Elementary Particle Physics: theory and experiments

Theory:
Preliminaries and recap on the Standard Model
Feynman diagrams and cross-section calculations

Some slides from M. A. Thomson lectures at Cambridge University

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Short recap on Standard Model

Feynmann diagrams and
Cross-section calculations

Preliminaries: Natural Units

- S.I. UNITS: **kg m s** are a natural choice for "everyday" objects

- not very natural in particle physics
- instead use Natural Units based on the language of particle physics
  - **From Quantum Mechanics** - the unit of action : $\hbar$
  - **From relativity** - the speed of light: c
  - **From Particle Physics** - unit of energy: GeV (1 GeV ~ proton rest mass energy)

★ **Units become (i.e. with the correct dimensions):**

Energy	GeV	Time	$(\mathrm{GeV}/\hbar)^{-1}$
Momentum	GeV/c	Length	$(\mathrm{GeV}/\hbar c)^{-1}$
Mass	GeV/c^2	Area	$(\mathrm{GeV}/\hbar c)^{-2}$

★ **Simplify algebra by setting:** $\hbar = c = 1$

- Now all quantities expressed in powers of GeV

Energy	GeV	Time	GeV^{-1}
Momentum	GeV	Length	GeV^{-1}
Mass	GeV	Area	GeV^{-2}

To convert back to S.I. units, need to restore missing factors of $\hbar$ and c

Preliminaries: Heaviside-Lorents Units

- Electron charge defined by Force equation: $F = \dfrac{e^2}{4\pi\varepsilon_0 r^2}$
- In Heaviside-Lorentz units set $\boxed{\varepsilon_0 = 1}$

  and $F \rightarrow \dfrac{e^2}{4\pi r^2}$

  NOW: electric charge has dimensions $[FL^2]^{\frac{1}{2}} = [EL]^{\frac{1}{2}} = [\hbar c]^{\frac{1}{2}}$
- Since $c = (\varepsilon_0\mu_0)^{-\frac{1}{2}} = 1$ ➡ $\boxed{\mu_0 = 1}$

$$\boxed{\hbar = c = \varepsilon_0 = \mu_0 = 1}$$

★ Unless otherwise stated, Natural Units are used throughout these handouts, $E^2 = p^2 + m^2$, $\vec{p} = \vec{k}$, etc.

Matter in the Standard Model

★ In the Standard Model the fundamental "matter" is described by point-like spin-1/2 fermions

	LEPTONS			QUARKS		
		q	m/GeV		q	m/GeV
First Generation	e^-	–1	0.0005	d	–1/3	0.3
	ν_1	0	≈0	u	+2/3	0.3
Second Generation	μ^-	–1	0.106	s	–1/3	0.5
	ν_2	0	≈0	c	+2/3	1.5
Third Generation	τ^-	–1	1.77	b	–1/3	4.5
	ν_3	0	≈0	t	+2/3	175

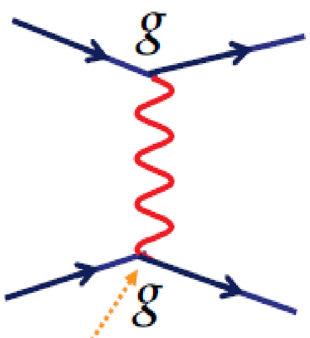
The masses quoted for the quarks are the "constituent masses", i.e. the effective masses for quarks confined in a bound state

- In the SM there are three generations – the particles in each generation are copies of each other differing only in mass. (not understood why three).
- The neutrinos are much lighter than all other particles (e.g. ν_1 has m<3 eV) – we now know that neutrinos have non-zero mass (don't understand why so small)

Forces in the Standard Model

★ Forces mediated by the exchange of spin-1 Gauge Bosons

Force	Boson(s)	J^P	*m*/GeV
EM (QED)	Photon γ	1^-	0
Weak	$W^\pm$ / Z	1^-	80 / 91
Strong (QCD)	8 Gluons g	1^-	0
Gravity (?)	Graviton?	2^+	0



- Fundamental interaction strength is given by charge g.
- Related to the **dimensionless** coupling "constant" α

e.g. QED $$g_{em} = e = \sqrt{4\pi\alpha\varepsilon_0\hbar c}$$

★ In Natural Units $$g = \sqrt{4\pi\alpha}$$

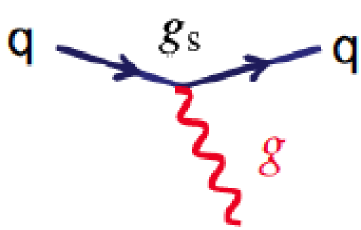
(both g and α are dimensionless, but g contains a "hidden" $\hbar c$)

★ Convenient to express couplings in terms of α which, being genuinely dimensionless does not depend on the system of units (this is not true for the numerical value for e)

