Elementary Particle Physics: theory and experiments

Electron-proton elastic and inelastic scattering

Form factors

Deep Inelastic Scattering (DIS) and proton structure functions (PDFs)

Slides taken from M. A. Thomson lectures at Cambridge University in 2011

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Electron-proton scattering

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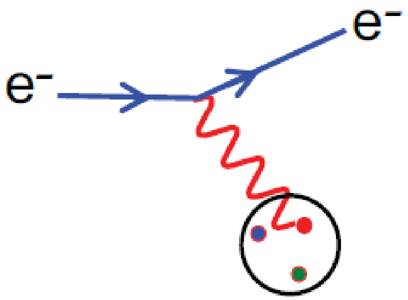
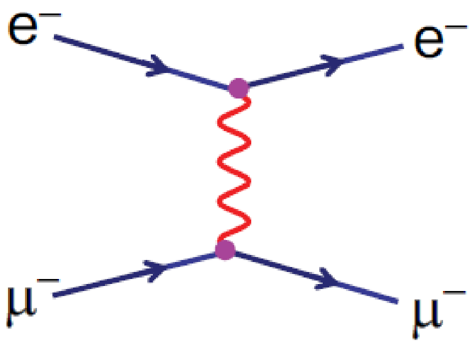


- **Two main topics:**
  - $e^-p \rightarrow e^-p$ **elastic scattering**
  - $e^-p \rightarrow e^-X$ **deep inelastic scattering**
- **But first consider scattering from a point-like particle e.g.**

  $e^-\mu^- \rightarrow e^-\mu^-$

  i.e. the QED part of

  $(e^-q \rightarrow e^-q)$



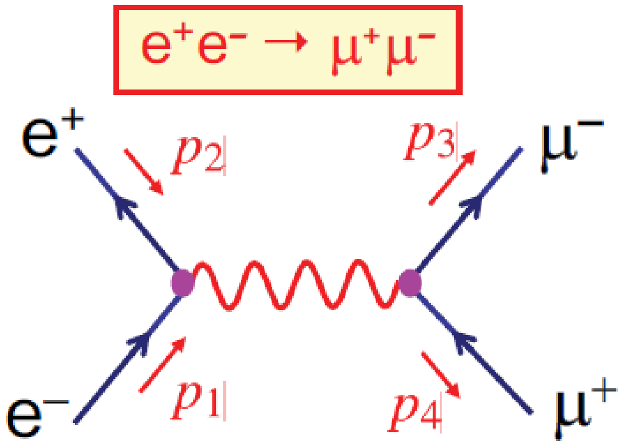
- **Two ways to proceed:**
  - **perform QED calculation from scratch**

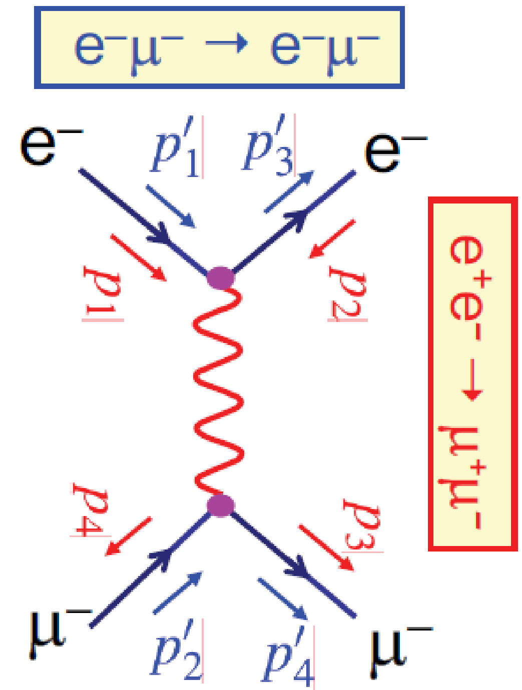
$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) \right] \quad (1)$$

  - **take results from $e^+e^- \rightarrow \mu^+\mu^-$ and use "Crossing Symmetry" to obtain the matrix element for $e^-\mu^- \rightarrow e^-\mu^-$**

Crossing symmetry

★ Having derived the Lorentz invariant matrix element for $e^+e^- \rightarrow \mu^+\mu^-$ "rotate" the diagram to correspond to $e^-\mu^- \rightarrow e^-\mu^-$ and apply the principle of crossing symmetry to write down the matrix element !





★ The transformation:

$$p_1 \rightarrow p_1'; \; p_2 \rightarrow -p_3'; \; p_3 \rightarrow p_4'; \; p_4 \rightarrow -p_2'$$

Changes the spin averaged matrix element for

$e^- e^+ \rightarrow \mu^- \mu^+$ ($p_1 \; p_2 \quad p_3 \; p_4$) → $e^- \mu^- \rightarrow e^- \mu^-$ ($p_1' \; p_2' \quad p_3' \; p_4'$)

• Take ME for $e^+e^- \rightarrow \mu^+\mu^-$ and apply crossing symmetry:

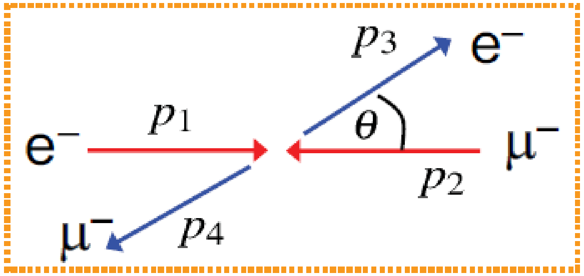
$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1.p_3)^2 + (p_1.p_4)^2}{(p_1.p_2)^2} \quad \Rightarrow \quad \boxed{\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p'_1.p'_4)^2 + (p'_1.p'_2)^2}{(p'_1.p'_3)^2}} \quad (1)$$

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1.p_4)^2 + (p_1.p_2)^2}{(p_1.p_3)^2} \quad (2)$$

$$\equiv 2e^4 \left(\frac{s^2+u^2}{t^2} \right)$$

- **Work in the C.o.M:**

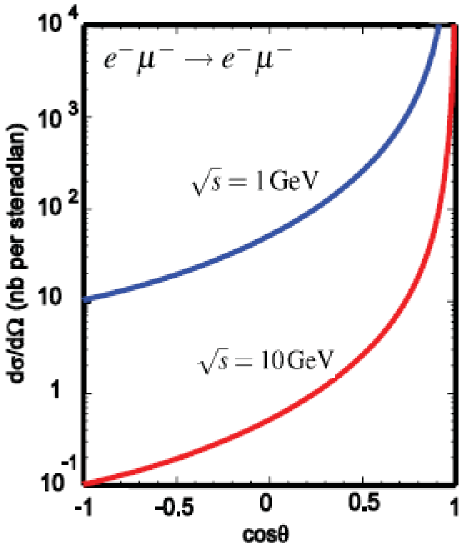
$$p_1 = (E,0,0,E) \qquad p_2 = (E,0,0,-E)$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta)$$
$$p_4 = (E, -E\sin\theta, 0, -E\cos\theta)$$



giving $p_1.p_2 = 2E^2; \quad p_1.p_3 = E^2(1-\cos\theta); \quad p_1.p_4 = E^2(1+\cos\theta)$

$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{E^4(1+\cos\theta)^2 + 4E^4}{E^4(1-\cos\theta)^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\langle |M_{fi}|^2 \rangle = \frac{e^4}{8\pi^2 s} \frac{\left[1+\frac{1}{4}(1+\cos\theta)^2\right]}{(1-\cos\theta)^2}$$



- The denominator arises from the propagator $-ig_{\mu\nu}/q^2$

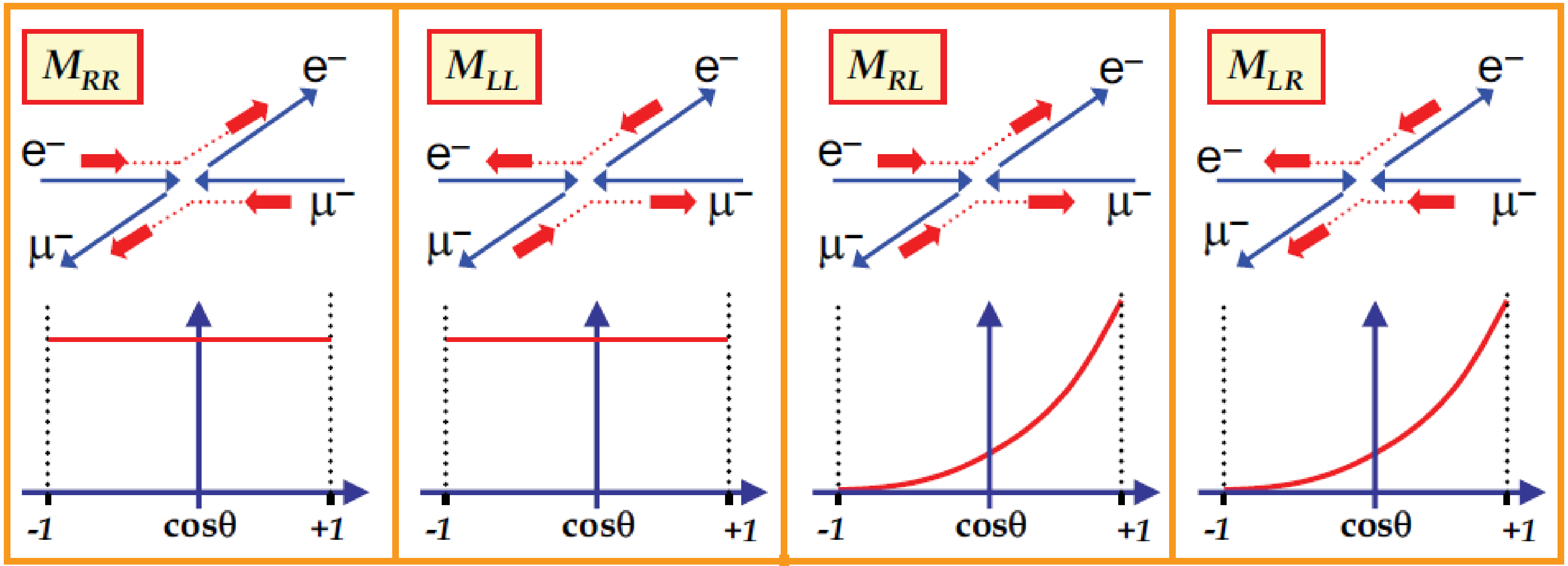
here $q^2 = (p_1 - p_3)^2 = E^2(1-\cos\theta)$

as $q^2 \rightarrow 0$ the cross section tends to infinity.

- What about the angular dependence of the numerator ?

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{8\pi^2 s}\frac{\left[1+\frac{1}{4}(1+\cos\theta)^2\right]}{(1-\cos\theta)^2}$$

- The factor $1+\frac{1}{4}(1+\cos\theta)^2$ reflects helicity (really chiral) structure of QED
- Of the 16 possible helicity combinations only 4 are non-zero:



$S_z = 0$

$$\frac{d\sigma}{d\Omega} \propto 1$$

i.e. no preferred polar angle

$S_z = +1$ $S_z = -1$

$$\frac{d\sigma}{d\Omega} \propto \frac{1}{4}(1+\cos\theta)^2$$

spin 1 rotation again

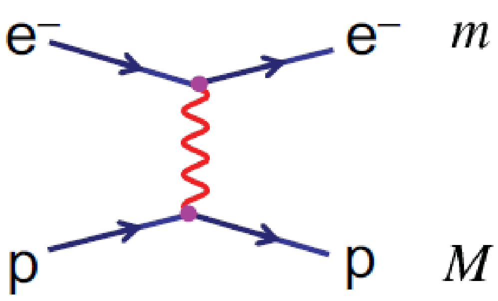
- The cross section calculated above is appropriate for the scattering of two spin half Dirac (i.e. point-like) particles in the ultra-relativistic limit (where both electron and muon masses can be neglected). In this case

$$\langle|M_{fi}|^2\rangle = 2e^4\frac{(p_1.p_4)^2+(p_1.p_2)^2}{(p_1.p_3)^2}$$

- We will use this again in the discussion of "Deep Inelastic Scattering" of electrons from the quarks within a proton

- Before doing so we will consider the scattering of electrons from the composite proton - i.e. how do we know the proton isn't fundamental "point-like" particle ?

- In this discussion we will not be able to use the relativistic limit and require the general expression for the matrix element

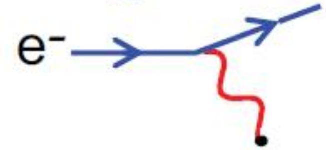


$$\langle|M_{fi}|^2\rangle = \frac{8e^4}{(p_1-p_3)^4}\left[(p_1.p_2)(p_3.p_4)+(p_1.p_4)(p_2.p_3)-(p_1.p_3)M^2-(p_1.p_4)m^2+2m^2M^2\right] \quad (3)$$

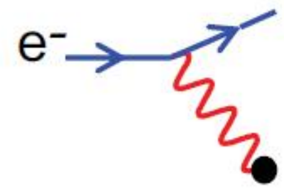
Probing the structure of the proton

★ In $e^-p \to e^-p$ scattering the nature of the interaction of the virtual photon with the proton depends strongly on wavelength

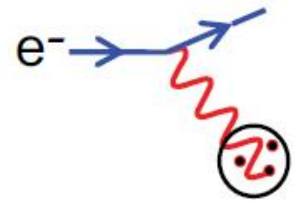
- At very low electron energies $\lambda \gg r_p$:
  the scattering is equivalent to that from a "point-like" spin-less object



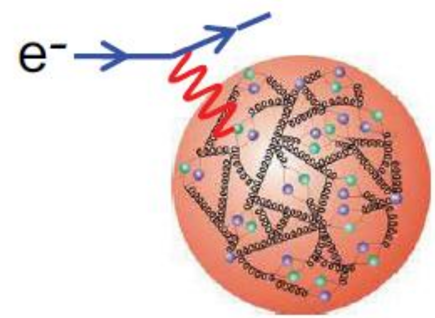
- At low electron energies $\lambda \sim r_p$:
  the scattering is equivalent to that from a extended charged object



- At high electron energies $\lambda < r_p$:
  the wavelength is sufficiently short to resolve sub-structure. Scattering from constituent quarks

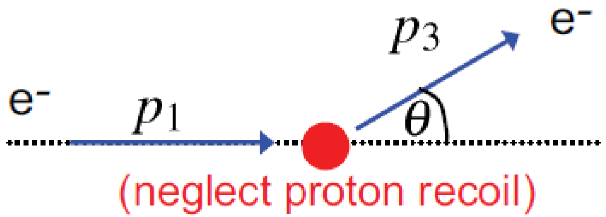


- At very high electron energies $\lambda \ll r_p$:
  the proton appears to be a sea of quarks and gluons.



