow{\hat{C}} \psi' = v_1 e^{-i(\vec{p}.\vec{r}-Et)}$

similarly $\psi = u_2 e^{i(\vec{p}.\vec{r}-Et)} \xrightarrow{\hat{C}} \psi' = v_2 e^{-i(\vec{p}.\vec{r}-Et)}$

★ Under the charge conjugation operator the particle spinors u_1 and u_2 transform to the anti-particle spinors v_1 and v_2

Using the anti-particle solutions

- There is a subtle but important point about the anti-particle solutions written as

$$\psi = v(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)}$$

Applying normal QM operators for momentum and energy $\hat{p} = -i\vec{\nabla}, \quad \hat{H} = i\partial/\partial t$

gives $\hat{H}v_1 = i\partial v_1/\partial t = -Ev_1$ and $\hat{p}v_1 = -i\vec{\nabla}v_1 = -\vec{p}v_1$

★ But have defined solutions to have E>0

★ Hence the quantum mechanical operators giving the physical energy and momenta of the anti-particle solutions are:

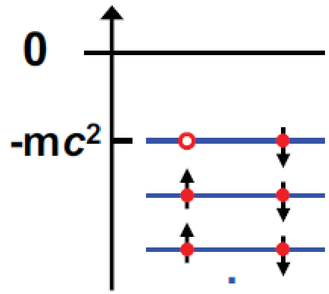
$$\hat{H}^{(v)} = -i\partial/\partial t \quad \text{and} \quad \hat{p}^{(v)} = i\vec{\nabla}$$

- Under the transformation $(E,\vec{p}) \rightarrow (-E,-\vec{p})$: $\vec{L} = \vec{r} \wedge \vec{p} \rightarrow -\vec{L}$

Conservation of total angular momentum $[H,\vec{L}+\vec{S}] = 0$ ➡ $\hat{S}^{(v)} \rightarrow -\hat{S}$

★ The physical spin of the anti-particle solutions is given by $\hat{S}^{(v)} = -\hat{S}$

In the hole picture:



A spin-up hole leaves the negative energy sea in a spin down state

Summary of Solutions to the Dirac Equation

- The normalised free **PARTICLE** solutions to the Dirac equation:

$$\psi = u(E,\vec{p})e^{+i(\vec{p}.\vec{r}-Et)} \quad \text{satisfy} \quad \boxed{(\gamma^\mu p_\mu - m)u = 0}$$

with

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}; \qquad u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

- The **ANTI-PARTICLE** solutions in terms of the physical energy and momentum:

$$\psi = v(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)} \quad \text{satisfy} \quad \boxed{(\gamma^\mu p_\mu + m)v = 0}$$

with

$$v_1 = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}; \qquad v_2 = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

For these states the spin is given by $\hat{S}^{(v)} = -\hat{S}$

- For both particle and anti-particle solutions: $E = \sqrt{|\vec{p}|^2 + m^2}$

Spin States

- **In general** the spinors u_1, u_2, v_1, v_2 **are not** Eigenstates of $\hat{S}_z$

$$\hat{S}_z = \tfrac{1}{2}\Sigma_z = \tfrac{1}{2}\begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

(Appendix II)

- However particles/anti-particles **travelling in the z-direction**: $p_z = \pm|\vec{p}|$

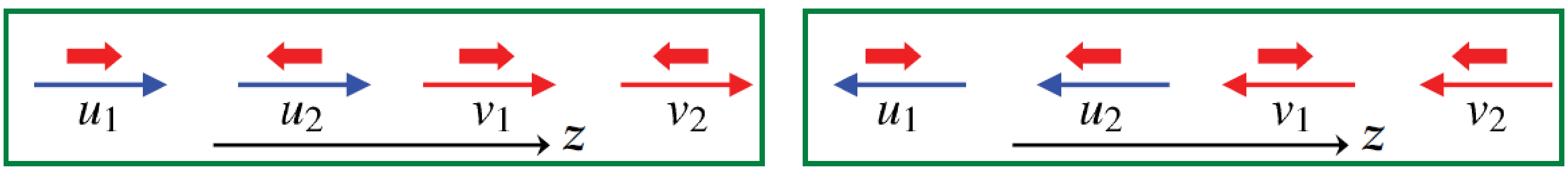
$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{\pm|\vec{p}|}{E+m} \\ 0 \end{pmatrix}; \quad u_2 = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\mp|\vec{p}|}{E+m} \end{pmatrix}; \quad v_1 = N\begin{pmatrix} 0 \\ \frac{\mp|\vec{p}|}{E+m} \\ 0 \\ 1 \end{pmatrix}; \quad v_2 = N\begin{pmatrix} \frac{\pm|\vec{p}|}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

are Eigenstates of $\hat{S}_z$

$$\hat{S}_z u_1 = +\tfrac{1}{2}u_1 \qquad \hat{S}_z^{(v)} v_1 = -\hat{S}_z v_1 = +\tfrac{1}{2}v_1$$

$$\hat{S}_z u_2 = -\tfrac{1}{2}u_2 \qquad \hat{S}_z^{(v)} v_2 = -\hat{S}_z v_2 = -\tfrac{1}{2}v_2$$