Standard Model vertices

The interaction of gauge bosons with fermions is described by the Standard Model

STRONG



Only quarks

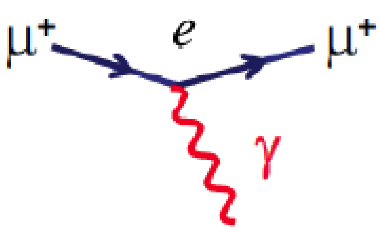
Never changes flavour

$\alpha_S \sim 1$

Gluons

massless

EM



All charged fermions

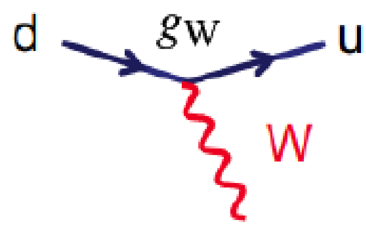
Never changes flavour

$\alpha \simeq 1/137$

Photon

massless

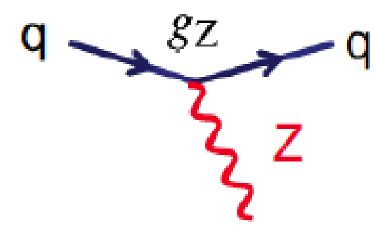
WEAK CC



All fermions

Always changes flavour

WEAK NC



All fermions

Never changes flavour

$\alpha_{W/Z} \sim 1/40$

W+, W-

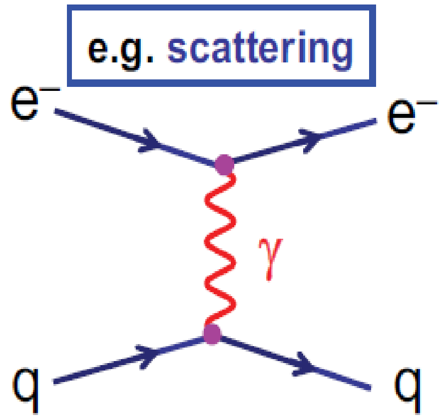
very massive

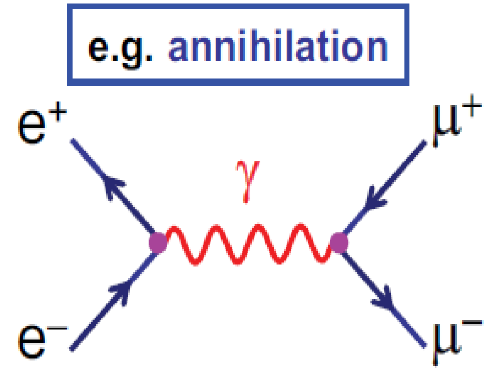
Z^0

very massive

Feynman diagrams

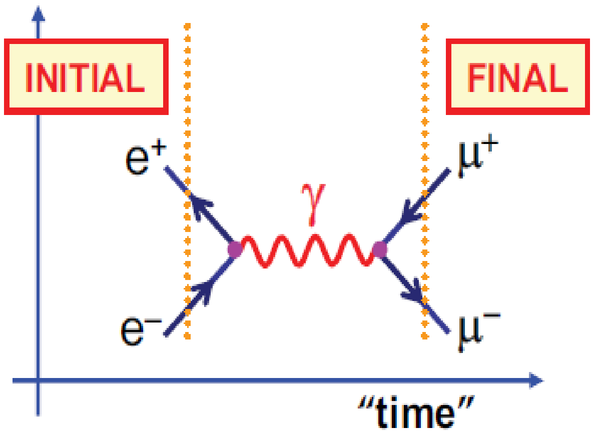
★ Particle interactions described in terms of Feynman diagrams





★ **IMPORTANT POINTS TO REMEMBER:**

- "time" runs from left – right, only in sense that:
  - LHS of diagram is initial state
  - RHS of diagram is final state
  - Middle is "how it happened"
- anti-particle arrows in –ve "time" direction
- Energy, momentum, angular momentum, etc. conserved at all interaction vertices
- All intermediate particles are "virtual" i.e. $E^2 \neq |\vec{p}|^2 + m^2$



Special relativity and 4-vector notation

- Will use 4-vector notation with p^0 as the time-like component, e.g.

$$p^\mu = \{E, \vec{p}\} = \{E, p_x, p_y, p_z\}$$ (contravariant)

$$p_\mu = g_{\mu\nu} p^\mu = \{E, -\vec{p}\} = \{E, -p_x, -p_y, -p_z\}$$ (covariant)

with

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- In particle physics, usually deal with relativistic particles. <u>Require</u> all calculations to be Lorentz Invariant. L.I. quantities formed from 4-vector scalar products, e.g.

$$p^\mu p_\mu = E^2 - p^2 = m^2$$ Invariant mass

$$x^\mu p_\mu = Et - \vec{p}.\vec{r}$$ Phase

- A few words on NOTATION

  Four vectors written as either: p^μ or p

  Four vector scalar product: $p^\mu q_\mu$ or $p.q$

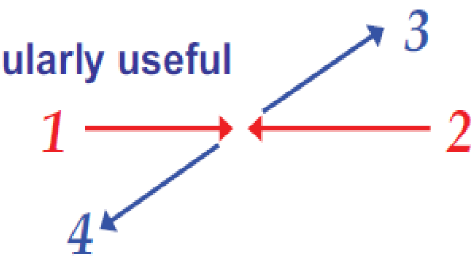
  Three vectors written as: $\vec{p}$

  Quantities evaluated in the centre of mass frame: $\vec{p}^*, p^*$ etc.