Rutherford scattering revisited

★ Rutherford scattering is the low energy limit where the recoil of the proton can be neglected and the electron is non-relativistic



- Start from RH and LH Helicity particle spinors

$$u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \frac{|\vec{p}|}{E+m}c \\ \frac{|\vec{p}|}{E+m}e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \frac{|\vec{p}|}{E+m}s \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}c \end{pmatrix} \quad \begin{aligned} N &= \sqrt{E+m}; \\ s &= \sin(\theta/2); \quad c = \cos(\theta/2) \end{aligned}$$

- Now write in terms of: $$\alpha = \frac{|\vec{p}|}{E+m_e}$$

Non-relativistic limit: $\alpha \to 0$

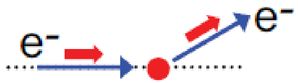
Ultra-relativistic limit: $\alpha \to 1$

$$\Rightarrow \quad u_\uparrow = N\begin{pmatrix} c \\ e^{i\phi}s \\ \alpha c \\ \alpha e^{i\phi}s \end{pmatrix} \quad u_\downarrow = N\begin{pmatrix} -s \\ e^{i\phi}c \\ \alpha s \\ -\alpha e^{i\phi}c \end{pmatrix}$$

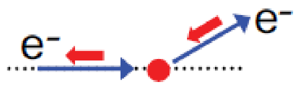
and the possible initial and final state electron spinors are:

$$u_\uparrow(p_1) = N_e\begin{pmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{pmatrix} \quad u_\downarrow(p_1) = N_e\begin{pmatrix} 0 \\ 1 \\ 0 \\ -\alpha \end{pmatrix} \quad u_\uparrow(p_3) = N_e\begin{pmatrix} c \\ s \\ \alpha c \\ \alpha s \end{pmatrix} \quad u_\downarrow(p_3) = N_e\begin{pmatrix} -s \\ c \\ \alpha s \\ -\alpha c \end{pmatrix}$$

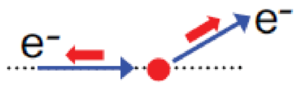
- Consider all four possible electron currents, i.e. Helicities **R→R, L→L, L→R, R→L**



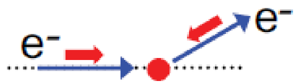
$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \quad (4)$$



$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \quad (5)$$



$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(1-\alpha^2)s, 0, 0, 0\right] \quad (6)$$



$$\overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(\alpha^2-1)s, 0, 0, 0\right] \quad (7)$$

- In the relativistic limit ($\alpha = 1$), i.e. $E \gg m$

  (6) and (7) are identically zero; only R→R and L→L combinations non-zero

- In the non-relativistic limit, $|\vec{p}| \ll E$ we have $\alpha = 0$

$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = \overline{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (2m_e)\left[c, 0, 0, 0\right]$$

$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = -\overline{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (2m_e)\left[s, 0, 0, 0\right]$$

All four electron helicity combinations have non-zero Matrix Element

i.e. Helicity eigenstates ≠ Chirality eigenstates

- The initial and final state **proton** spinors (assuming no recoil) are:

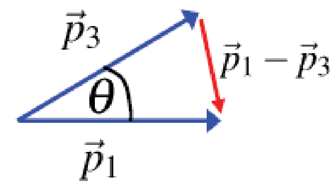
$$u_{\uparrow}(0) = \sqrt{2M_p}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} \qquad u_{\downarrow}(0) = \sqrt{2M_p}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

Solutions of Dirac equation for a particle at rest

giving the proton currents:

$$j_{p\uparrow\uparrow} = j_{p\downarrow\downarrow} = 2M_p(1,0,0,0)$$
$$j_{p\uparrow\downarrow} = j_{p\downarrow\uparrow} = 0$$

- The spin-averaged ME summing over the 8 allowed helicity states



$$\langle|M_{fi}^2|\rangle = \frac{1}{4}\frac{e^4}{q^4}(16M_p^2 m_e^2)(4c^2+4s^2) = \frac{16M_p^2 m_e^2 e^4}{q^4}$$

where $q^2 = (p_1 - p_3)^2 = (0, \vec{p}_1 - \vec{p}_3)^2 = -4|\vec{p}|^2 \sin^2(\theta/2)$

$$\langle|M_{fi}^2|\rangle = \frac{M_p^2 m_e^2 e^4}{|\vec{p}|^4 \sin^4(\theta/2)}$$

Note: in this limit all angular dependence is in the propagator

- The formula for the differential cross-section in the lab. frame

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2 \qquad (8)$$

- Here the electron is non-relativistic so $E \sim m_e \ll M_p$ and we can neglect E_1 in the denominator of equation (8)

$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 M_p^2}|M_{fi}|^2 = \frac{m_e^2 e^4}{64\pi^2 |\vec{p}|^4 \sin^4(\theta/2)}$$

- Writing $e^2 = 4\pi\alpha$ and the kinetic energy of the electron as $E_K = p^2/2m_e$

$$\Rightarrow \quad \boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{\alpha^2}{16 E_K^2 \sin^4 \theta/2}} \qquad (9)$$

QED coupling

★ This is the normal expression for the Rutherford cross section. It could have been derived by considering the scattering of a non-relativistic particle in the static Coulomb potential of the proton $V(\vec{r})$, **without any consideration of the interaction due to the intrinsic magnetic moments** of the electron or proton. From this we can conclude, that in this non-relativistic limit only the interaction between the electric charges of the particles matters.

The Mott scattering cross-section

- For Rutherford scattering we are in the limit where the target recoil is neglected and the scattered particle is non-relativistic $E_K \ll m_e$
- The limit where the target recoil is neglected and the scattered particle is relativistic (i.e. just neglect the electron mass) is called Mott Scattering
- In this limit the electron currents, equations (4) and (6), become:

$$\overline{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = 2E\,[c, s, -is, c] \qquad \overline{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = E\,[0,0,0,0]$$

Relativistic ➡ Electron "helicity conserved"

- It is then straightforward to obtain the result:

$$\rightarrow \quad \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2\sin^4\theta/2}\cos^2\frac{\theta}{2} \qquad (10)$$

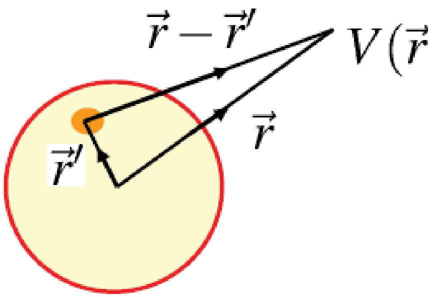
Rutherford formula with $E_K = E \quad (E \gg m_e)$

Overlap between initial/final state electron wave-functions. Just QM of spin ½

★ NOTE: we could have derived this expression from scattering of electrons in a static potential from a fixed point in space $V(\vec{r})$. The interaction is ELECTRIC rather than magnetic (spin-spin) in nature.

★ Still haven't taken into account the charge distribution of the proton…..

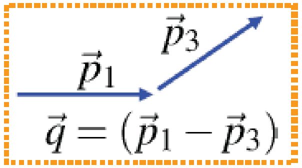
Form factors



- Consider the scattering of an electron in the static potential due to an extended charge distribution.
- The potential at $\vec{r}$ from the centre is given by:

$$V(\vec{r}) = \int \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} d^3\vec{r}' \quad \text{with} \quad \int \rho(\vec{r}) \mathrm{d}^3\vec{r} = 1$$

- In first order perturbation theory the matrix element is given by:



$$M_{fi} = \langle \psi_f | V(\vec{r}) | \psi_i \rangle = \int e^{-i\vec{p}_3.\vec{r}} V(\vec{r}) e^{i\vec{p}_1.\vec{r}} \mathrm{d}^3\vec{r}$$

$$= \int\int e^{i\vec{q}.\vec{r}} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} \mathrm{d}^3\vec{r}' \mathrm{d}^3\vec{r} = \int\int e^{i\vec{q}.(\vec{r}-\vec{r}')} e^{i\vec{q}.\vec{r}'} \frac{Q\rho(\vec{r}')}{4\pi|\vec{r}-\vec{r}'|} \mathrm{d}^3\vec{r}' \mathrm{d}^3\vec{r}$$

- Fix $\vec{r}'$ and integrate over $\mathrm{d}^3\vec{r}$ with substitution $\vec{R} = \vec{r} - \vec{r}'$

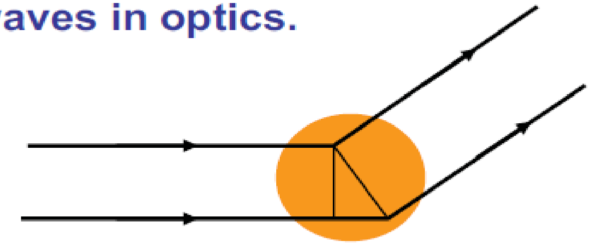
$$M_{fi} = \int e^{i\vec{q}.\vec{R}} \frac{Q}{4\pi|\vec{R}|} \mathrm{d}^3\vec{R} \int \rho(\vec{r}') e^{i\vec{q}.\vec{r}'} \mathrm{d}^3\vec{r}' = (M_{fi})_{point} F(\vec{q}^2)$$

★ The resulting matrix element is equivalent to the matrix element for scattering from a point source multiplied by the form factor

$$F(\vec{q}^2) = \int \rho(\vec{r}) e^{i\vec{q}.\vec{r}} \mathrm{d}^3\vec{r}$$

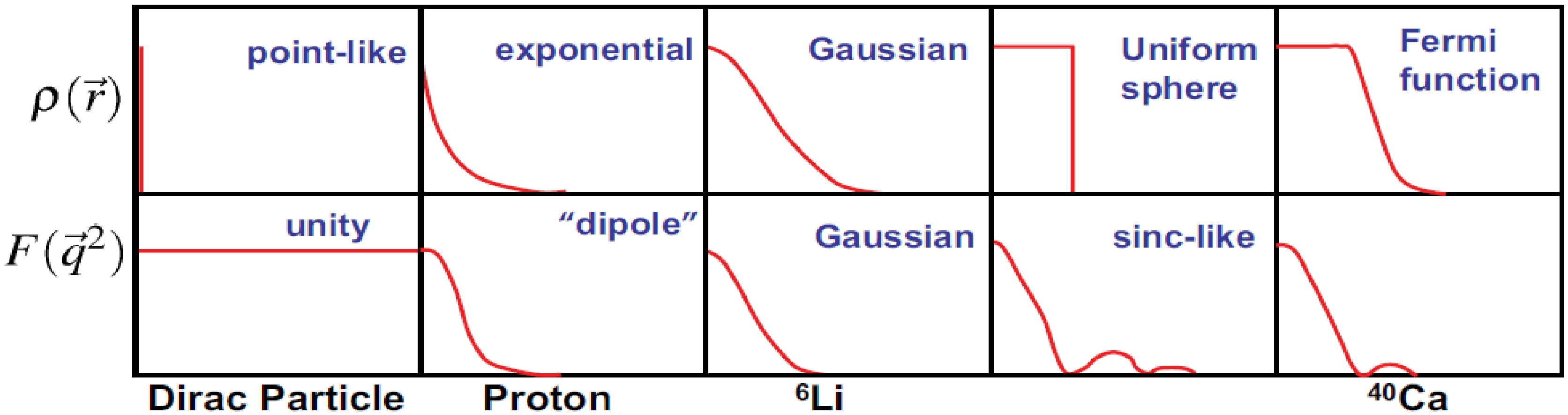
$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Mott}} \rightarrow \frac{\alpha^2}{4E^2\sin^4\theta/2}\cos^2\frac{\theta}{2}|F(\vec{q}^2)|^2$$

- There is nothing mysterious about form factors – similar to diffraction of plane waves in optics.



- The finite size of the scattering centre introduces a phase difference between plane waves "scattered from different points in space". If wavelength is long compared to size all waves in phase and $F(\vec{q}^2) = 1$

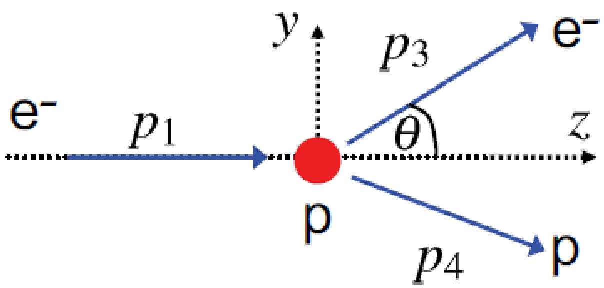
For example:



- NOTE that for a point charge the form factor is unity.