Note the change of sign of $\hat{S}$ when dealing with antiparticle spinors



★ Spinors u_1, u_2, v_1, v_2 are only eigenstates of $\hat{S}_z$ for $p_z = \pm|\vec{p}|$

Pause for Breath …

- Have found solutions to the Dirac equation which are also eigenstates $\hat{S}_z$ but only for particles travelling along the z axis.
- Not a particularly useful basis
- More generally, want to label our states in terms of "good quantum numbers", i.e. a set of commuting observables.
- Can't use z component of spin: $[\hat{H}, \hat{S}_z] \neq 0$ (Appendix II)
- Introduce a new concept "**HELICITY**"

Helicity plays an important role in much that follows

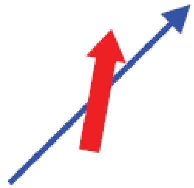
Helicity

★ The component of a particles spin along its direction of flight is a good quantum number:

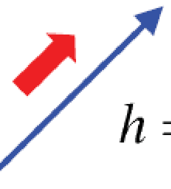
$$[\hat{H}, \hat{S}.\hat{p}] = 0$$

★ Define the component of a particles spin along its direction of flight as **HELICITY**:

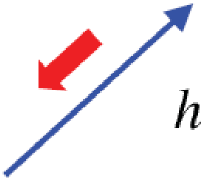
$$h \equiv \frac{\vec{S}.\vec{p}}{|\vec{S}||\vec{p}|} = \frac{2\vec{S}.\vec{p}}{|\vec{p}|} = \frac{\vec{\Sigma}.\vec{p}}{|\vec{p}|}$$



- If we make a measurement of the component of spin of a spin-half particle along any axis it can take two values $\pm 1/2$, consequently the eigenvalues of the helicity operator for a spin-half particle are: ± 1



$h = +1$



$h = -1$

Often termed: "right-handed" "left-handed"

★ **NOTE**: these are "RIGHT-HANDED" and LEFT-HANDED **HELICITY** eigenstates

★ In **Lecture 3** we will discuss RH and LH **CHIRAL** eigenstates. **Only in the limit** $v \approx c$ are the **HELICITY** eigenstates the same as the **CHIRAL** eigenstates

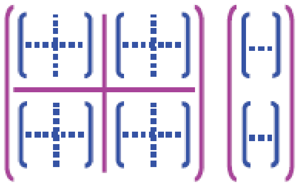
Helicity Eigenstates

★ **Wish to find solutions of Dirac equation which are also eigenstates of Helicity:**

$$(\vec{\Sigma}.\hat{p})u_\uparrow = +u_\uparrow \qquad\qquad (\vec{\Sigma}.\hat{p})u_\downarrow = -u_\downarrow$$

where $u_\uparrow$ and $u_\downarrow$ are right and left handed helicity states and here $\hat{p}$ is the unit vector in the direction of the particle.

- **The eigenvalue equation:**

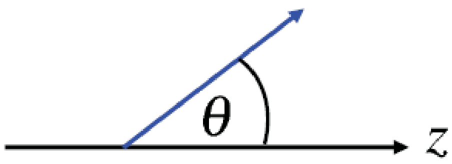


$$\begin{pmatrix} \vec{\sigma}.\hat{p} & 0 \\ 0 & \vec{\sigma}.\hat{p} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \pm \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

gives the coupled equations:

$$\left.\begin{aligned} (\vec{\sigma}.\hat{p})u_A &= \pm u_A \\ (\vec{\sigma}.\hat{p})u_B &= \pm u_B \end{aligned}\right\} \qquad \textbf{(D15)}$$

- **Consider a particle propagating in (θ,ϕ) direction**



$$\hat{p} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\vec{\sigma}.\hat{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix}$$

$$\vec{\sigma}.\hat{p} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

- Writing <u>either</u> $u_A = \begin{pmatrix} a \\ b \end{pmatrix}$ or $u_B = \begin{pmatrix} a \\ b \end{pmatrix}$ then (D15) gives the relation

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix}$$

(For helicity ± 1)

So for the components of <u>BOTH</u> u_A and u_B

$$\frac{b}{a} = \frac{\pm 1 - \cos\theta}{\sin\theta} e^{i\phi}$$

- For the right-handed helicity state, i.e. helicity +1:

$$\frac{b}{a} = \frac{1-\cos\theta}{\sin\theta} e^{i\phi} = \frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} e^{i\phi} = e^{i\phi}\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \quad u_{A\uparrow} \propto \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad u_{B\uparrow} \propto \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