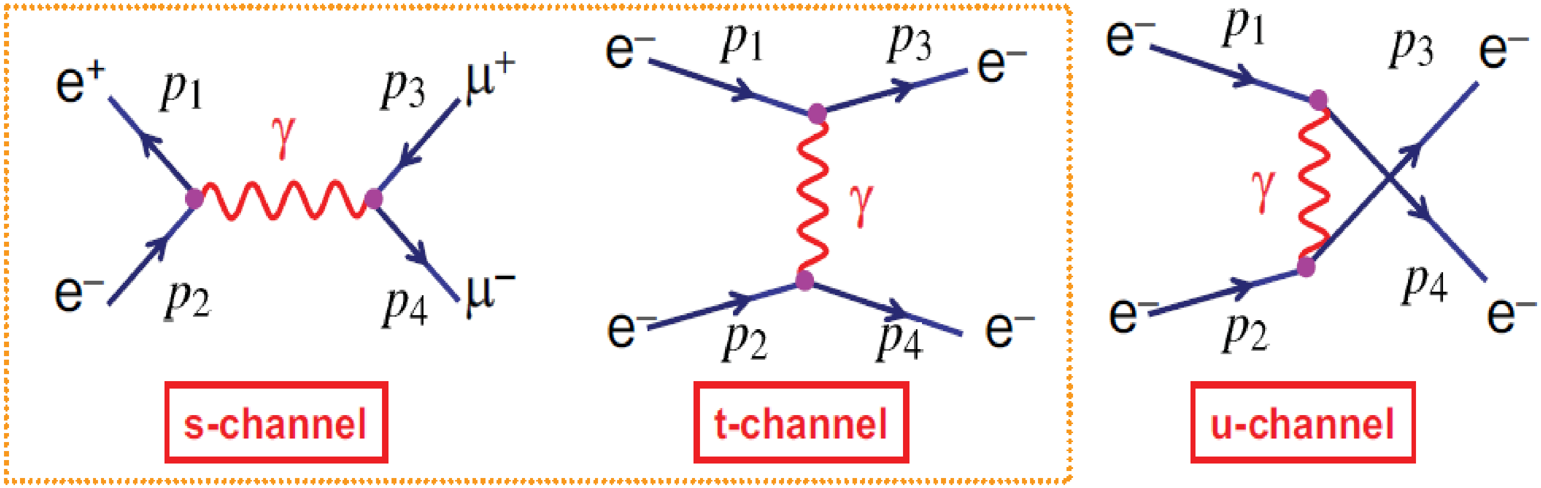
Mandelstam s, t and u

★ In particle scattering/annihilation there are three particularly useful Lorentz Invariant quantities: s, t and u

★ Consider the scattering process $1+2 \rightarrow 3+4$



★ (Simple) Feynman diagrams can be categorised according to the four-momentum of the exchanged particle



• Can define three kinematic variables: s, t and u from the following four vector scalar products (squared four-momentum of exchanged particle)

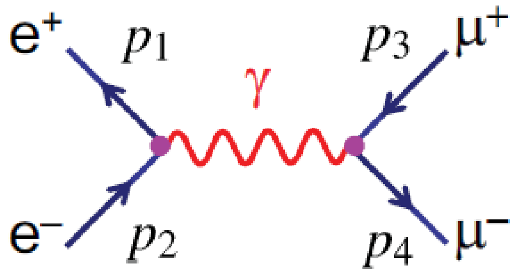
$$s=(p_1+p_2)^2, \quad t=(p_1-p_3)^2, \quad u=(p_1-p_4)^2$$

Example: Mandelstam s, t and u

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2$$

Note: $$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$

★ e.g. Centre-of-mass energy, s:



$$s = (p_1 + p_2)^2 = (E_1 + E_2)^2 - (\vec{p}_1 + \vec{p}_2)^2$$

- This is a scalar product of two four-vectors ➡ Lorentz Invariant
- Since this is a L.I. quantity, can evaluate in any frame. Choose the most convenient, i.e. the centre-of-mass frame:

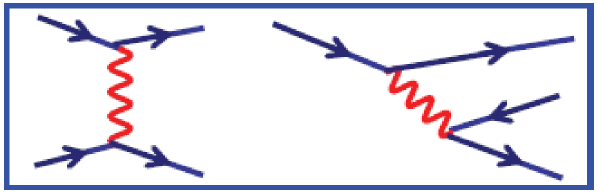
$$p_1^* = (E_1^*, \vec{p}^*) \quad p_2 = (E_2^*, -\vec{p}^*)$$

➡ $$s = (E_1^* + E_2^*)^2$$

★ Hence $\sqrt{s}$ is the total energy of collision in the centre-of-mass frame

Cross-sections and decay rates

- In particle physics we are mainly concerned with particle **interactions** and **decays**, i.e. **transitions between states**
  - these are the experimental observables of particle physics



- Calculate transition rates from **Fermi's Golden Rule**

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

Γ_{fi} is number of transitions per unit time from initial state $|i\rangle$ to final state $\langle f|$ – not Lorentz Invariant !

T_{fi} is Transition **Matrix Element**

$$T_{fi} = \langle f|\hat{H}|i\rangle + \sum_{j\neq i} \frac{\langle f|\hat{H}|j\rangle\langle j|\hat{H}|i\rangle}{E_i - E_j} + \ldots$$

$\hat{H}$ is the perturbing Hamiltonian

$\rho(E_f)$ is density of final states

★ Rates depend on MATRIX ELEMENT and DENSITY OF STATES

the ME contains the fundamental particle physics

just kinematics

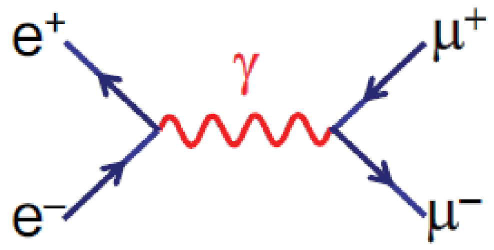
The first two lectures

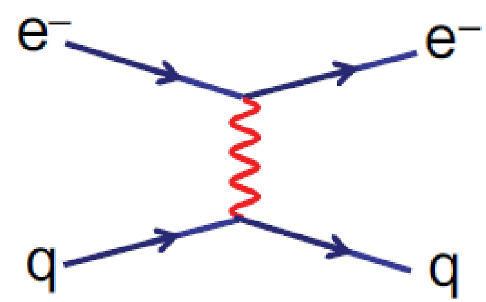
★ Aiming towards a proper calculation of decay and scattering processes

Will concentrate on:

- $e^+e^- \to \mu^+\mu^-$
- $e^-q \to e^-q$

(e⁻q→e⁻q to probe proton structure)





⅄ Need relativistic calculations of particle decay rates and cross sections:

$$\sigma = \frac{|M_{fi}|^2}{\text{flux}} \times (\text{phase space})$$

⅄ Need relativistic treatment of spin-half particles:

Dirac Equation *(this I will skip)*

⅄ Need relativistic calculation of interaction Matrix Element:

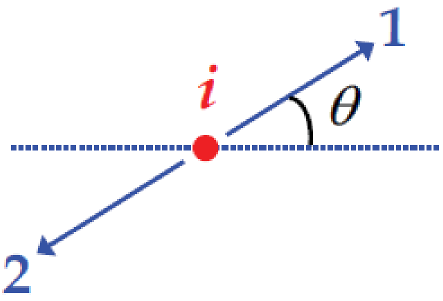
Interaction by particle exchange and Feynman rules

\+ and a few mathematical tricks along, e.g. the Dirac Delta Function

Paricle decay rates

- Consider the two-body decay

$$i \rightarrow 1 + 2$$



- Want to calculate the decay rate in first order perturbation theory using plane-wave descriptions of the particles (Born approximation):

$$\psi_1 = N e^{i(\vec{p}.\vec{r} - Et)} = N e^{-ip.x}$$

$(\vec{k}.\vec{r} = \vec{p}.\vec{r}$ as $\hbar = 1)$

where N is the normalisation and $p.x = p^{\mu} x_{\mu}$

For decay rate calculation need to know:

- Wave-function normalisation
- Transition matrix element from perturbation theory
- Expression for the density of states

} All in a Lorentz Invariant form

★ First consider wave-function normalisation

- Previously have used a non-relativistic formulation
- Non-relativistic: normalised to one particle in a cube of side a

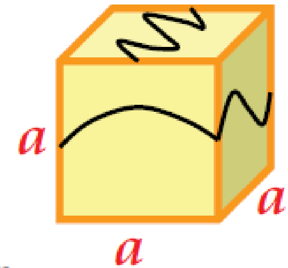
$$\int \psi \psi^* \mathrm{d}V = N^2 a^3 = 1 \quad \Rightarrow \quad N^2 = 1/a^3$$

Non-relativistic phase-space (revision)

- Apply boundary conditions $(\vec{p} = \hbar\vec{k})$:
- Wave-function vanishing at box boundaries

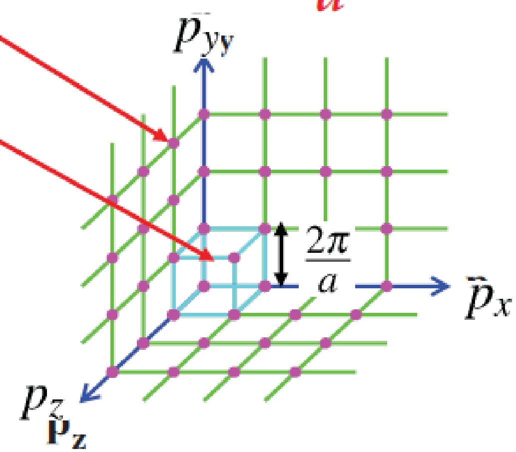
  ➡ quantised particle momenta:

$$p_x = \frac{2\pi n_x}{a};\; p_y = \frac{2\pi n_y}{a};\; p_z = \frac{2\pi n_z}{a}$$



- Volume of single state in momentum space:

$$\left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$



- Normalising to **one particle/unit volume** gives **number of states** in element: $\mathrm{d}^3\vec{p} = \mathrm{d}p_x\mathrm{d}p_y\mathrm{d}p_z$

$$\mathrm{d}n = \frac{\mathrm{d}^3\vec{p}}{\frac{(2\pi)^3}{V}} \times \frac{1}{V} = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}$$

- Therefore density of states in Golden rule:

$$\rho(E_f) = \left|\frac{\mathrm{d}n}{dE}\right|_{E_f} = \left|\frac{\mathrm{d}n}{\mathrm{d}|\vec{p}|}\frac{\mathrm{d}|\vec{p}|}{\mathrm{d}E}\right|_{E_f}$$

with $p = \beta E$

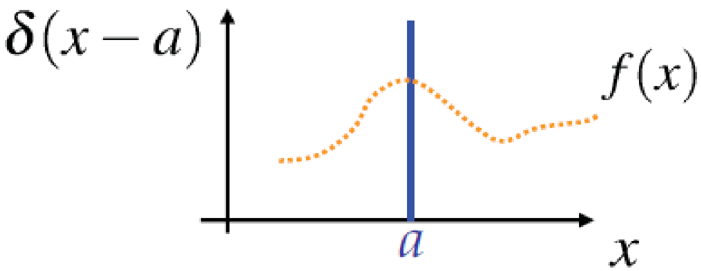
- Integrating over an elemental shell in momentum-space gives

$$(\mathrm{d}^3\vec{p} = 4\pi p^2 \mathrm{d}p)$$

$$\rho(E_f) = \frac{4\pi p^2}{(2\pi)^3} \times \beta$$

Dirac δ function

- In the relativistic formulation of decay rates and cross sections we will make use of the Dirac δ function: "infinitely narrow spike of unit area"



$$\int_{-\infty}^{+\infty} \delta(x-a)\mathrm{d}x = 1$$

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)\mathrm{d}x = f(a)$$

- Any function with the above properties can represent $\delta(x)$

e.g. $$\delta(x) = \lim_{\sigma\to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x^2}{2\sigma^2}\right)}$$ (an infinitesimally narrow Gaussian)