Point-like electron-proton scattering

- So far have only considered the case we the proton does not recoil... For $E_1 \gg m_e$ the general case is



$$
\begin{aligned}
p_1 &= (E_1, 0, 0, E_1) \\
p_2 &= (M, 0, 0, 0) \\
p_3 &= (E_3, 0, E_3 \sin\theta, E_3 \cos\theta) \\
p_4 &= (E_4, \vec{p}_4)
\end{aligned}
$$

- From Eqn. (2) with $m = m_e = 0$ the matrix element for this process is:

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1.p_2)(p_3.p_4) + (p_1.p_4)(p_2.p_3) - (p_1.p_3)M^2 \right] \quad (11)$$

- Experimentally observe scattered electron so eliminate p_4
- The scalar products not involving p_4 are:

$$p_1.p_2 = E_1 M \qquad p_1.p_3 = E_1 E_3 (1 - \cos\theta) \qquad p_2.p_3 = E_3 M$$

- From momentum conservation can eliminate p_4 : $p_4 = p_1 + p_2 - p_3$

$$p_3.p_4 = p_3.p_1 + p_3.p_2 - \cancel{p_3.p_3} = E_1 E_3 (1 - \cos\theta) + E_3 M$$

$$p_1.p_4 = \cancel{p_1.p_1} + p_1.p_2 - p_1.p_3 = E_1 M - E_1 E_3 (1 - \cos\theta)$$

$$p_1.p_1 = E_1^2 - |\vec{p}_1|^2 = m_e^2 \approx 0$$ i.e. neglect m_e

- Substituting these scalar products in Eqn. (11) gives

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1-p_3)^4} M E_1 E_3 \left[(E_1-E_3)(1-\cos\theta) + M(1+\cos\theta)\right]$$
$$= \frac{8e^4}{(p_1-p_3)^4} 2M E_1 E_3 \left[(E_1-E_3)\sin^2(\theta/2) + M\cos^2(\theta/2)\right] \quad (12)$$

- Now obtain expressions for $q^4 = (p_1-p_3)^4$ and (E_1-E_3)

$$q^2 = (p_1-p_3)^2 = p_1^2 + p_3^2 - 2p_1.p_3 = -2E_1E_3(1-\cos\theta) \quad (13)$$
$$= -4E_1E_3\sin^2\theta/2 \quad (14)$$

NOTE: $q^2 < 0$ Space-like

- For (E_1-E_3) start from

$$q.p_2 = (p_1-p_3).p_2 = M(E_1-E_3)$$

and use

$$(q+p_2)^2 = p_4^2 \qquad q = (p_1-p_3) = (p_4-p_2)$$
$$q^2 + p_2^2 + 2q.p_2 = p_4^2$$
$$q^2 + M^2 + 2q.p_2 = M^2$$
$$\rightarrow \quad q.p_2 = -q^2/2$$

- Hence the energy transferred to the proton:

$$E_1 - E_3 = -\frac{q^2}{2M} \quad (15)$$

Because q^2 is always negative $E_1 - E_3 > 0$ and the scattered electron is always lower in energy than the incoming electron

- Combining equations (11), (13) and (14):

$$\begin{aligned}\langle |M_{fi}|^2 \rangle &= \frac{8e^4}{16E_1^2 E_3^2 \sin^4\theta/2} 2ME_1E_3 \left[M\cos^2\theta/2 - \frac{q^2}{2M}\sin^2\theta/2 \right] \\ &= \frac{M^2 e^4}{E_1 E_3 \sin^4\theta/2} \left[\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2 \right]\end{aligned}$$

- For $E \gg m_e$ we have

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{ME_1} \right)^2 |M_{fi}|^2$$

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \frac{E_3}{E_1} \left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2 \right) \quad (16)$$

Interpretation

- So far have derived the differential cross-section for $e^-p \to e^-p$ elastic scattering assuming point-like Dirac spin ½ particles. How should we interpret the equation?

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right)$$

- Compare with $$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2\sin^4\theta/2}\cos^2\frac{\theta}{2}$$

the important thing to note about the Mott cross-section is that it is equivalent to scattering of spin ½ electrons in a fixed electro-static potential. Here the term E_3/E_1 is due to the proton recoil.

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \underbrace{\frac{q^2}{2M^2}\sin^2\theta/2}\right)$$

- the new term: $\propto \sin^2\frac{\theta}{2}$ ⟷ Magnetic interaction : due to the spin-spin interaction

Interpretation

- The above differential cross-section depends on a single parameter. For an electron scattering angle θ, both q^2 and the energy, E_3, are fixed by kinematics

- Equating (13) and (15)

$$-2M(E_1 - E_3) = -2E_1E_3(1-\cos\theta)$$

$$\Rightarrow \quad \frac{E_3}{E_1} = \frac{M}{M + E_1(1-\cos\theta)}$$

- Substituting back into (13):

$$\Rightarrow \quad q^2 = -\frac{2ME_1^2(1-\cos\theta)}{M + E_1(1-\cos\theta)}$$

- e.g. $e^-p \rightarrow e^-p$ at E_{beam} = 529.5 MeV, look at scattered electrons at θ = 75°

For elastic scattering expect:

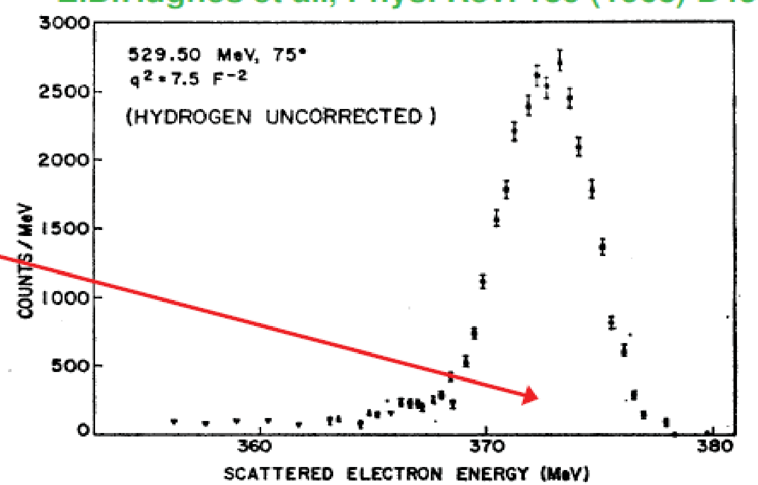
$$E_3 = \frac{ME_1}{M + E_1(1-\cos\theta)}$$

$$E_3 = \frac{938 \times 529}{938 + 529(1-\cos 75^\circ)} = 373 \text{ MeV}$$

The energy identifies the scatter as elastic.
Also know squared four-momentum transfer

$$|q^2| = \frac{2 \times 938 \times 529^2(1-\cos 75^\circ)}{938 + 529(1-\cos 75^\circ)} = 294\,\text{MeV}^2$$

E.B.Hughes et al., Phys. Rev. 139 (1965) B458



Elastic scattering from a finite size proton

★ **In general** the finite size of the proton can be accounted for by introducing two structure functions. One related to the charge distribution in the proton, $G_E(q^2)$ and the other related to the distribution of the magnetic moment of the proton, $G_M(q^2)$

- It can be shown that equation (16) generalizes to the **ROSENBLUTH FORMULA.**

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \theta/2} \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} \cos^2 \frac{\theta}{2} + 2\tau G_M^2 \sin^2 \frac{\theta}{2} \right)$$

with the Lorentz Invariant quantity:

$$\tau = -\frac{q^2}{4M^2} > 0$$

- Unlike our previous discussion of form factors, here the form factors are a function of q^2 rather than $\vec{q}^2$ and cannot simply be considered in terms of the FT of the charge and magnetic moment distributions.

But $q^2 = (E_1 - E_3)^2 - \vec{q}^2$ and from eq (15) obtain

$$\Rightarrow \quad -\vec{q}^2 = q^2 \left[1 - \left(\frac{q}{2M} \right)^2 \right]$$

So for $\frac{q^2}{4M^2} \ll 1$ we have $q^2 \approx -\vec{q}^2$ and $G(q^2) \approx G(\vec{q}^2)$

- Hence in the limit $q^2/4M^2 \ll 1$ we can interpret the structure functions in terms of the Fourier transforms of the charge and magnetic moment distributions

$$G_E(q^2) \approx G_E(\vec{q}^2) = \int e^{i\vec{q}.\vec{r}} \rho(\vec{r}) \mathrm{d}^3\vec{r}$$

$$G_M(q^2) \approx G_M(\vec{q}^2) = \int e^{i\vec{q}.\vec{r}} \mu(\vec{r}) \mathrm{d}^3\vec{r}$$

- Note in deriving the Rosenbluth formula we assumed that the proton was a spin-half Dirac particle, i.e.

$$\vec{\mu} = \frac{e}{M}\vec{S}$$

- However, the experimentally measured value of the proton magnetic moment is larger than expected for a point-like Dirac particle:

$$\vec{\mu} = 2.79\frac{e}{M}\vec{S}$$

So for the proton expect

$$G_E(0) = \int \rho(\vec{r}) \mathrm{d}^3\vec{r} = 1 \qquad G_M(0) = \int \mu(\vec{r}) \mathrm{d}^3\vec{r} = \mu_p = +2.79$$

- Of course the anomalous magnetic moment of the proton is already evidence that it is not point-like !

Measuring from-factors

- Express the Rosenbluth formula as:

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right)$$

where $$\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\cos^2\frac{\theta}{2}$$

i.e. the Mott cross-section including the proton recoil. It corresponds to scattering from a spin-0 proton.

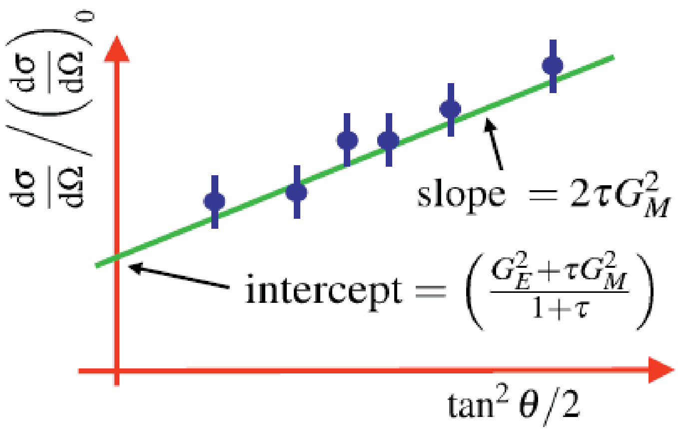
- At very low q^2: $\tau = -q^2/4M^2 \approx 0$

$$\frac{d\sigma}{d\Omega} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0 \approx G_E^2(q^2)$$

- At high q^2: $\tau \gg 1$

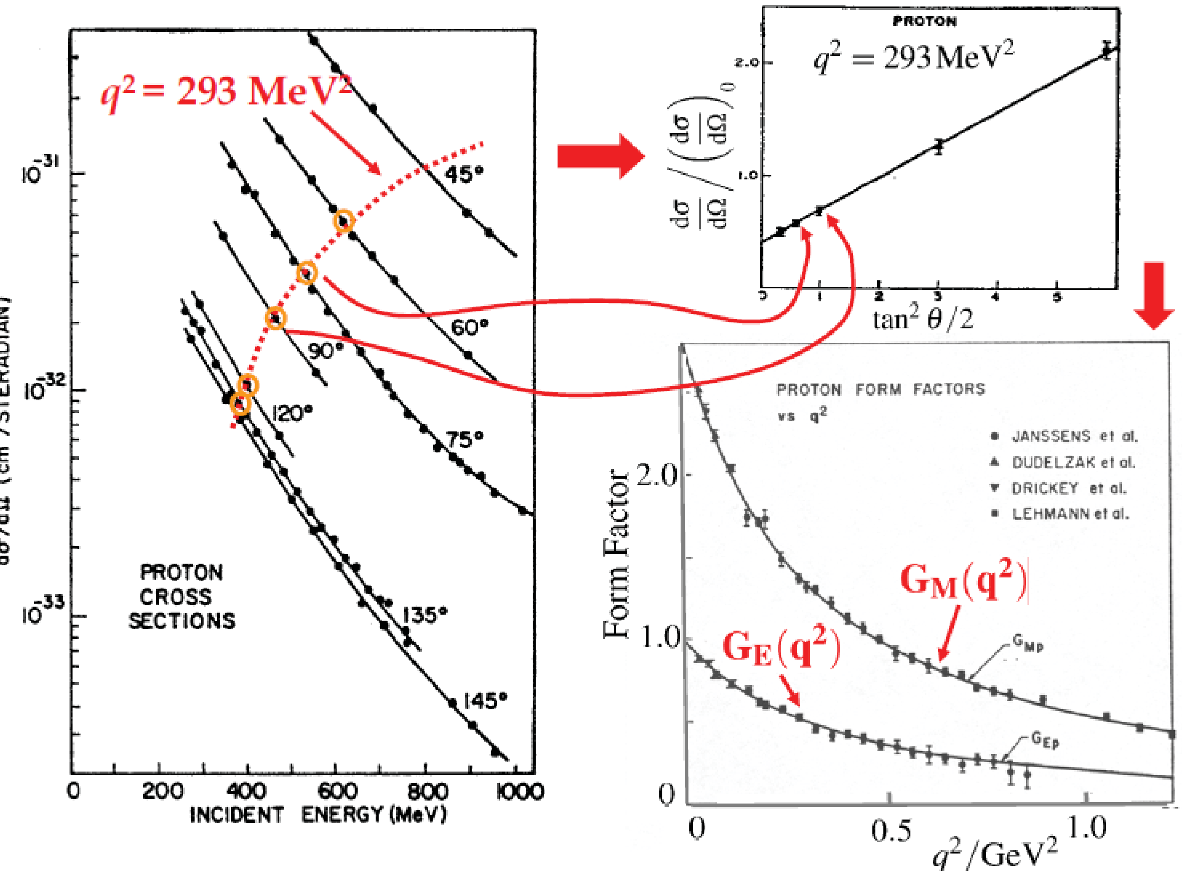
$$\frac{d\sigma}{d\Omega} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0 \approx \left(1 + 2\tau \tan^2\frac{\theta}{2}\right) G_M^2(q^2)$$

- In general we are sensitive to both structure functions! These can be resolved from the angular dependence of the cross section at FIXED q^2



- **EXAMPLE:** $e^-p \to e^-p$ at E_{beam} = 529.5 MeV
  - Electron beam energies chosen to give certain values of q^2
  - Cross sections measured to 2-3 %

E.B.Hughes et al., Phys. Rev. 139 (1965) B458



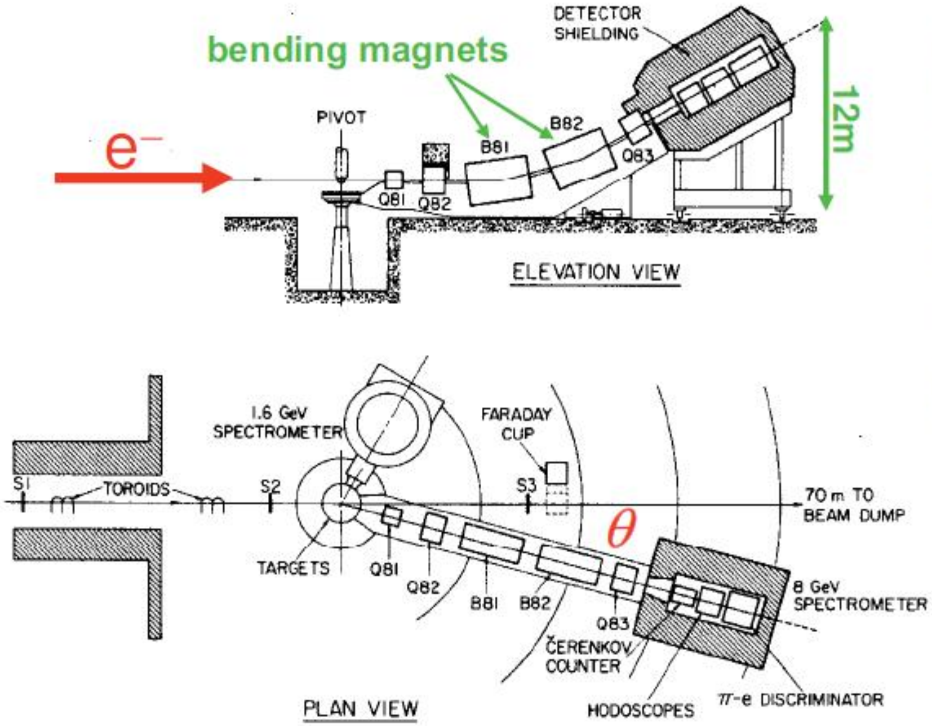
NOTE

Experimentally find $G_M(q^2) = 2.79G_E(q^2)$, i.e. the electric and and magnetic form factors have same distribution