- Putting in the constants of proportionality gives:

$$u_\uparrow = \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} \kappa_1 \cos\left(\frac{\theta}{2}\right) \\ \kappa_1 e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \kappa_2 \cos\left(\frac{\theta}{2}\right) \\ \kappa_2 e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

- From the Dirac Equation (D12) we also have

$$(\vec{\sigma}.\vec{p})u_A = (E+m)u_B$$

$$u_B = \frac{\vec{\sigma}.\vec{p}}{E+m}u_A = \frac{|\vec{p}|}{E+m}\underbrace{(\vec{\sigma}.\hat{p})}_{\text{Helicity}}u_A = \pm\frac{|\vec{p}|}{E+m}u_A \qquad \text{(D16)}$$

★ (D15) determines the relative normalisation of u_A and u_B, i.e. here

$$u_B = +1\frac{|\vec{p}|}{E+m}u_A$$

$$\Rightarrow \quad u_\uparrow = N\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

- The negative helicity particle state is obtained in the same way.
- The anti-particle states can also be obtained in the same manner although it must be remembered that $\hat{S}^{(v)} = -\hat{S}$

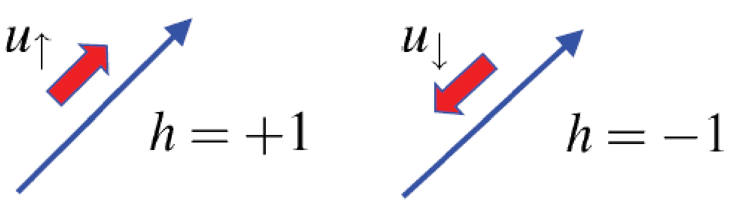
i.e. $\hat{h}^{(v)} = -(\vec{\Sigma}.\hat{p}) \quad \Rightarrow \quad (\vec{\Sigma}.\hat{p})v_\uparrow = -v_\uparrow$

- The particle and anti-particle helicity eigenstates states are:

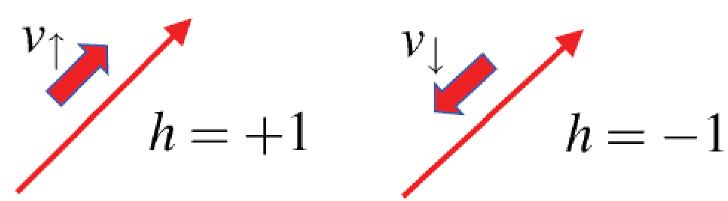
$$u_\uparrow = N\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad u_\downarrow = N\begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

particles



anti-particles



- For all four states, normalising to $2E$ particles/Volume again gives $N=\sqrt{E+m}$
- The helicity eigenstates will be used extensively in the calculations that follow.

Intrinsic Parity of Dirac Particles

★ Before leaving the Dirac equation, consider parity

★ The parity operation is defined as spatial inversion through the origin:

$$x' \equiv -x; \quad y' \equiv -y; \quad z' \equiv -z; \quad t' \equiv t$$

- Consider a Dirac spinor, $\psi(x,y,z,t)$, which satisfies the Dirac equation

$$i\gamma^1\frac{\partial\psi}{\partial x}+i\gamma^2\frac{\partial\psi}{\partial y}+i\gamma^3\frac{\partial\psi}{\partial z}-m\psi=-i\gamma^0\frac{\partial\psi}{\partial t} \qquad \text{(D17)}$$

- Under the parity transformation: $\psi'(x',y',z',t')=\hat{P}\psi(x,y,z,t)$

Try $\hat{P}=\gamma^0$ $\qquad \psi'(x',y',z',t')=\gamma^0\psi(x,y,z,t)$

$(\gamma^0)^2=1$ so $\psi(x,y,z,t)=\gamma^0\psi'(x',y',z',t')$

(D17) ➡ $$i\gamma^1\gamma^0\frac{\partial\psi'}{\partial x}+i\gamma^2\gamma^0\frac{\partial\psi'}{\partial y}+i\gamma^3\gamma^0\frac{\partial\psi'}{\partial z}-m\gamma^0\psi'=-i\gamma^0\gamma^0\frac{\partial\psi'}{\partial t}$$

- Expressing derivatives in terms of the primed system:

$$-i\gamma^1\gamma^0\frac{\partial\psi'}{\partial x'}-i\gamma^2\gamma^0\frac{\partial\psi'}{\partial y'}-i\gamma^3\gamma^0\frac{\partial\psi'}{\partial z'}-m\gamma^0\psi'=-i\gamma^0\gamma^0\frac{\partial\psi'}{\partial t'}$$