- In relativistic quantum mechanics delta functions prove extremely useful for integrals over phase space, e.g. in the decay $a \to 1+2$

$$\int \ldots \delta(E_a - E_1 - E_2)\mathrm{d}E \quad \text{and} \quad \int \ldots \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2)\mathrm{d}^3\vec{p}$$

express energy and momentum conservation

★ We will soon need an expression for the **delta function of a function** $\delta(f(x))$

- Start from the definition of a delta function

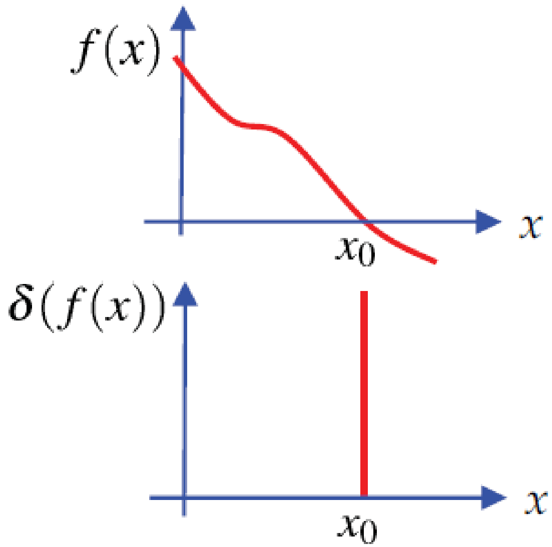
$$\int_{y_1}^{y_2} \delta(y)\mathrm{d}y = \begin{cases} 1 & \text{if } y_1 < 0 < y_2 \\ 0 & \text{otherwise} \end{cases}$$

- Now express in terms of $y = f(x)$ where $f(x_0) = 0$ and then change variables

$$\int_{x_1}^{x_2} \delta(f(x))\frac{\mathrm{d}f}{\mathrm{d}x}\mathrm{d}x = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

- From properties of the delta function (i.e. here only non-zero at x_0)

$$\left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0} \int_{x_1}^{x_2} \delta(f(x))\mathrm{d}x = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$



- Rearranging and expressing the **RHS** as a delta function

$$\int_{x_1}^{x_2} \delta(f(x))\mathrm{d}x = \frac{1}{|\mathrm{d}f/\mathrm{d}x|_{x_0}} \int_{x_1}^{x_2} \delta(x - x_0)\mathrm{d}x$$

$$\Rightarrow \quad \boxed{\delta(f(x)) = \left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x_0}^{-1} \delta(x - x_0)} \qquad (1)$$

The Golden Rule revisited

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

- Rewrite the expression for density of states using a delta-function

$$\rho(E_f) = \left|\frac{dn}{dE}\right|_{E_f} = \int \frac{dn}{dE}\delta(E - E_i)dE \qquad \text{since } E_f = E_i$$

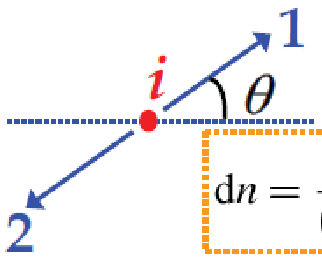
Note : integrating over all final state energies but energy conservation now taken into account explicitly by delta function

- Hence the golden rule becomes: $\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E)dn$

the integral is over **all** "allowed" final states of any energy

- For dn in a two-body decay, only need to consider one particle : mom. conservation fixes the other

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E_1 - E_2)\frac{d^3\vec{p}_1}{(2\pi)^3}$$



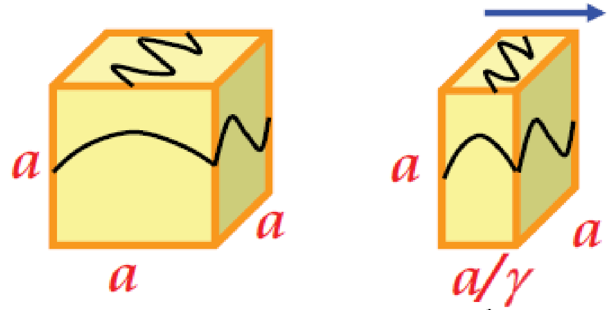
$$dn = \frac{d^3\vec{p}}{(2\pi)^3}$$

- However, can include momentum conservation explicitly by integrating over the momenta of both particles and using another δ-fn

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \underbrace{\delta(E_i - E_1 - E_2)}_{\text{Energy cons.}}\underbrace{\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)}_{\text{Mom. cons.}}\underbrace{\frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d^3\vec{p}_2}{(2\pi)^3}}_{\text{Density of states}}$$

Lorentz invariant phase-space

- In non-relativistic QM normalise to one particle/unit volume: $\int \psi^* \psi \mathrm{d}V = 1$
- When considering relativistic effects, volume contracts by $\gamma = E/m$



- Particle density therefore increases by $\gamma = E/m$
  - ★ Conclude that a relativistic invariant wave-function normalisation needs to be **proportional to E particles per unit volume**
- Usual convention: **Normalise to 2E particles/unit volume** $\int \psi'^* \psi' \mathrm{d}V = 2E$
- Previously used ψ normalised to 1 particle per unit volume $\int \psi^* \psi \mathrm{d}V = 1$
- Hence $\psi' = (2E)^{1/2}\psi$ is normalised to $2E$ per unit volume
- Define Lorentz Invariant Matrix Element, M_{fi} , in terms of the wave-functions normalised to $2E$ particles per unit volume

$$M_{fi} = \langle \psi'_1.\psi'_2...|\hat{H}|...\psi'_{n-1}\psi'_n\rangle = (2E_1.2E_2.2E_3....2E_n)^{1/2} T_{fi}$$