Higher energy electron-proton scattering

★ **Use electron beam from SLAC LINAC:** $5 < E_{beam} < 20$ GeV

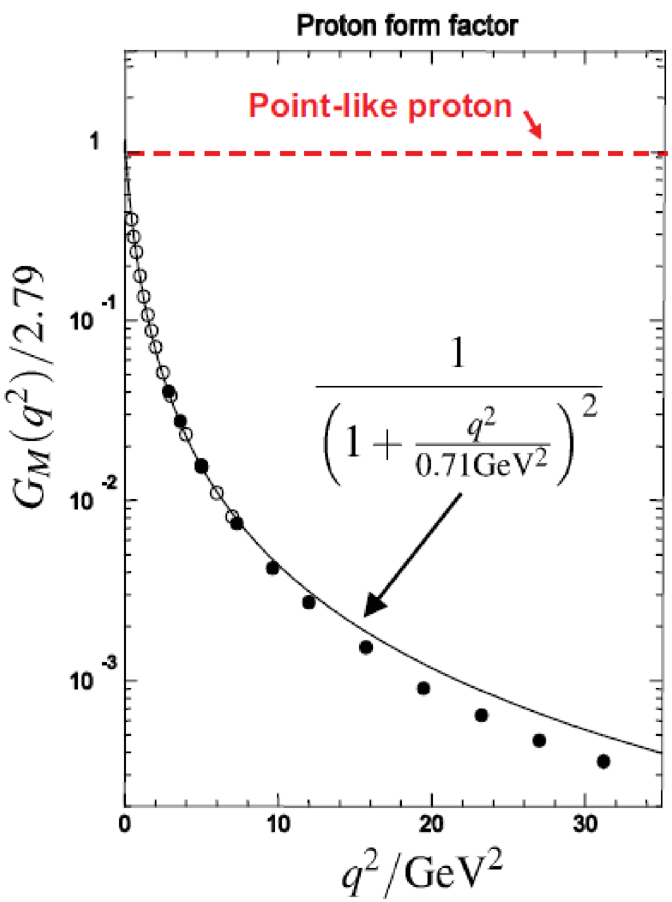
- **Detect scattered electrons using the "8 GeV Spectrometer"**



P.N.Kirk et al., Phys Rev D8 (1973) 63

High q² → Measure $G_M(q^2)$

High q^2 results



R.C.Walker et al., Phys. Rev. D49 (1994) 5671
A.F.Sill et al., Phys. Rev. D48 (1993) 29

★ Form factor falls rapidly with q^2
- Proton is not point-like
- Good fit to the data with "dipole form":

$$G_E^p(q^2) \approx \frac{G_M^p}{2.79} \approx \frac{1}{(1+q^2/0.71\mathrm{GeV}^2)^2}$$

★ Taking FT find spatial charge and magnetic moment distribution

$$\rho(r) \approx \rho_0 e^{-r/a}$$

with $a \approx 0.24$ fm

- Corresponds to a rms charge radius

$$r_{rms} \approx 0.8 \text{ fm}$$

★ Although suggestive, does not imply proton is composite !

★ **Note:** so far have only considered **ELASTIC scattering;**

Summary: elastic scattering

★ For elastic scattering of relativistic electrons from a point-like Dirac proton:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \frac{E_3}{E_1} \left(\cos^2\frac{\theta}{2} - \frac{q^2}{2M^2} \sin^2\frac{\theta}{2} \right)$$

Rutherford | Proton recoil | Electric/ Magnetic scattering | Magnetic term due to spin

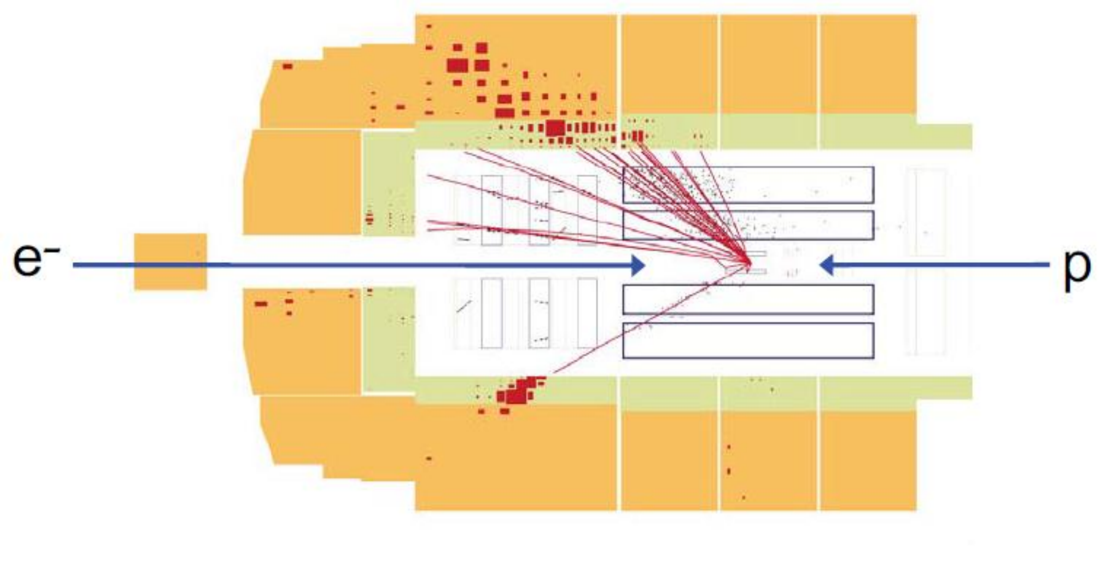
★ For elastic scattering of relativistic electrons from an extended proton:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} \cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2} \right)$$

Rosenbluth Formula

★ Electron elastic scattering from protons demonstrates that the proton is an extended object with rms charge radius of ~0.8 fm

Deep-Inelastic scattering



e^- p Elastic Scattering at very high q^2

★ At high q^2 the Rosenbluth expression for elastic scattering becomes

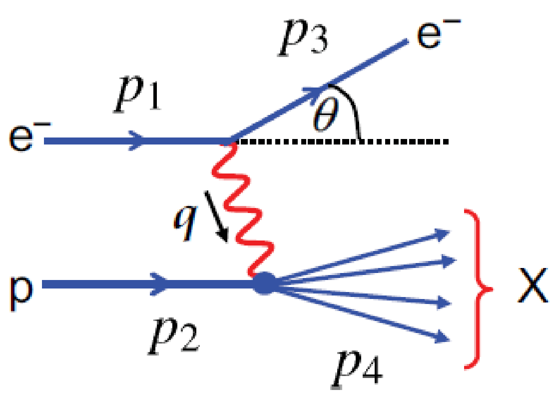
$$\left(\frac{d\sigma}{d\Omega}\right)_{elastic} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\left(\frac{q^2}{2M^2}G_M^2\sin^2\frac{\theta}{2}\right) \qquad \tau = -\frac{q^2}{4M^2} \gg 1$$

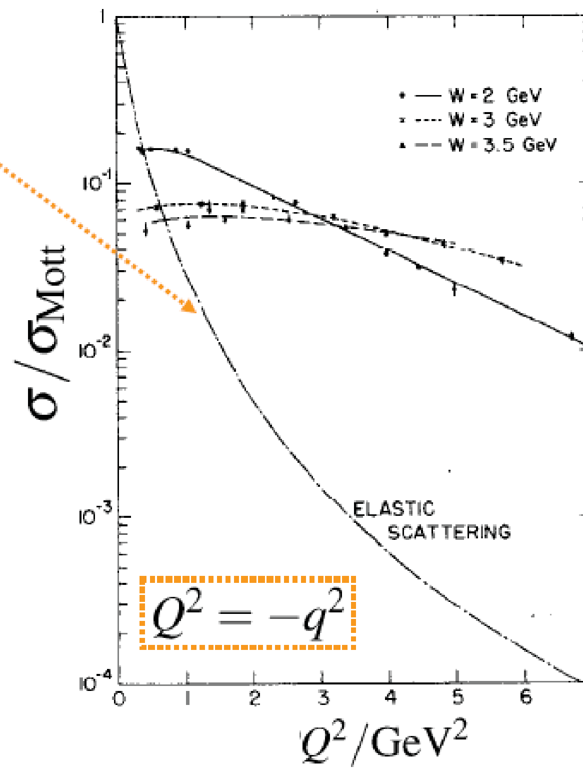
- From e^- p elastic scattering, the proton magnetic form factor is

$$G_M(q^2) \approx \frac{1}{(1+q^2/0.71\mathrm{GeV}^2)^2} \Rightarrow G_M(q^2) \propto q^{-4} \quad \text{at high } q^2$$

$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{elastic} \propto q^{-6}$$

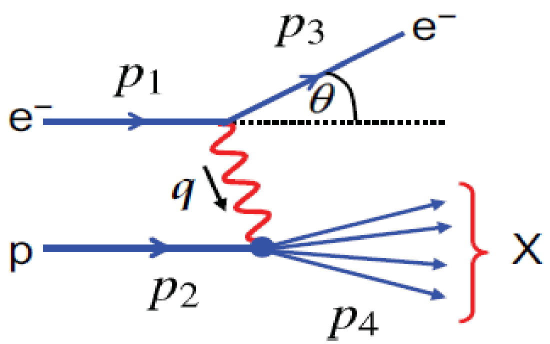
- Due to the finite proton size, elastic scattering at high q^2 is unlikely and inelastic reactions where the proton breaks up dominate.





M.Breidenbach et al., Phys. Rev. Lett. 23 (1969) 935

Kinematics of inelastic scattering



- For inelastic scattering the mass of the final state hadronic system is no longer the proton mass, M
- The final state hadronic system must contain at least one **baryon** which implies the final state invariant mass $M_X > M$

$$M_X^2 = p_4^2 = (E_4^2 - |\vec{p}_4|^2)$$

✶ For inelastic scattering introduce four new kinematic variables: x, y, ν, Q^2

★ **Define:**

$$x \equiv \frac{Q^2}{2p_2.q}$$

Bjorken x (Lorentz Invariant)

where $Q^2 \equiv -q^2$ $\qquad Q^2 > 0$

- Here $M_X^2 = p_4^2 = (q + p_2)^2 = -Q^2 + 2p_2.q + M^2$

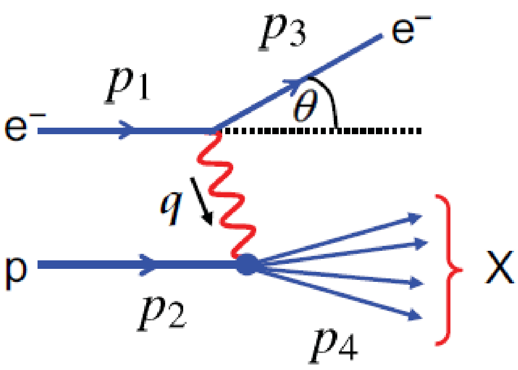
⇨ $Q^2 = 2p_2.q + M^2 - M_X^2$ ➡ $Q^2 \leq 2p_2.q$

Note: in many text books W is often used in place of M_X

hence $0 < x < 1$ inelastic $\qquad x = 1$ elastic

Proton intact $M_X = M$

★ **Define:** $$y \equiv \frac{p_2 . q}{p_2 . p_1}$$ (Lorentz Invariant)



• In the Lab. Frame:

$$p_1 = (E_1, 0, 0, E_1) \quad p_2 = (M, 0, 0, 0)$$
$$q = (E_1 - E_3, \vec{p}_1 - \vec{p}_3)$$

$$\Rightarrow \quad y = \frac{M(E_1 - E_3)}{ME_1} = 1 - \frac{E_3}{E_1}$$

So y is the fractional energy loss of the incoming particle

$$0 < y < 1$$

• In the C.o.M. Frame (neglecting the electron and proton masses):

$$p_1 = (E, 0, 0, E); \quad p_2 = (E, 0, 0, -E); \quad p_3 = (E, E\sin\theta^*, 0, E\cos\theta^*)$$

$$\Rightarrow \quad y = \tfrac{1}{2}(1 - \cos\theta^*) \qquad \text{for } E \gg M$$

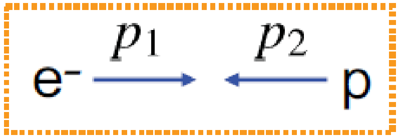
★ **Finally Define:** $$\nu \equiv \frac{p_2 . q}{M}$$ (Lorentz Invariant)

• In the Lab. Frame: $\nu = E_1 - E_3$

ν is the energy lost by the incoming particle

Relationship between kinematic variables

- Can rewrite the new kinematic variables in terms of the squared centre-of-mass energy, s, for the electron-proton collision



$$s = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1.p_2 = 2p_1.p_2 + M^2 + \cancel{m_e^2}$$

$$2p_1.p_2 = s - M^2$$

Neglect mass of electron

- For a fixed centre-of-mass energy, it can then be shown that the four kinematic variables

$$Q^2 \equiv -q^2 \qquad x \equiv \frac{Q^2}{2p_2.q} \qquad y \equiv \frac{p_2.q}{p_2.p_1} \qquad \nu \equiv \frac{p_2.q}{M}$$

are not independent.