Since γ^0 anti-commutes with γ^1, γ^2, γ^3:

$$+i\gamma^0\gamma^1\frac{\partial\psi'}{\partial x'}+i\gamma^0\gamma^2\frac{\partial\psi'}{\partial y'}+i\gamma^0\gamma^3\frac{\partial\psi'}{\partial z'}-m\gamma^0\psi'=-i\frac{\partial\psi'}{\partial t'}$$

Pre-multiplying by γ^0 ⇨ $$i\gamma^1\frac{\partial\psi'}{\partial x'}+i\gamma^2\frac{\partial\psi'}{\partial y'}+i\gamma^3\frac{\partial\psi'}{\partial z'}-m\psi'=-i\gamma^0\frac{\partial\psi'}{\partial t'}$$

- Which is the Dirac equation in the new coordinates.

★ There for under parity transformations the form of the Dirac equation is unchanged provided Dirac spinors transform as

$$\psi\rightarrow\hat{P}\psi=\pm\gamma^0\psi$$

(note the above algebra doesn't depend on the choice of $\hat{P}=\pm\gamma^0$)

- For a particle/anti-particle at rest the solutions to the Dirac Equation are:

$$\psi=u_1e^{-imt};\ \psi=u_2e^{-imt};\ \psi=v_1e^{+imt};\psi=v_2e^{+imt}$$

with $$u_1=N\begin{pmatrix}1\\0\\0\\0\end{pmatrix};\ u_2=N\begin{pmatrix}0\\1\\0\\0\end{pmatrix};\ v_1=N\begin{pmatrix}0\\0\\0\\1\end{pmatrix};\ v_2=N\begin{pmatrix}0\\0\\1\\0\end{pmatrix};$$

$$\hat{P}u_1=\pm\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}=\pm u_1$$ etc. ➡ $$\hat{P}u_1=\pm u_1\quad \hat{P}v_1=\mp v_1$$ $$\hat{P}u_2=\pm u_2\quad \hat{P}v_2=\mp v_2$$

★ Hence an anti-particle at rest has opposite intrinsic parity to a particle at rest.

★ Convention: particles are chosen to have +ve parity; corresponds to choosing

$$\hat{P}=+\gamma^0$$

Summary

★ The formulation of relativistic quantum mechanics starting from the linear Dirac equation

$$\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\frac{\partial \psi}{\partial t}$$

➡ New degrees of freedom : found to describe Spin ½ particles

★ In terms of 4x4 gamma matrices the Dirac Equation can be written:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

★ Introduces the 4-vector current and adjoint spinor:

$$j^\mu = \psi^\dagger \gamma^0 \gamma^\mu \psi = \overline{\psi}\gamma^\mu \psi$$

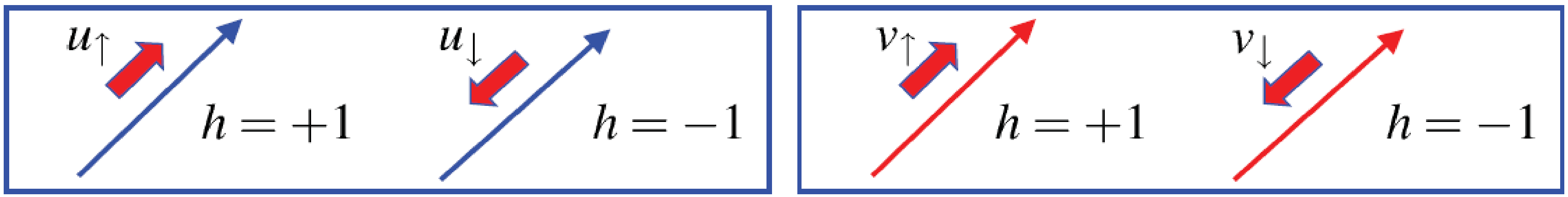
★ With the Dirac equation: forced to have two positive energy and two negative energy solutions

★ Feynman-Stückelberg interpretation: -ve energy particle solutions propagating backwards in time correspond to physical +ve energy anti-particles propagating forwards in time

$$u_1,\ u_2,\ v_1,\ v_2$$

Summary

★ **Most useful basis: particle and anti-particle helicity eigenstates**



★ **In terms of 4-component spinors, the charge conjugation and parity operations are:**

$$\psi \rightarrow \hat{C}\psi = i\gamma^2\psi^\dagger \qquad \psi \rightarrow \hat{P}\psi = \gamma^0\psi$$

★ **Now have all we need to know about a relativistic description of particles… next discuss particle interactions and QED.**

Appendix I: Dimensions of the Dirac Matrices

Starting from $\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\frac{\partial \psi}{\partial t}$