Decay rate calculations

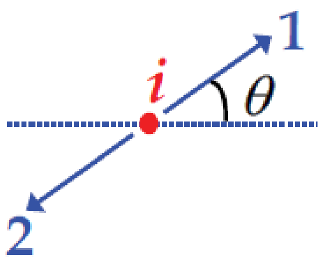
$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i}\int |M_{fi}|^2 \delta(E_i - E_1 - E_2)\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)\frac{d^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}$$

★ **Because the integral is Lorentz invariant (i.e. frame independent) it can be evaluated in any frame we choose. The C.o.M. frame is most convenient**

- **In the C.o.M. frame** $E_i = m_i$ **and** $\vec{p}_i = 0$ $\Rightarrow$

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i}\int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\delta^3(\vec{p}_1 + \vec{p}_2)\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}$$

- **Integrating over** $\vec{p}_2$ **using the δ-function:**



$$\Rightarrow \Gamma_{fi} = \frac{1}{8\pi^2 E_i}\int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\frac{d^3\vec{p}_1}{4E_1 E_2}$$

now $E_2^2 = (m_2^2 + |\vec{p}_1|^2)$ **since the δ-function imposes** $\vec{p}_2 = -\vec{p}_1$

- **Writing** $d^3\vec{p}_1 = p_1^2 dp_1 \sin\theta d\theta d\phi = p_1^2 dp_1 d\Omega$

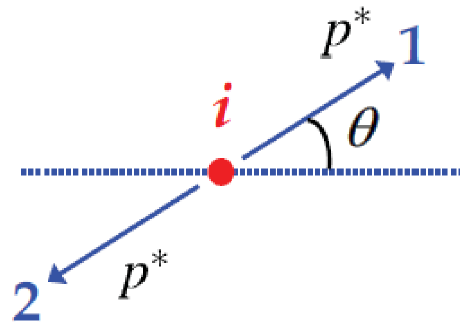
For convenience, here $|\vec{p}_1|$ **is written as** p_1

$$\Rightarrow \Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \delta\left(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right)\frac{p_1^2 dp_1 d\Omega}{E_1 E_2}$$

- Which can be written in the form $$\Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 g(p_1)\delta(f(p_1))\mathrm{d}p_1\mathrm{d}\Omega \qquad (2)$$

where $g(p_1) = p_1^2/(E_1E_2) = p_1^2(m_1^2+p_1^2)^{-1/2}(m_2^2+p_1^2)^{-1/2}$

and $f(p_1) = m_i - (m_1^2+p_1^2)^{1/2} - (m_2^2+p_1^2)^{1/2}$



Note:
- $\delta(f(p_1))$ imposes energy conservation.
- $f(p_1)=0$ determines the C.o.M momenta of the two decay products

  i.e. $f(p_1)=0$ for $p_1 = p^*$

★ Eq. (2) can be integrated using the property of δ – function derived earlier (eq. (1))

$$\int g(p_1)\delta(f(p_1))\mathrm{d}p_1 = \frac{1}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}}\int g(p_1)\delta(p_1-p^*)\mathrm{d}p_1 = \frac{g(p^*)}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}}$$

where p^* is the value for which $f(p^*)=0$

- All that remains is to evaluate $\mathrm{d}f/\mathrm{d}p_1$

$$\frac{\mathrm{d}f}{\mathrm{d}p_1} = -\frac{p_1}{(m_1^2+p_1^2)^{1/2}} - \frac{p_1}{(m_2^2+p_1^2)^{1/2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1\frac{E_1+E_2}{E_1E_2}$$

giving:

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \left|\frac{E_1E_2}{p_1(E_1+E_2)}\,\frac{p_1^2}{E_1E_2}\right|_{p_1=p^*} \mathrm{d}\Omega$$

$$= \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \left|\frac{p_1}{E_1+E_2}\right|_{p_1=p^*} \mathrm{d}\Omega$$

- But from $f(p_1)=0$, i.e. energy conservation: $E_1+E_2=m_i$

$$\Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 E_i m_i}\int |M_{fi}|^2 \mathrm{d}\Omega$$

In the particle's rest frame $E_i = m_i$

$$\frac{1}{\tau} = \Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2}\int |M_{fi}|^2 \mathrm{d}\Omega \qquad (3)$$

VALID FOR ALL TWO-BODY DECAYS !

- p^* can be obtained from $f(p_1)=0$

$$(m_1^2+p^{*2})^{1/2} + (m_2^2+p^{*2})^{1/2} = m_i$$

$$p^* = \frac{1}{2m_i}\sqrt{\left[(m_i^2-(m_1+m_2)^2\right]\left[m_i^2-(m_1-m_2)^2\right]}$$

Cross-section definition

$$\sigma = \frac{\text{no of interactions per unit time per target}}{\text{incident flux}}$$

Flux = number of incident particles/ unit area/unit time

- The "cross section", σ, can be thought of as the <u>effective</u> cross-sectional area of the target particles for the interaction to occur.
- In general this has nothing to do with the physical size of the target although there are exceptions, e.g. neutron absorption

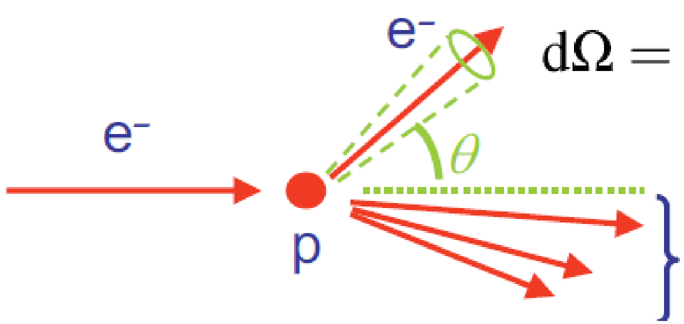
here (σ) is the projective area of nucleus