- i.e. the scaling variables x and y can be expressed as

$$x = \frac{Q^2}{2M\nu} \qquad y = \frac{2M}{s - M^2}\nu$$

Note the simple relationship between y and ν

and $$xy = \frac{Q^2}{s - M^2} \quad \Rightarrow \quad Q^2 = (s - M^2)xy$$

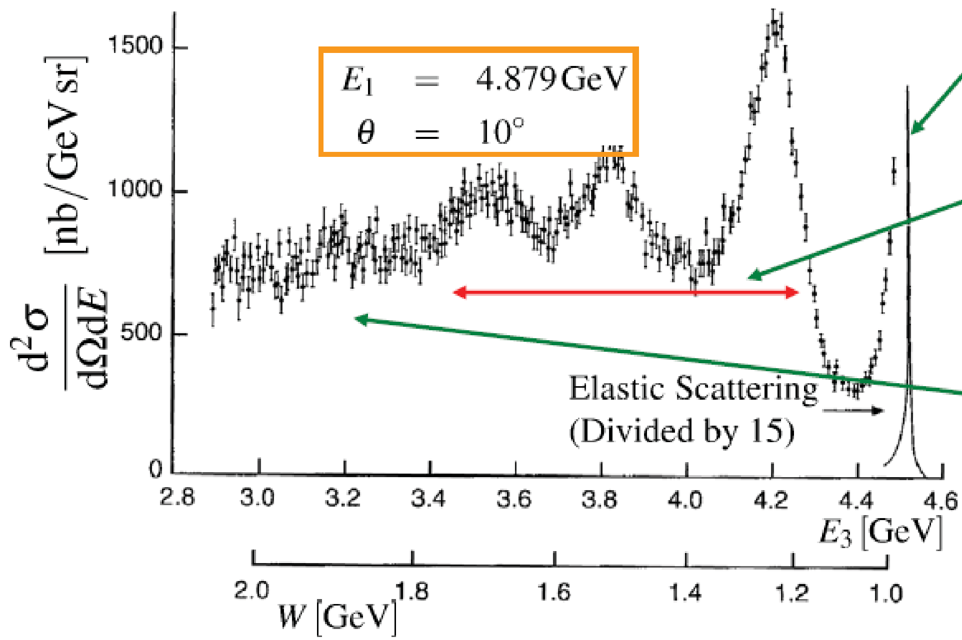
- For a fixed centre of mass energy, the interaction kinematics are completely defined by any <u>two</u> of the above kinematic variables (except y and ν)

- For elastic scattering $(x = 1)$ there is only one independent variable. As we saw previously if you measure electron scattering angle know everything else.

Inelastic scattering

Example: Scattering of 4.879 GeV electrons from protons at rest

- Place detector at 10° to beam and measure the energies of scattered e^{-}
- Kinematics fully determined from the electron energy and angle !
- e.g. for this energy and angle : the invariant mass of the final state hadronic system $W^2 = M_X^2 = 10.06 - 2.03E_3$ (try and show this)



Elastic Scattering

proton remains intact

$$W = M$$

Inelastic Scattering

produce "excited states" of proton e.g. $\Delta^{+}(1232)$

$$W = M_{\Delta}$$

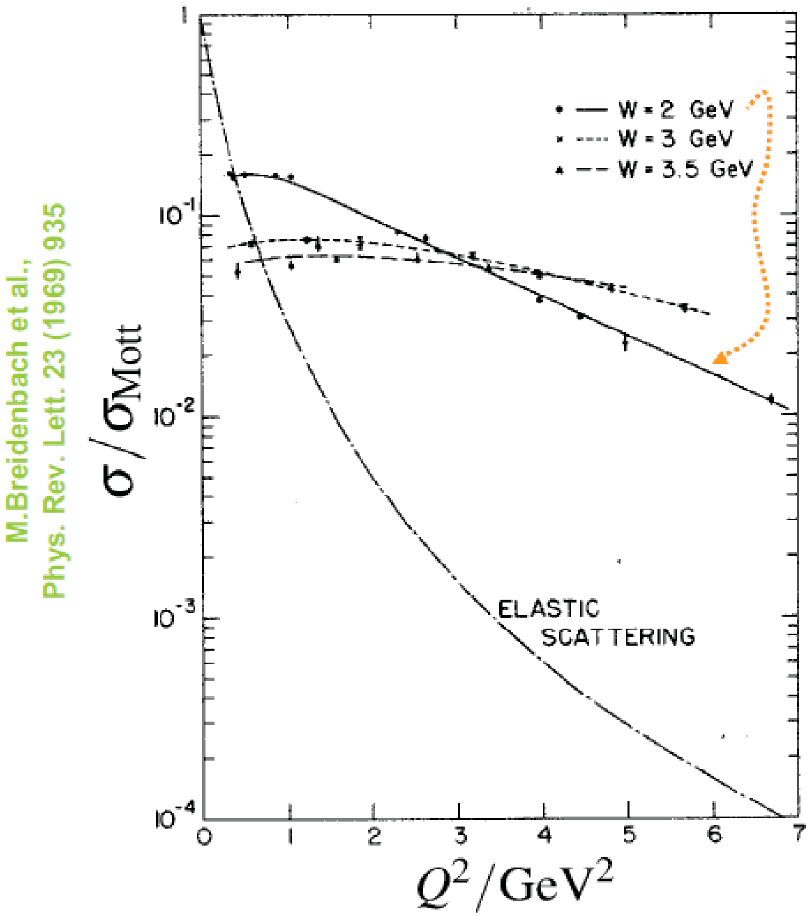
Deep Inelastic Scattering

proton breaks up resulting in a many particle final state

DIS = large W

Inelastic cross-sections

- Repeat experiments at different angles/beam energies and determine q^2 dependence of elastic and inelastic cross-sections



- Elastic scattering falls of rapidly with q^2 due to the proton not being point-like (i.e. form factors)
- Inelastic scattering cross sections only weakly dependent on q^2
- Deep Inelastic scattering cross sections almost independent of q^2 !

i.e. "Form factor" → 1

→ **Scattering from point-like objects within the proton !**

Elastic -> Inelastic scattering

★ **Recall:** Elastic scattering

- Only one independent variable. In Lab. frame express differential cross section in terms of the electron scattering angle (Rosenbluth formula)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4 \theta/2} \frac{E_3}{E_1} \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} \cos^2 \frac{\theta}{2} + 2\tau G_M^2 \sin^2 \frac{\theta}{2} \right) \qquad \tau = \frac{Q^2}{4M^2}$$

Note: here the energy of the scattered electron is determined by the angle.

- In terms of the Lorentz invariant kinematic variables can express this differential cross section in terms of Q^2

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4} \left[\frac{G_E^2 + \tau G_M^2}{(1+\tau)} \left(1 - y - \frac{M^2 y^2}{Q^2}\right) + \frac{1}{2} y^2 G_M^2 \right]$$

which can be written as:

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4} \left[f_2(Q^2) \left(1 - y - \frac{M^2 y^2}{Q^2}\right) + \frac{1}{2} y^2 f_1(Q^2) \right]$$

★ Inelastic scattering

- For Deep Inelastic Scattering have two independent variables. Therefore need a double differential cross section

Deep Inelastic scattering

★ It can be shown that **the most general Lorentz Invariant expression** for $e^-p \to e^-X$ inelastic scattering (via a single exchanged photon is):

$$\frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{M^2y^2}{Q^2}\right)\frac{F_2(x,Q^2)}{x} + y^2F_1(x,Q^2)\right] \quad (1)$$

INELASTIC SCATTERING

c.f.
$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1 - y - \frac{M^2y^2}{Q^2}\right)f_2(Q^2) + \frac{1}{2}y^2f_1(Q^2)\right]$$

ELASTIC SCATTERING

We will soon see how this connects to the quark model of the proton

- **NOTE:** The form factors have been replaced by the **STRUCTURE FUNCTIONS**

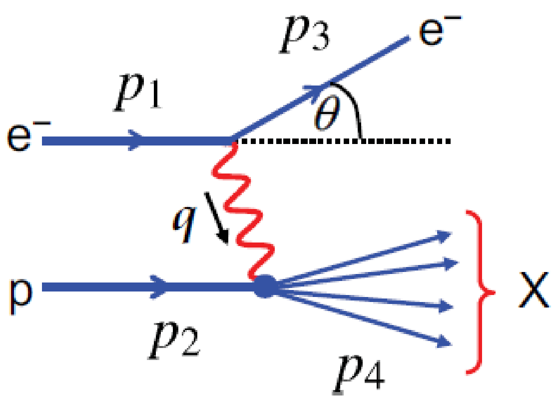
$$F_1(x,Q^2) \quad \text{and} \quad F_2(x,Q^2)$$

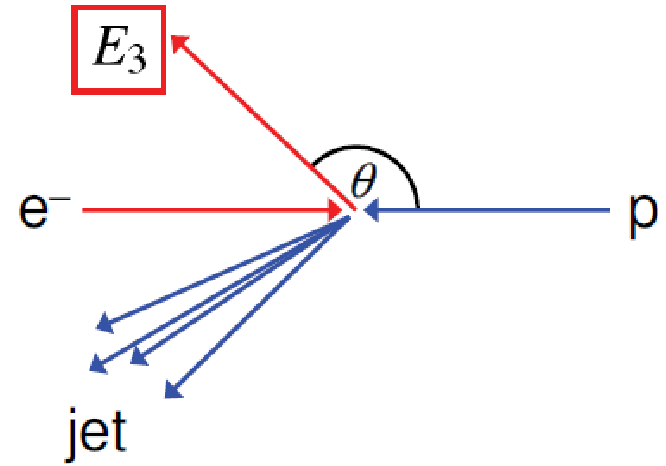
which are a function of x and Q^2: can not be interpreted as the Fourier transforms of the charge and magnetic moment distributions. We shall soon see that they describe the **momentum distribution** of the quarks within the proton

★ In the limit of high energy (or more correctly $Q^2 \gg M^2y^2$) eqn. (1) becomes:

$$\frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2F_1(x,Q^2)\right] \quad (2)$$

- In the Lab. frame it is convenient to express the cross section in terms of the angle, θ, and energy, E_3, of the scattered electron – **experimentally well measured.**





$$Q^2 = 4E_1E_3\sin^2\theta/2; \quad x = \frac{Q^2}{2M(E_1 - E_3)}; \quad y = 1 - \frac{E_3}{E_1}; \quad \nu = E_1 - E_3$$

- In the Lab. frame, Equation (2) becomes:

$$\frac{d^2\sigma}{dE_3 d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\left[\frac{1}{\nu}F_2(x,Q^2)\cos^2\frac{\theta}{2} + \frac{2}{M}F_1(x,Q^2)\sin^2\frac{\theta}{2}\right] \quad (3)$$

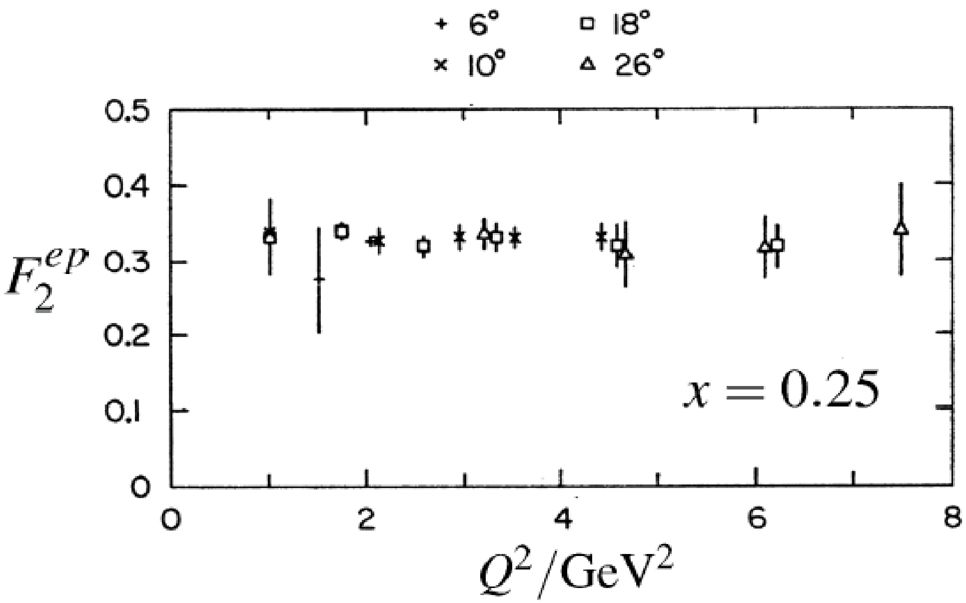
Electromagnetic Structure Function

Pure Magnetic Structure Function

Measuring the structure functions

★ To determine $F_1(x,Q^2)$ and $F_2(x,Q^2)$ for a given x and Q^2 need measurements of the differential cross section at several different scattering angles and incoming electron beam energies

Example: electron-proton scattering F_2 vs. Q^2 at fixed x



J.T.Friedman + H.W.Kendall, Ann. Rev. Nucl. Sci. 22 (1972) 203

♦ Experimentally it is observed that both F_1 and F_2 are (almost) independent of Q^2

Bjorken scalling and Callan-Gross relation

★ The near (see later) independence of the structure functions on Q^2 is known as Bjorken Scaling, i.e.

$$F_1(x,Q^2) \rightarrow F_1(x) \qquad F_2(x,Q^2) \rightarrow F_2(x)$$

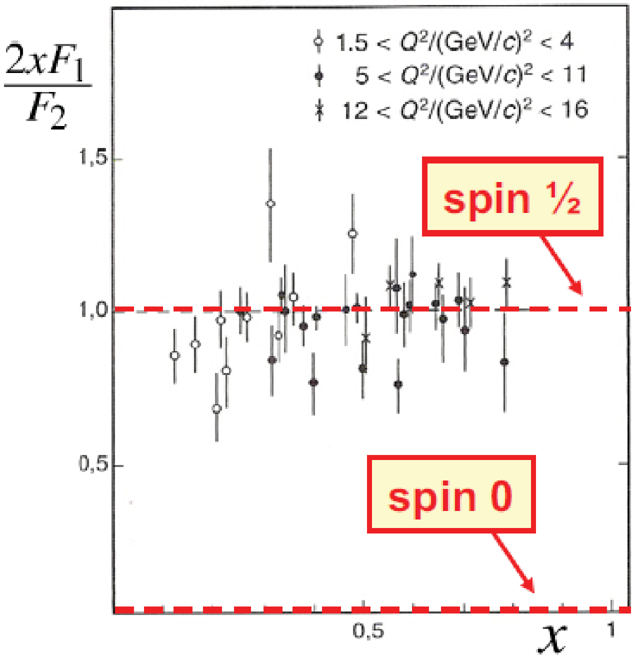
- It is strongly suggestive of scattering from point-like constituents within the proton

★ It is also observed that $F_1(x)$ and $F_2(x)$ are not independent but satisfy the Callan-Gross relation

$$F_2(x) = 2xF_1(x)$$

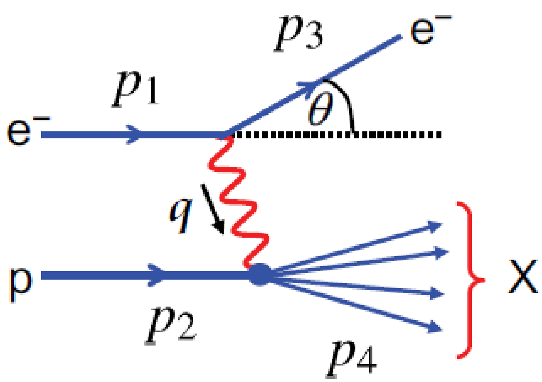
- As we shall soon see this is exactly what is expected for scattering from spin-half quarks.