For $\hat{H}$ **to be Hermitian for all** $\vec{p}$ **requires** $\alpha_i = \alpha_i^\dagger \quad \beta = \beta^\dagger$

To recover the KG equation: $\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1$

$$\beta\alpha_j + \alpha_j\beta = 0$$

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 0 \quad (j \neq k)$$

Consider

with $B^\dagger B = 1$

$$\begin{aligned} Tr(B^\dagger AB) &= B^\dagger_{ij}A_{jk}B_{ki} \\ &= B_{ki}B^\dagger_{ij}A_{jk} \\ &= \delta_{jk}A_{jk} \\ &= Tr(A) \end{aligned}$$

Therefore

$$\begin{aligned} Tr(\alpha) &= Tr(\alpha_j^\dagger \alpha_i \alpha_j) \\ &= -Tr(\alpha_j^\dagger \alpha_j \alpha_i) \quad \text{(using commutation relation)} \\ &= -Tr(\alpha_i) \\ \Rightarrow \quad Tr(\alpha_i) &= 0 \end{aligned}$$

similarly

$$Tr(\beta) = 0$$

We can now show that the matrices are of even dimension by considering the eigenvalue equation, e.g. $\alpha\vec{x} = \lambda\vec{x}$

$$\vec{x}^\dagger\vec{x} = \vec{x}\alpha^\dagger\alpha\vec{x} = \lambda^*\lambda\vec{x}^\dagger\vec{x}$$

Eigenvalues of a Hermitian matrix are real so $\lambda^2 = 1 \quad \rightarrow \quad \lambda = \pm 1$

but $$Tr(\alpha) = \sum_i \lambda_i$$

Since the α_i, β are trace zero Hermitian matrices with eigenvalues of ± 1 they must be of even dimension

For **N=2** the 3 Pauli spin matrices satisfy

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 0 \quad (j \neq i)$$

But we require 4 anti-commuting matrices. Consequently the α_i, β of the Dirac equation must be of dimension **4**, **6**, **8**,….. The simplest choice for is to assume that the α_i, β are of dimension **4**.

Appendix II: Spin

- For a Dirac spinor is orbital angular momentum a good quantum number? i.e. does $L = \vec{r} \wedge \vec{p}$ commute with the Hamiltonian?

$$
\begin{aligned}
[H, \vec{L}] &= [\vec{\alpha}.\vec{p} + \beta m, \vec{r} \wedge \vec{p}] \\
&= [\vec{\alpha}.\vec{p}, \vec{r} \wedge \vec{p}]
\end{aligned}
$$

Consider the x component of L:

$$
\begin{aligned}
[H, L_x] &= [\vec{\alpha}.\vec{p}, (\vec{r} \wedge \vec{p})_x] \\
&= [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, y p_z - z p_y]
\end{aligned}
$$

The only non-zero contributions come from: $[x, p_x] = [y, p_y] = [z, p_z] = i$

$$
\begin{aligned}
[H, L_x] &= \alpha_y p_z [p_y, y] - \alpha_z p_y [p_z, z] \\
&= -i(\alpha_y p_z - \alpha_z p_y) \\
&= -i(\vec{\alpha} \wedge \vec{p})_x
\end{aligned}
$$

Therefore $$\boxed{[H, \vec{L}] = -i\vec{\alpha} \wedge \vec{p}} \qquad \text{(A.1)}$$

★ Hence the angular momentum does not commute with the Hamiltonian and is not a constant of motion

Introduce a new 4x4 operator:

$$\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

where $\vec{\sigma}$ are the Pauli spin matrices: i.e.

$$\Sigma_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad \Sigma_y = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}; \quad \Sigma_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now consider the commutator

$$[H, \vec{\Sigma}] = [\vec{\alpha}.\vec{p} + \beta m, \vec{\Sigma}]$$

here $$[\beta, \vec{\Sigma}] = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} - \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} = 0$$

and hence $$[H, \vec{\Sigma}] = [\vec{\alpha}.\vec{p}, \vec{\Sigma}]$$

Consider the x comp:

$$\begin{aligned} [H, \Sigma_x] &= [\alpha_x p_x + \alpha_y p_y + \alpha_z p_z, \Sigma_x] \\ &= p_x[\alpha_x, \Sigma_x] + p_y[\alpha_y, \Sigma_x] + p_z[\alpha_z, \Sigma_x] \end{aligned}$$

Taking each of the commutators in turn:

$$[\alpha_x,\Sigma_x] = \begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix} = 0$$

$$\begin{aligned}
[\alpha_y,\Sigma_x] &= \begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} - \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}\begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & \sigma_y\sigma_y - \sigma_y\sigma_x \\ \sigma_y\sigma_x - \sigma_x\sigma_y & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -2i\sigma_z \\ -2i\sigma_z & 0 \end{pmatrix} \\
&= -2i\alpha_z \\
[\alpha_z,\Sigma_x] &= 2i\alpha_y
\end{aligned}$$