Differential Cross section

$$\frac{d\sigma}{d\Omega} = \frac{\text{no of particles per sec/per target into } d\Omega}{\text{incident flux}}$$

or generally

$$\frac{d\sigma}{d\ldots}$$



$$d\Omega = d(\cos\theta)d\phi$$

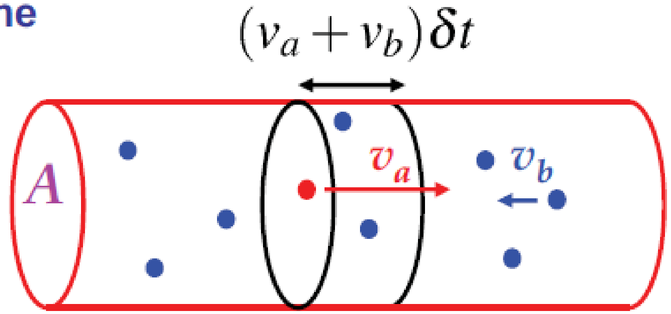
integrate over all other particles

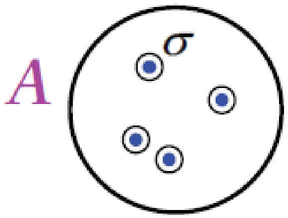
with $\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$

example

- Consider a single particle of type a with velocity, v_a, traversing a region of area A containing n_b particles of type b per unit volume

In time δt a particle of type a traverses region containing $n_b(v_a + v_b)A\delta t$ particles of type b





★Interaction probability obtained from effective cross-sectional area occupied by the $n_b(v_a + v_b)A\delta t$ particles of type b

- Interaction Probability = $\dfrac{n_b(v_a + v_b)A\delta t\sigma}{A} = n_b v \delta t \sigma$ $\quad [v = v_a + v_b]$

➡ Rate per particle of type a = $n_b v \sigma$

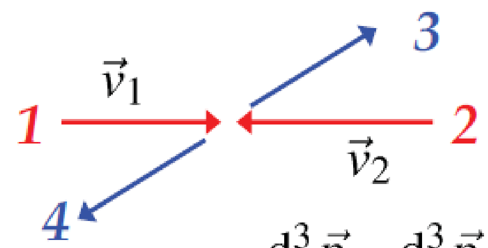
- Consider volume V, total reaction rate = $(n_b v \sigma).(n_a V) = (n_b V)(n_a v)\sigma$
  = $N_b \phi_a \sigma$

- As anticipated: **Rate = Flux x Number of targets x cross section**

Cross-section calculations

- Consider scattering process

$$1+2\rightarrow 3+4$$



- Start from Fermi's Golden Rule:

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3}$$

where T_{fi} is the transition matrix for a normalisation of 1/unit volume

- Now

$$\begin{aligned}\text{Rate/Volume} &= (\text{flux of } 1)\times(\text{number density of } 2)\times\sigma\\ &= n_1(v_1+v_2)\times n_2\times\sigma\end{aligned}$$

- For 1 target particle per unit volume $\text{Rate} = (v_1+v_2)\sigma$

$$\sigma = \frac{\Gamma_{fi}}{(v_1+v_2)}$$

$$\sigma = \frac{(2\pi)^4}{v_1+v_2}\int |T_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3}$$

the parts are not Lorentz Invariant

- To obtain a Lorentz Invariant form use wave-functions normalised to $2E$ particles per unit volume

$$\psi' = (2E)^{1/2}\psi$$

- Again define L.I. Matrix element $M_{fi} = (2E_1\, 2E_2\, 2E_3\, 2E_4)^{1/2} T_{fi}$

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2 (v_1+v_2)} \int |M_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{2E_3}\frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

- The integral is now written in a Lorentz invariant form
- The quantity $F = 2E_1 2E_2(v_1+v_1)$ can be written in terms of a four-vector scalar product and is therefore also Lorentz Invariant (the Lorentz Inv. Flux)

$$F = 4\left[(p_1^{\mu} p_{2\mu})^2 - m_1^2 m_2^2\right]^{1/2}$$

(see appendix I)

- Consequently cross section is a Lorentz Invariant quantity

Two special cases of Lorentz Invariant Flux:

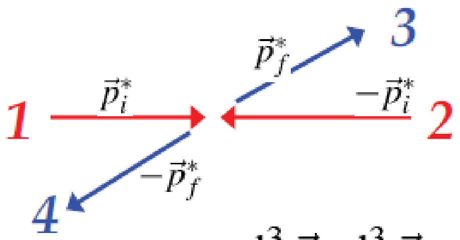
- Centre-of-Mass Frame

$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1E_2(|\vec{p}^*|/E_1 + |\vec{p}^*|/E_2) \\ &= 4|\vec{p}^*|(E_1+E_2) \\ &= 4|\vec{p}^*|\sqrt{s} \end{aligned}$$

- Target (particle 2) at rest

$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1 m_2 v_1 \\ &= 4E_1 m_2(|\vec{p}_1|/E_1) \\ &= 4m_2|\vec{p}_1| \end{aligned}$$

2-> 2 Body Scattering in C.o.M. Frame



- We will now apply above Lorentz Invariant formula for the interaction cross section to the most common cases used in the course. First consider 2→2 scattering in C.o.M. frame
- Start from

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1+v_2)} \int |M_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{d^3\vec{p}_3}{2E_3}\frac{d^3\vec{p}_4}{2E_4}$$