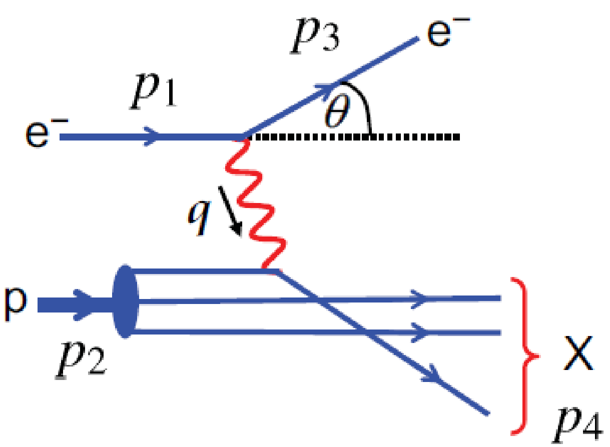
- <u>Note</u> if quarks were spin zero particles we would expect the purely magnetic structure function to be zero, i.e. $F_1(x) = 0$



The quark-parton model

- Before quarks and gluons were generally accepted Feynman proposed that the proton was made up of point-like constituents "partons"
- Both Bjorken Scaling and the Callan-Gross relationship can be explained by assuming that Deep Inelastic Scattering is dominated by the scattering of a single virtual photon from point-like spin-half constituents of the proton.

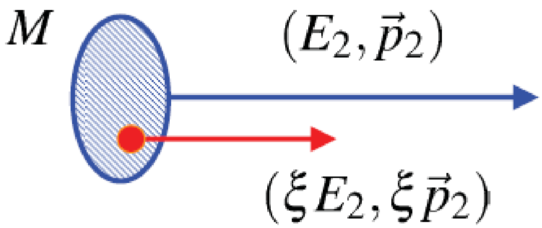


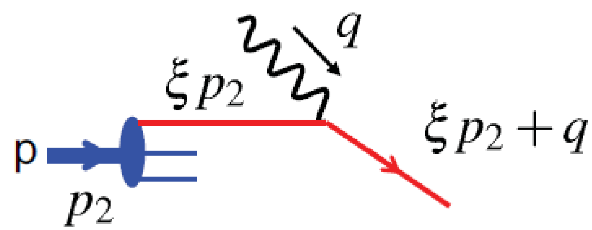


Scattering from a proton with structure functions → Scattering from a point-like quark within the proton

★ How do these two pictures of the interaction relate to each other?

- In the parton model the basic interaction is **ELASTIC** scattering from a **"quasi-free"** spin-½ quark in the proton, i.e. treat the quark as a free particle!
- The parton model is most easily formulated in a frame where the proton has very high energy, often referred to as the **"infinite momentum frame"**, where we can neglect the proton mass and $p_2 = (E_2, 0, 0, E_2)$
- In this frame can also neglect the mass of the quark and any momentum transverse to the direction of the proton.
- Let the quark carry a fraction ξ of the proton's four-momentum.





- After the interaction the struck quark's four-momentum is $\xi p_2 + q$

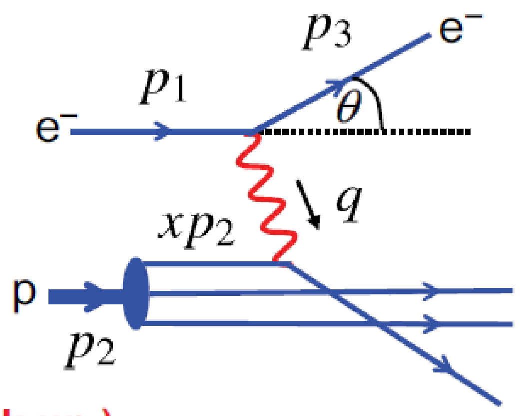
$$(\xi p_2 + q)^2 = m_q^2 \approx 0 \quad \Rightarrow \quad \cancel{\xi^2 p_2^2} + q^2 + 2\xi p_2 . q = 0 \qquad (\xi^2 p_2^2 = m_q^2 \approx 0)$$

$$\Rightarrow \quad \xi = \frac{Q^2}{2 p_2 . q} = x$$

Bjorken x can be identified as the fraction of the proton momentum carried by the struck quark (in a frame where the proton has very high energy)

- In terms of the proton momentum

$$s = (p_1 + p_2)^2 \simeq 2p_1.p_2 \quad y = \frac{p_2.q}{p_2.p_1} \quad x = \frac{Q^2}{2p_2.q}$$



- But for the underlying quark interaction

$$s^q = (p_1 + xp_2)^2 = 2xp_1.p_2 = xs$$

$$y_q = \frac{p_q.q}{p_q.p_1} = \frac{xp_2.q}{xp_2.p_1} = y$$

$$x_q = 1$$ (elastic, i.e. assume quark does not break up)

- Previously derived the Lorentz Invariant cross section for $e^-\mu^- \to e^-\mu^-$ elastic scattering in the ultra-relativistic limit

Now apply this to $e^-q \to e^-q$

$$\frac{d\sigma}{dq^2} = \frac{2\pi\alpha^2 e_q^2}{q^4}\left[1 + \left(1 + \frac{q^2}{s_q}\right)^2\right]$$

e_q is quark charge, i.e. $e_u = +2/3; \quad e_d = -1/3$

- Using $-q^2 = Q^2 = (s_q - m^2)x_q y_q$ ➡ $\frac{q^2}{s_q} = -y_q = -y$

$$\frac{d\sigma}{dQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}\left[1 + (1-y)^2\right]$$

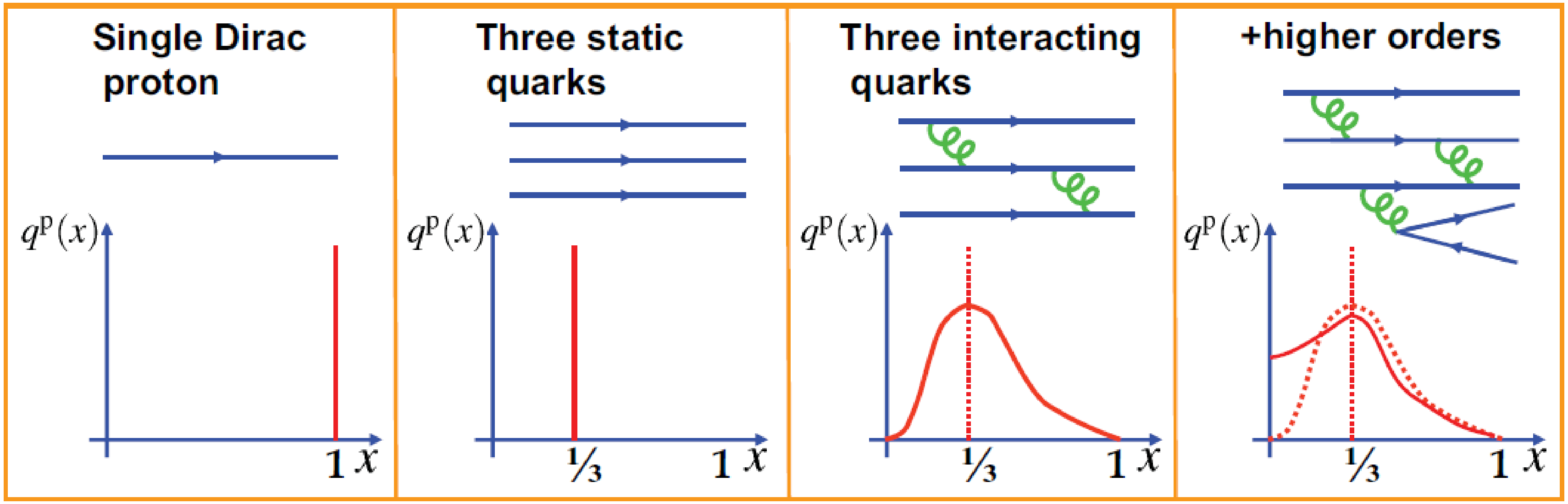
$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2 e_q^2}{Q^4}\left[(1-y)+\frac{y^2}{2}\right] \qquad (3)$$

★ This is the expression for the differential cross-section for elastic e^-q scattering from a quark carrying a fraction x of the proton momentum.

- Now need to account for distribution of quark momenta within proton

★ Introduce parton distribution functions such that $q^p(x)dx$ is the number of quarks of type q within a proton with momenta between $x \rightarrow x + dx$

- Expected form of the parton distribution function ?



★ The cross section for scattering from a **particular quark type** within the proton which in the range $x \to x + \mathrm{d}x$ is

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y) + \frac{y^2}{2}\right] \times e_q^2 q^{\mathrm{p}}(x)\mathrm{d}x$$

★ Summing over all types of quark within the proton gives the expression for the electron-proton scattering cross section

$$\frac{\mathrm{d}^2\sigma^{\mathrm{ep}}}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y) + \frac{y^2}{2}\right]\sum_q e_q^2 q^{\mathrm{p}}(x) \qquad (5)$$

★ Compare with the electron-proton scattering cross section in terms of structure functions (equation (2)):

$$\frac{\mathrm{d}^2\sigma}{\mathrm{d}x\mathrm{d}Q^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right] \qquad (6)$$

★ By comparing (5) and (6) obtain the parton model prediction for the structure functions in the general L.I. form for the differential cross section

$$F_2^{\mathrm{p}}(x,Q^2) = 2xF_1^{\mathrm{p}}(x,Q^2) = x\sum_q e_q^2 q^{\mathrm{p}}(x)$$

⇒ Can relate measured structure functions to the underlying quark distributions

The parton model predicts:

- **Bjorken Scaling** $F_1(x,Q^2) \rightarrow F_1(x) \qquad F_2(x,Q^2) \rightarrow F_2(x)$
  - ✶ **Due to scattering from point-like particles within the proton**
- **Callan-Gross Relation** $F_2(x) = 2xF_1(x)$
  - ✶ **Due to scattering from spin half Dirac particles where the magnetic moment is directly related to the charge; hence the "electro-magnetic" and "pure magnetic" terms are fixed with respect to each other.**

★ **At present parton distributions cannot be calculated from QCD**

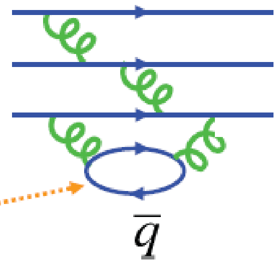
- **Can't use perturbation theory due to large coupling constant**

★ **Measurements of the structure functions enable us to determine the parton distribution functions !**

★ **For electron-proton scattering we have:**

$$F_2^{\mathrm{p}}(x) = x\sum_q e_q^2 q^{\mathrm{p}}(x)$$

- **Due to higher orders, the proton contains not only up and down quarks but also anti-up and anti-down quarks**
  (will neglect the small contributions from heavier quarks)



- For electron-proton scattering have:

$$F_2^{\mathrm{ep}}(x) = x\sum_q e_q^2 q^{\mathrm{p}}(x) = x\left(\frac{4}{9}u^{\mathrm{p}}(x) + \frac{1}{9}d^{\mathrm{p}}(x) + \frac{4}{9}\overline{u}^{\mathrm{p}}(x) + \frac{1}{9}\overline{d}^{\mathrm{p}}(x)\right)$$

- For electron-neutron scattering have:

$$F_2^{\mathrm{en}}(x) = x\sum_q e_q^2 q^{\mathrm{n}}(x) = x\left(\frac{4}{9}u^{\mathrm{n}}(x) + \frac{1}{9}d^{\mathrm{n}}(x) + \frac{4}{9}\overline{u}^{\mathrm{n}}(x) + \frac{1}{9}\overline{d}^{\mathrm{n}}(x)\right)$$

★ Now assume "isospin symmetry", i.e. that the neutron (ddu) is the same as a proton (uud) with up and down quarks interchanged, i.e.

$$d^{\mathrm{n}}(x) = u^{\mathrm{p}}(x); \qquad u^{\mathrm{n}}(x) = d^{\mathrm{p}}(x)$$

and define the neutron distributions functions in terms of those of the proton

$$u(x) \equiv u^{\mathrm{p}}(x) = d^{\mathrm{n}}(x); \qquad d(x) \equiv d^{\mathrm{p}}(x) = u^{\mathrm{n}}(x)$$

$$\overline{u}(x) \equiv \overline{u}^{\mathrm{p}}(x) = \overline{d}^{\mathrm{n}}(x); \qquad \overline{d}(x) \equiv \overline{d}^{\mathrm{p}}(x) = \overline{u}^{\mathrm{n}}(x)$$

giving:

$$F_2^{\mathrm{ep}}(x) = 2xF_1^{\mathrm{ep}}(x) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x) + \frac{4}{9}\overline{u}(x) + \frac{1}{9}\overline{d}(x)\right) \qquad (7)$$

$$F_2^{\mathrm{en}}(x) = 2xF_1^{\mathrm{en}}(x) = x\left(\frac{4}{9}d(x) + \frac{1}{9}u(x) + \frac{4}{9}\overline{d}(x) + \frac{1}{9}\overline{u}(x)\right) \qquad (8)$$

- Integrating (7) and (8) :

$$\int_0^1 F_2^{ep}(x)\mathrm{d}x = \int_0^1 x\left(\frac{4}{9}[u(x)+\overline{u}(x)] + \frac{1}{9}[d(x)+\overline{d}(x)]\right)\mathrm{d}x = \frac{4}{9}f_u + \frac{1}{9}f_d$$

$$\int_0^1 F_2^{en}(x)\mathrm{d}x = \int_0^1 x\left(\frac{4}{9}[d(x)+\overline{d}(x)] + \frac{1}{9}[u(x)+\overline{u}(x)]\right)\mathrm{d}x = \frac{4}{9}f_d + \frac{1}{9}f_u$$