Hence

$$\begin{aligned}
[H,\Sigma_x] &= p_x[\alpha_x,\Sigma_x] + p_y[\alpha_y,\Sigma_x] + p_z[\alpha_z,\Sigma_x] \\
&= -2ip_y\alpha_x + 2ip_z\alpha_y \\
&= 2i(\vec{\alpha}\wedge\vec{p})_x
\end{aligned}$$

$$[H,\vec{\Sigma}] = 2i\vec{\alpha}\wedge\vec{p}$$

- Hence the observable corresponding to the operator $\vec{\Sigma}$ is also not a constant of motion. However, referring back to (A.1)

$$[H,\vec{S}] = \frac{1}{2}[H,\vec{\Sigma}] = i\vec{\alpha}\wedge\vec{p} = -[H,\vec{L}]$$

Therefore: $$\boxed{[H,\vec{L}+\vec{S}] = 0}$$

- Because $$\vec{S} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

the commutation relationships for $\vec{S}$ are the same as for the $\vec{\sigma}$, e.g. $[S_x,S_y] = iS_z$. Furthermore both S^2 and S_z are diagonal

$$S^2 = \tfrac{1}{4}(\Sigma_x^2+\Sigma_y^2+\Sigma_z^2) = \tfrac{3}{4}\begin{pmatrix} 1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}; \quad S_z = \tfrac{1}{2}\begin{pmatrix} 1&0&0&0\\0&-1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$$

- Consequently $S^2\psi = S(S+1)\psi = \frac{3}{4}$ and for a particle travelling along the z direction $S_z\psi = \pm\frac{1}{2}\psi$

★ S has all the properties of spin in quantum mechanics and therefore the Dirac equation provides a natural account of the intrinsic angular momentum of fermions

Appendix III: Magnetic Moment

- In the part II Relativity and Electrodynamics course it was shown that the motion of a charged particle in an electromagnetic field $A^{\mu} = (\phi, \vec{A})$ can be obtained by making the *minimal substitution*

$$\vec{p} \rightarrow \vec{p} - q\vec{A}; \quad E \rightarrow E - q\phi$$

- Applying this to equations (D12)

$$\begin{aligned}(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_B &= (E - m - q\phi)u_A \\ (\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A &= (E + m - q\phi)u_B\end{aligned} \qquad \text{(A.2)}$$

Multiplying (A.2) by $(E + m - q\phi)$

$$\begin{aligned}(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})(E + m - q\phi)u_B &= (E - m - q\phi)(E + m - q\phi)u_A \\ (\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A &= (T - q\phi)(T + 2m - q\phi)u_A \qquad \text{(A.3)}\end{aligned}$$

where kinetic energy $T = E - m$

- In the non-relativistic limit $T \ll m$ (A.3) becomes

$$(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{p} - q\vec{\sigma}.\vec{A})u_A \approx 2m(T - q\phi)u_A$$

$$\left[(\vec{\sigma}.\vec{p})^2 - q(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{p}) - q(\vec{\sigma}.\vec{p})(\vec{\sigma}.\vec{A}) + q^2(\vec{\sigma}.\vec{A})^2\right]u_A \approx 2m(T - q\phi)u_A \qquad \text{(A.4)}$$

- **Now** $\vec{\sigma}.\vec{A} = \begin{pmatrix} A_z & A_x - iA_y \\ A_x + iA_y & -A_z \end{pmatrix}$; $\vec{\sigma}.\vec{B} = \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix}$;

which leads to $(\vec{\sigma}.\vec{A})(\vec{\sigma}.\vec{B}) = \vec{A}.\vec{B} + i\vec{\sigma}.(\vec{A} \wedge \vec{B})$

and $(\vec{\sigma}.\vec{A})^2 = |\vec{A}|^2$

- **The operator on the LHS of (A.4):**

$$\begin{aligned}
&= \vec{p}^2 - q\left[\vec{A}.\vec{p} + i\vec{\sigma}.\vec{A} \wedge \vec{p} + \vec{p}.\vec{A} + i\vec{\sigma}.\vec{p} \wedge \vec{A}\right] + q^2\vec{A}^2 \\
&= (\vec{p} - q\vec{A})^2 - iq\vec{\sigma}.\left[\vec{A} \wedge \vec{p} + \vec{p} \wedge \vec{A}\right] \\
&= (\vec{p} - q\vec{A})^2 - q^2\vec{\sigma}.\left[\vec{A}.\vec{\nabla} + \vec{\nabla}.\vec{A}\right] && \vec{p} = -i\vec{\nabla} \\
&= (\vec{p} - q\vec{A})^2 - q\vec{\sigma}.(\vec{\nabla} \wedge \vec{A}) && (\vec{\nabla} \wedge \vec{A})\psi = \vec{\nabla} \wedge (\vec{A}\psi) + \vec{A} \wedge (\vec{\nabla}\psi) \\
&= (\vec{p} - q\vec{A})^2 - q\vec{\sigma}.\vec{B} && \vec{B} = \vec{\nabla} \wedge \vec{A}
\end{aligned}$$

★ **Substituting back into (A.4) gives the Schrödinger-Pauli equation for the motion of a non-relativisitic spin ½ particle in an EM field**

$$\left[\frac{1}{2m}(\vec{p} - q\vec{A})^2 - \frac{q}{2m}\vec{\sigma}.\vec{B} + q\phi\right] u_A = T u_A$$

$$\left[\frac{1}{2m}(\vec{p}-q\vec{A})^2 - \frac{q}{2m}\vec{\sigma}.\vec{B} + q\phi\right] u_A = T u_A$$

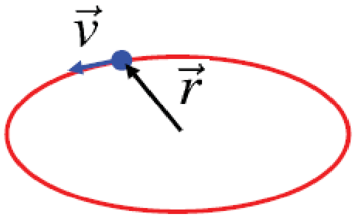
- Since the energy of a magnetic moment in a field $\vec{B}$ is $-\vec{\mu}.\vec{B}$ we can identify the intrinsic magnetic moment of a spin ½ particle to be:

$$\vec{\mu} = \frac{q}{2m}\vec{\sigma}$$

In terms of the spin: $\vec{S} = \frac{1}{2}\vec{\sigma}$

$$\vec{\mu} = \frac{q}{m}\vec{S}$$

- Classically, for a charged particle current loop



$$\mu = \frac{q}{2m}\vec{L}$$

- The intrinsic magnetic moment of a spin half Dirac particle is twice that expected from classical physics. This is often expressed in terms of the gyromagnetic ratio is g=2.

$$\vec{\mu} = g\frac{q}{2m}\vec{S}$$

Appendix IV: Covariance of Dirac Equation

- For a Lorentz transformation we wish to demonstrate that the Dirac Equation is covariant i.e.

$$i\gamma^{\mu}\partial_{\mu}\psi = m\psi \qquad \text{(A.5)}$$

transforms to

$$i\gamma^{\mu}\partial'_{\mu}\psi' = m\psi' \qquad \text{(A.6)}$$

where

$$\partial'_{\mu} \equiv \frac{\partial}{\partial x'^{\mu}} = \left(\frac{\partial}{\partial t'}, \frac{\partial}{\partial x'}, \frac{\partial}{\partial y'}, \frac{\partial}{\partial z'}\right)$$

and $\psi'(x') = S\psi(x)$ is the transformed spinor.

- The covariance of the Dirac equation will be established if the 4x4 matrix S exists.
- Consider a Lorentz transformation with the primed frame moving with velocity v along the x axis

$$\partial'_{\mu} = \Lambda^{\nu}_{\mu}\partial_{\nu}$$

where

$$\Lambda^{\mu}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

With this transformation equation (A.6)

$$\begin{aligned} i\gamma^{\nu}\partial'_{\nu}\psi' &= m\psi' \\ \Rightarrow \quad i\gamma^{\nu}\Lambda^{\mu}_{\nu}\partial_{\mu}S\psi &= mS\psi \end{aligned}$$

which should be compared to the matrix S multiplying (A.5)

$$iS\gamma^{\mu}\partial_{\mu}\psi = mS\psi$$

★ Therefore the covariance of the Dirac equation will be demonstrated if we can find a matrix S such that

$$\begin{aligned} i\gamma^{\nu}\Lambda^{\mu}_{\nu}\partial_{\mu}S\psi &= iS\gamma^{\mu}\partial_{\mu}\psi \\ \Rightarrow \quad \gamma^{\nu}\Lambda^{\mu}_{\nu}S\partial_{\mu}\psi &= S\gamma^{\mu}\partial_{\mu}\psi \\ \Rightarrow \quad \boxed{S\gamma^{\mu} = \gamma^{\nu}S\Lambda^{\mu}_{\nu}} \end{aligned} \qquad \text{(A.7)}$$