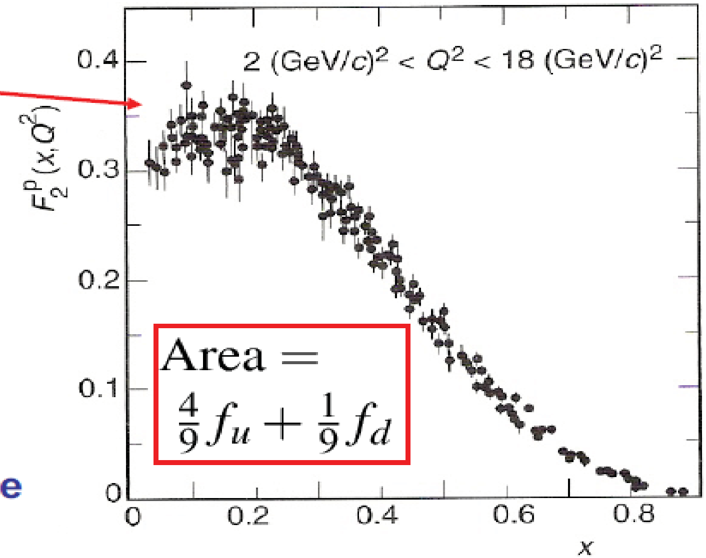
★ $f_u = \int_0^1 [xu(x) + x\overline{u}(x)]\mathrm{d}x$ is the fraction of the proton momentum carried by the up and anti-up quarks

Experimentally

$$\int F_2^{ep}(x)\mathrm{d}x \approx 0.18$$

$$\int F_2^{en}(x)\mathrm{d}x \approx 0.12$$

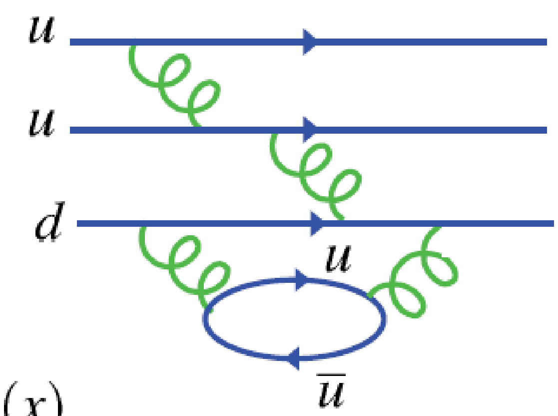
➡ $f_u \approx 0.36 \qquad f_d \approx 0.18$



★ In the proton, as expected, the up quarks carry twice the momentum of the down quarks

★ The quarks carry just over 50 % of the total proton momentum. The rest is carried by gluons (which being neutral doesn't contribute to electron-nucleon scattering).

Valence and Sea Quarks



- As we are beginning to see the proton is complex…
- The parton distribution function $u^{\mathrm{p}}(x) = u(x)$ includes contributions from the "valence" quarks and the virtual quarks produced by gluons: the "sea"
- Resolving into valence and sea contributions:

$$u(x) = u_{\mathrm{V}}(x) + u_{\mathrm{S}}(x) \qquad d(x) = d_{\mathrm{V}}(x) + d_{\mathrm{S}}(x)$$
$$\overline{u}(x) = \overline{u}_{\mathrm{S}}(x) \qquad \overline{d}(x) = \overline{d}_{\mathrm{S}}(x)$$

- The proton contains two valence up quarks and one valence down quark and would expect: $\int_0^1 u_{\mathrm{V}}(x)\mathrm{d}x = 2 \qquad \int_0^1 d_{\mathrm{V}}(x)\mathrm{d}x = 1$
- But no *a priori* expectation for the total number of sea quarks !
- But sea quarks arise from gluon quark/anti-quark pair production and with $m_u = m_d$ it is reasonable to expect

$$u_{\mathrm{S}}(x) = d_{\mathrm{S}}(x) = \overline{u}_{\mathrm{S}}(x) = \overline{d}_{\mathrm{S}}(x) = S(x)$$

- With these relations (7) and (8) become

$$F_2^{\mathrm{ep}}(x) = x\left(\frac{4}{9}u_{\mathrm{V}}(x) + \frac{1}{9}d_{\mathrm{V}}(x) + \frac{10}{9}S(x)\right) \qquad F_2^{\mathrm{en}}(x) = x\left(\frac{4}{9}d_{\mathrm{V}}(x) + \frac{1}{9}u_{\mathrm{V}}(x) + \frac{10}{9}S(x)\right)$$

Giving the ratio

$$\frac{F_2^{\text{en}}(x)}{F_2^{\text{ep}}(x)} = \frac{4d_V(x) + u_V(x) + 10S(x)}{4u_V(x) + d_V(x) + 10S(x)}$$

- The sea component arises from processes such as $g \to \overline{u}u$. Due to the $1/q^2$ dependence of the gluon propagator, much more likely to produce low energy gluons. Expect the sea to comprise of low energy $q/\overline{q}$
- Therefore at low x expect the sea to dominate:

$$\frac{F_2^{\text{en}}(x)}{F_2^{\text{ep}}(x)} \to 1 \quad \text{as} \quad x \to 0$$

Observed experimentally

- At high x expect the sea contribution to be small

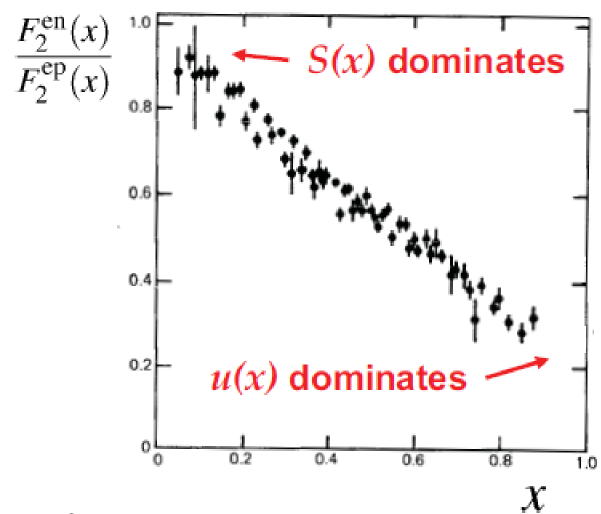
$$\frac{F_2^{\text{en}}(x)}{F_2^{\text{ep}}(x)} \to \frac{4d_V(x) + u_V(x)}{4u_V(x) + d_V(x)} \quad \text{as} \quad x \to 1$$

Note: $u_V = 2d_V$ would give ratio 2/3 as $x \to 1$

Experimentally $F_2^{\text{en}}(x)/F_2^{\text{ep}}(x) \to 1/4 \quad \text{as} \quad x \to 1$

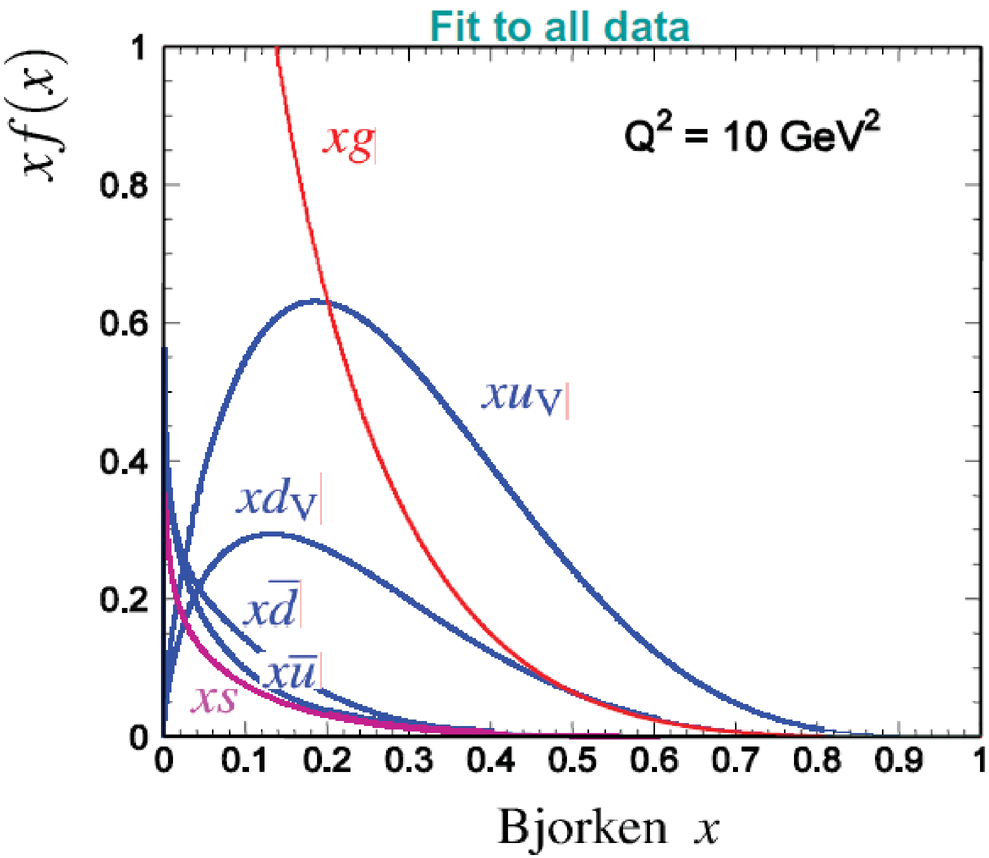
➡ $d(x)/u(x) \to 0 \quad \text{as} \quad x \to 1$

This behaviour is not understood.



Parton Distribution Functions (PDFs)

- Ultimately the parton distribution functions are obtained from a fit to all experimental data including neutrino scattering
  - Hadron-hadron collisions give information on gluon pdf $g(x)$

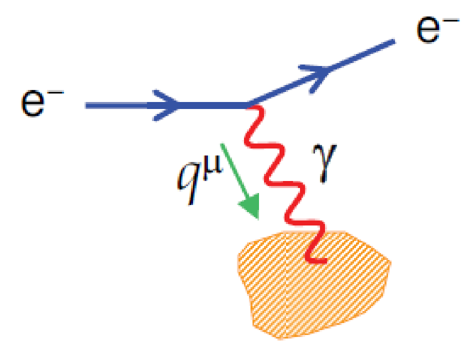


Note:

- Apart from at large x
  $u_V(x) \approx 2d_V(x)$
- For $x < 0.2$
  gluons dominate
- In fits to data assume
  $u_s(x) = \bar{u}(x)$
- $\bar{d}(x) > \bar{u}(x)$
  not understood – exclusion principle?
- Small strange quark component $s(x)$

Scaling violations

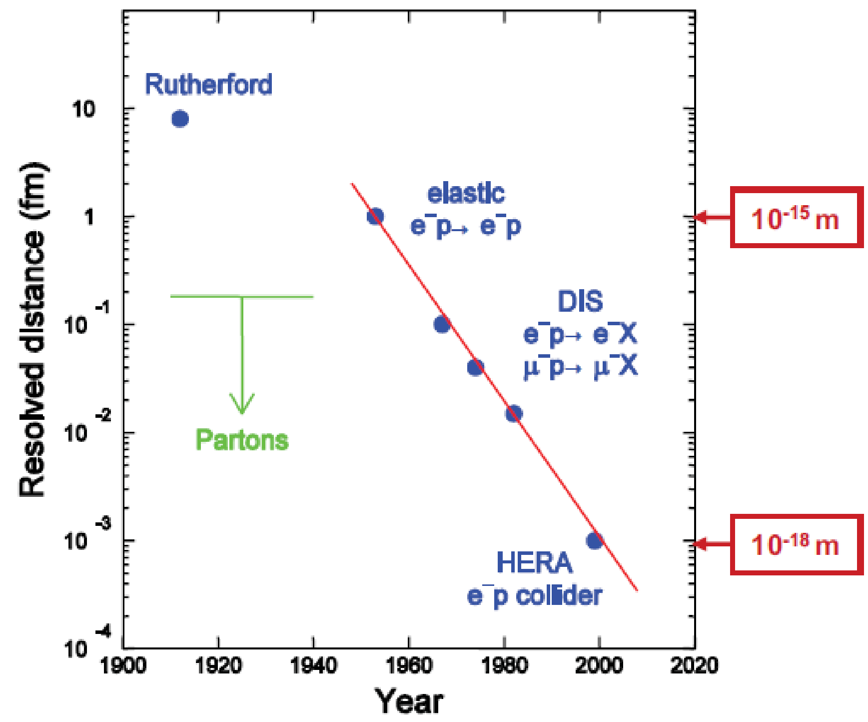
- In last 40 years, experiments have probed the proton with virtual photons of ever increasing energy
- Non-point like nature of the scattering becomes apparent when λ_γ ~ size of scattering centre



$$\lambda_\gamma = \frac{h}{|\vec{q}|} \sim \frac{1\,\mathrm{GeV\,fm}}{|\vec{q}|(\mathrm{GeV})}$$

- Scattering from point-like quarks gives rise to Bjorken scaling: no q^2 cross section dependence
- IF quarks were not point-like, at high q^2 (when the wavelength of the virtual photon ~ size of quark) would observe rapid decrease in cross section with increasing q^2.
- To search for quark sub-structure want to go to highest q^2