- Considering each value of $\mu = 0,\ 1,\ 2,\ 3$

$$\left.\begin{aligned} S\gamma^{0} &= \gamma\gamma^{0}S - \beta\gamma\gamma^{1}S \\ S\gamma^{1} &= -\beta\gamma\gamma^{0}S + \gamma\gamma^{1}S \\ S\gamma^{2} &= \gamma^{2}S \\ S\gamma^{3} &= \gamma^{3}S. \end{aligned}\right\}$$

where $\gamma = (1-\beta^2)^{-1/2}$

and $\beta = v/c$

- It is easy (although tedious) to demonstrate that the matrix:

$$S = aI + b\gamma^0\gamma^1 \quad \text{with} \quad a = \sqrt{\tfrac{1}{2}(\gamma+1)}, \quad b = \sqrt{\tfrac{1}{2}(\gamma-1)}$$

satisfies the above simultaneous equations

NOTE: For a transformation along in the $-x$ direction $b = -\sqrt{\tfrac{1}{2}(\gamma-1)}$

★ To summarise, under a Lorentz transformation a spinor $\psi(x)$ transforms to $\psi'(x') = S\psi(x)$. This transformation preserves the mathematical form of the Dirac equation

Appendix V: Transformation of Dirac current

★ The Dirac current $j^\mu = \overline{\psi}\gamma^\mu\psi$ plays an important rôle in the description of particle interactions. Here we consider its transformation properties.

- Under a Lorentz transformation we have $\psi' = S\psi$
  and for the adjoint spinor: $\overline{\psi}' = \psi'^\dagger\gamma^0 = S\psi^\dagger\gamma^0 = \psi^\dagger S^\dagger\gamma^0$
- First consider the transformation properties of $\overline{\psi}'\psi'$

$$\overline{\psi}'\psi' = \psi^\dagger S^\dagger\gamma^0 S\psi$$

where $$S^\dagger = aI + b\gamma^{1\dagger}\gamma^{0\dagger} = aI - b\gamma^1\gamma^0$$

giving
$$\begin{aligned} S^\dagger\gamma^0 S &= (aI - b\gamma^1\gamma^0)\gamma^0(aI + b\gamma^0\gamma^1) \\ &= a^2\gamma^0 - b^2\gamma^1\gamma^0\gamma^0\gamma^0\gamma^1 + ab\gamma^0\gamma^0\gamma^1 - b\gamma^1\gamma^0\gamma^0 \\ &= a^2\gamma^0 + b^2\gamma^0(\gamma^0)^2(\gamma^1)^2 + ab\gamma^1 - ab\gamma^1 \\ &= (a^2 - b^2)\gamma^0 \\ &= \gamma^0 \end{aligned}$$

hence $$\overline{\psi}'\psi' = \psi^\dagger S^\dagger\gamma^0 S\psi = \psi^\dagger\gamma^0\psi = \overline{\psi}\psi$$

★ The product $\overline{\psi}\psi$ is therefore a Lorentz invariant. More generally, the product $\overline{\psi_1}\psi_2$ is Lorentz covariant

★ Now consider
$$j'^{\mu} = \overline{\psi'}\gamma^{\mu}\psi' = (\psi^{\dagger}S^{\dagger}\gamma^{0})\gamma^{\mu}S\psi$$

• To evaluate this wish to express $\gamma^{\mu}S$ in terms of $S\gamma^{\mu}$

(A.7) $$S\gamma^{\mu} = \gamma^{\nu}S\Lambda^{\mu}_{\nu}$$

➡ $$S\gamma^{\mu}\Lambda^{\rho}_{\mu} = \gamma^{\nu}S\Lambda^{\mu}_{\nu}\Lambda^{\rho}_{\mu} = \gamma^{\nu}S\delta^{\rho}_{\nu} = \gamma^{\rho}S$$

where we used $\Lambda^{\mu}_{\nu}\Lambda^{\rho}_{\mu} = \delta^{\rho}_{\nu}$

• Rearranging the labels and reordering gives:

$$\boxed{\gamma^{\mu}S = \Lambda^{\mu}_{\nu}S\gamma^{\nu}}$$

$$\begin{aligned} j'^{\mu} &= (\psi^{\dagger}S^{\dagger}\gamma^{0})\gamma^{\mu}S\psi = \psi^{\dagger}S^{\dagger}\gamma^{0}(\Lambda^{\mu}_{\nu}S\gamma^{\nu})\psi \\ &= \Lambda^{\mu}_{\nu}\psi^{\dagger}(S^{\dagger}\gamma^{0}S)\gamma^{\nu}\psi = \Lambda^{\mu}_{\nu}\psi^{\dagger}\gamma^{0}\gamma^{\nu}\psi \\ &= \Lambda^{\mu}_{\nu}\overline{\psi}\gamma^{\nu}\psi = \Lambda^{\mu}_{\nu}j^{\nu} \end{aligned}$$

➡ $$\boxed{\overline{\psi'}\gamma^{\mu}\psi = \Lambda^{\mu}_{\nu}\overline{\psi}\gamma^{\nu}\psi}$$

★ Hence the Dirac current, $\overline{\psi}\gamma^{\mu}\psi$, transforms as a four-vector