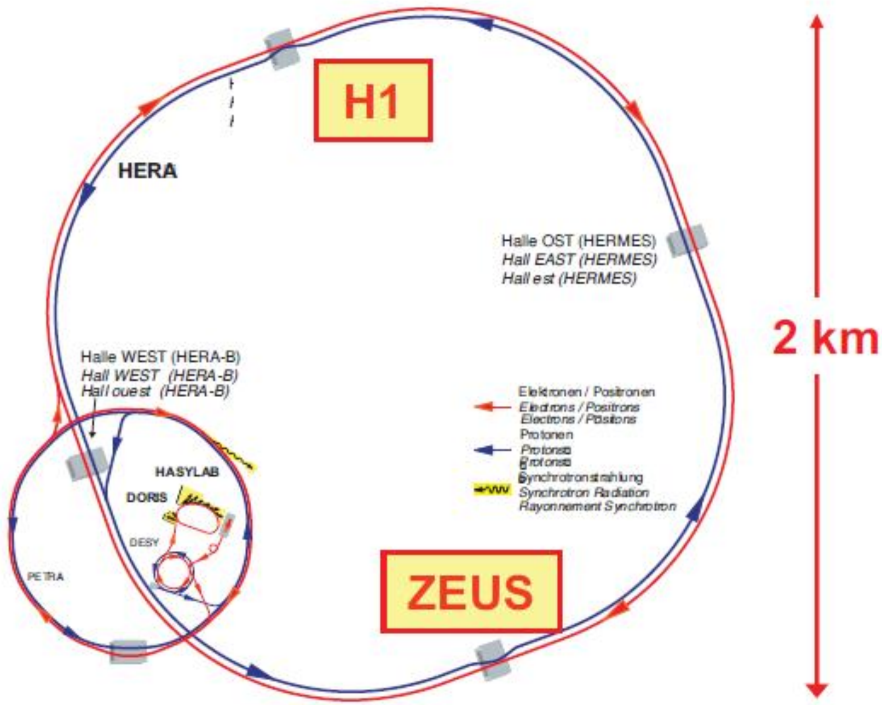
HERA



HERA e$^{\pm}$p Collider: 1991 - 2007

★ DESY (Deutsches Elektronen-Synchroton) Laboratory, Hamburg, Germany

$e^{\pm}$ 27.5 GeV → ← 820 GeV p $\sqrt{s} = 300$ GeV

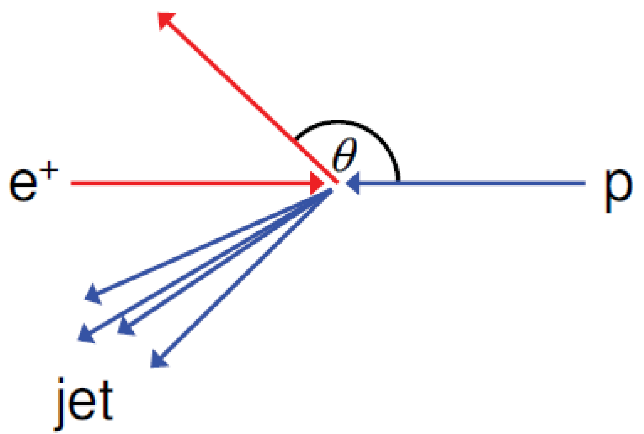


★ Two large experiments : H1 and ZEUS

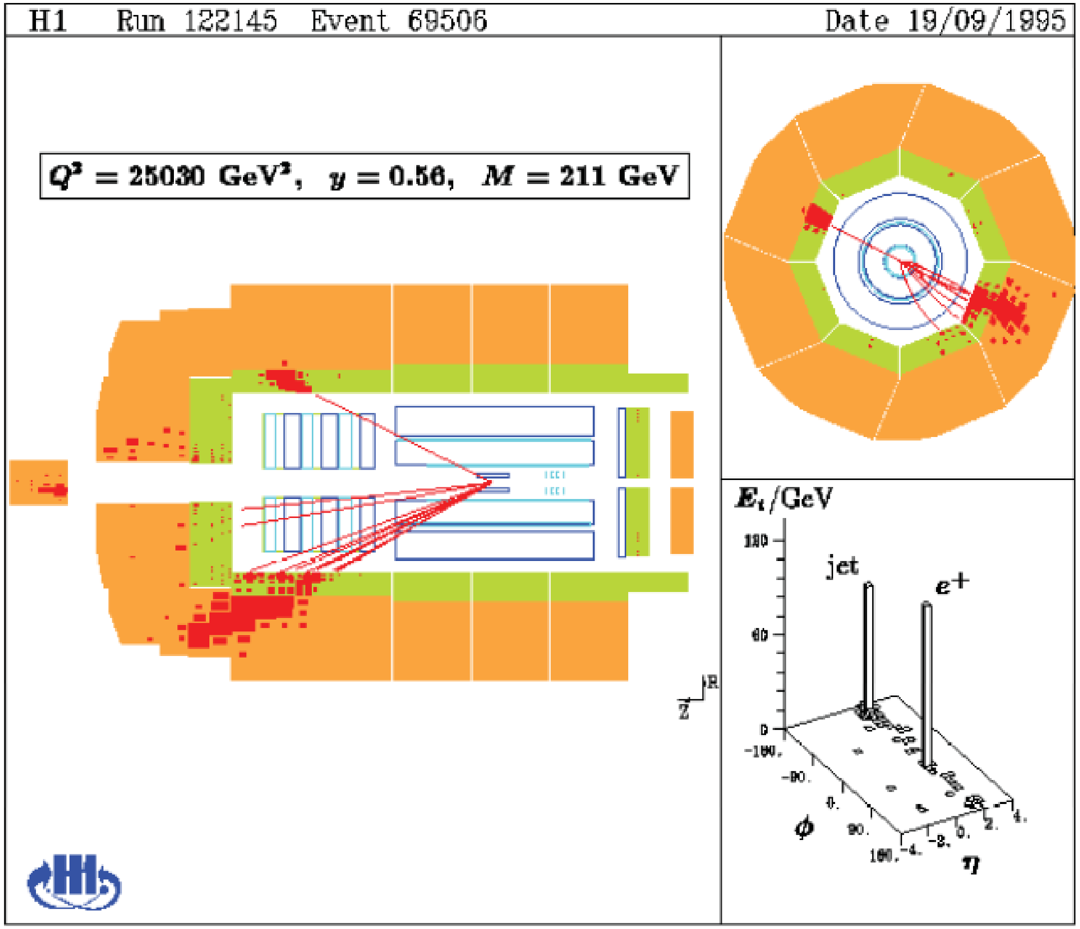
★ Probe proton at very high Q^2 and very low x

Example of a High Q^2 event in H1

- Event kinematics determined from electron angle and energy



- Also measure hadronic system (although not as precisely) – gives some redundancy



$F_2(x, Q^2)$ results

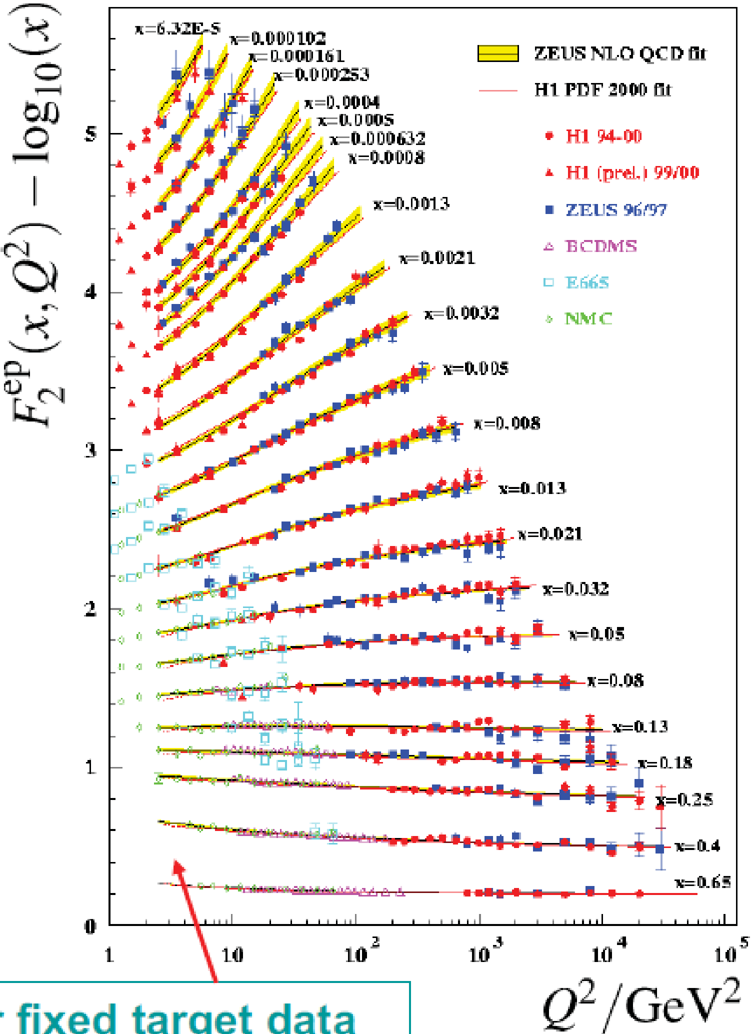
★ No evidence of rapid decrease of cross section at highest Q^2

➡ $R_{\text{quark}} < 10^{-18}\,\text{m}$

★ For $x > 0.05$, only weak dependence of F_2 on Q^2 : consistent with the expectation from the quark-parton model

★ But observe clear scaling violations, particularly at low x

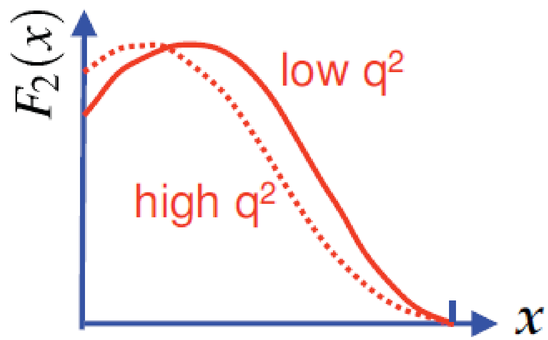
$$F_2(x,Q^2) \neq F_2(x)$$

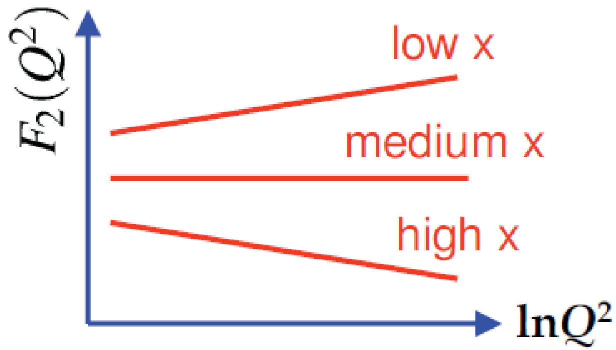


Earlier fixed target data

Origin of Scaling Violations

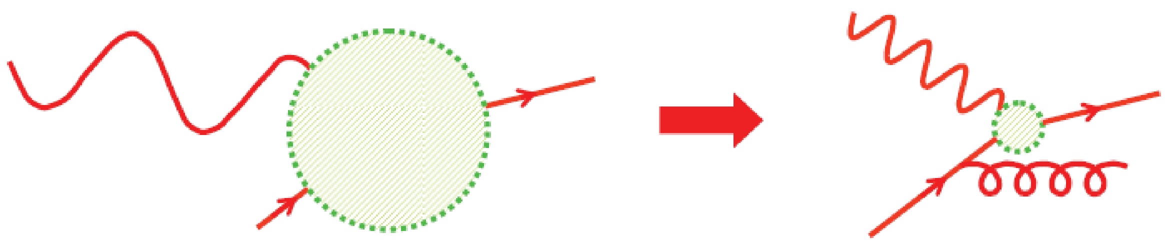
★ Observe "small" deviations from exact Bjorken scaling $F_2(x) \rightarrow F_2(x, Q^2)$





★ At high Q^2 observe more low x quarks

★ "**Explanation**": at high Q^2 (shorter wave-length) resolve finer structure: i.e. reveal quark is sharing momentum with gluons. At higher Q^2 expect to "see" more low x quarks



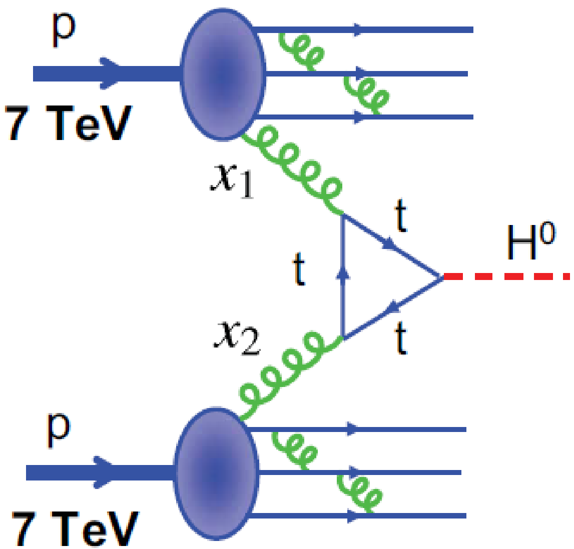
★ QCD cannot predict the x dependence of $F_2(x, Q^2)$

★ But QCD **can** predict the Q^2 dependence of $F_2(x, Q^2)$

Proton-proton colisions at the LHC

★ Measurements of structure functions not only provide a powerful test of QCD, the parton distribution functions are essential for the calculation of cross sections at pp and $p\bar{p}$ colliders.

- **Example:** Higgs production at the Large Hadron Collider LHC **Year 2012**
  - The LHC will collide 7 TeV protons on 7 TeV protons
  - However underlying collisions are between partons
  - Higgs production the LHC dominated by "gluon-gluon fusion"



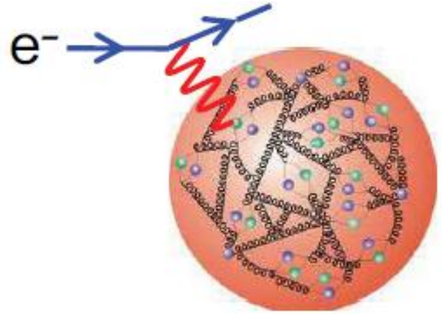
- Cross section depends on gluon PDFs

$$\sigma(pp \to HX) \sim \int_0^1 \int_0^1 g(x_1) g(x_2) \sigma(gg \to H) \mathrm{d}x_1 \mathrm{d}x_2$$

- Uncertainty in gluon PDFs lead to a ±5 % uncertainty in Higgs production cross section

- Prior to HERA data uncertainty was ±25 %

Summary



- At very high electron energies $\lambda \ll r_p$: the proton appears to be a sea of quarks and gluons.
- Deep Inelastic Scattering = Elastic scattering from the quasi-free constituent quarks
  - ➡ Bjorken Scaling $F_1(x, Q^2) \rightarrow F_1(x)$ — point-like scattering
  - ➡ Callan-Gross $F_2(x) = 2xF_1(x)$ — Scattering from spin-1/2
- Describe scattering in terms of parton distribution functions $u(x), d(x), \ldots$ which describe momentum distribution inside a nucleon
- The proton is much more complex than just uud - sea of anti-quarks/gluons
- Quarks carry only 50 % of the protons momentum – the rest is due to low energy gluons
- We will come back to this topic when we discuss neutrino scattering…