- Here $\vec{p}_1+\vec{p}_2=0$ and $E_1+E_2=\sqrt{s}$

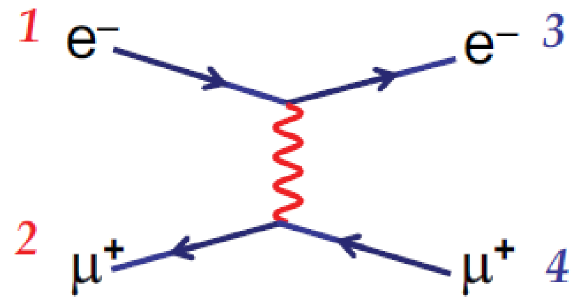
$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \int |M_{fi}|^2 \delta(\sqrt{s}-E_3-E_4)\delta^3(\vec{p}_3+\vec{p}_4)\frac{d^3\vec{p}_3}{2E_3}\frac{d^3\vec{p}_4}{2E_4}$$

★ The integral is exactly the same integral that appeared in the particle decay calculation but with m_a replaced by $\sqrt{s}$

$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}}\frac{|\vec{p}_f^*|}{4\sqrt{s}} \int |M_{fi}|^2 d\Omega^*$$

$$\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 d\Omega^* \qquad (4)$$

- In the case of elastic scattering $|\vec{p}_i^*| = |\vec{p}_f^*|$



$$\sigma_{\text{elastic}} = \frac{1}{64\pi^2 s}\int |M_{fi}|^2 \mathrm{d}\Omega^*$$

- For calculating the total cross-section (which is Lorentz Invariant) the result on the previous page (eq. (4)) is sufficient. However, it is not so useful for calculating the differential cross section in a rest frame other than the C.o.M:

$$\mathrm{d}\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}|M_{fi}|^2 \mathrm{d}\Omega^*$$

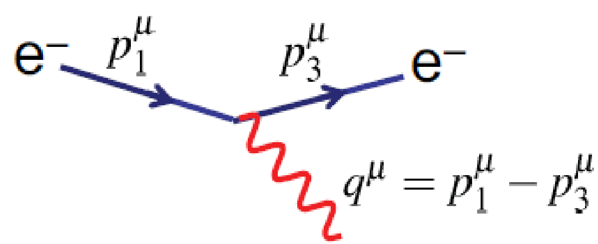
because the angles in $\mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^*$ refer to the C.o.M frame

- For the last calculation in this section, we need to find a L.I. expression fo $\mathrm{d}\sigma$

★ Start by expressing $\mathrm{d}\Omega^*$ in terms of Mandelstam t
i.e. the square of the four-momentum transfer

$$t = q^2 = (p_1 - p_3)^2$$

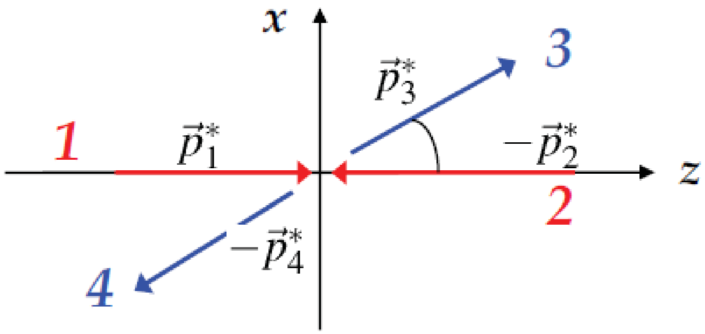
Product of four-vectors therefore L.I.



- Want to express $d\Omega^*$ in terms of Lorentz Invariant dt

  where $t \equiv (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1.p_3 = m_1^2 + m_3^2 - 2p_1.p_3$

- In C.o.M. frame:



$$
\begin{aligned}
p_1^{*\mu} &= (E_1^*, 0, 0, |\vec{p}_1^*|) \\
p_3^{*\mu} &= (E_3^*, |\vec{p}_3^*|\sin\theta^*, 0, |\vec{p}_3^*|\cos\theta^*) \\
p_1^\mu p_{3\mu} &= E_1^* E_3^* - |\vec{p}_1^*||\vec{p}_3^*|\cos\theta^* \\
t &= m_1^2 + m_3^3 - E_1^* E_3^* + 2|\vec{p}_1^*||\vec{p}_3^*|\cos\theta^*
\end{aligned}
$$

giving $$dt = 2|\vec{p}_1^*||\vec{p}_3^*|d(\cos\theta^*)$$

therefore $$d\Omega^* = d(\cos\theta^*)d\phi^* = \frac{dt\,d\phi^*}{2|\vec{p}_1^*||\vec{p}_3^*|}$$

hence $$d\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_3^*|}{|\vec{p}_1^*|}|M_{fi}|^2 d\Omega^* = \frac{1}{2\cdot 64\pi^2 s|\vec{p}_1^*|^2}|M_{fi}|^2 d\phi^* dt$$

- Finally, integrating over $d\phi^*$ (assuming no ϕ^* dependence of $|M_{fi}|^2$) gives:

$$\boxed{\frac{d\sigma}{dt} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2}$$

Lorentz invariant differential cross-section

- All quantities in the expression for $\mathrm{d}\sigma/\mathrm{d}t$ are Lorentz Invariant and therefore, it applies to any rest frame. It should be noted that $|\vec{p}_i^*|^2$ is a constant, fixed by energy/momentum conservation

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$$

- As an example of how to use the invariant expression $\mathrm{d}\sigma/\mathrm{d}t$ we will consider elastic scattering in the laboratory frame in the limit where we can neglect the mass of the incoming particle $E_1 \gg m_1$

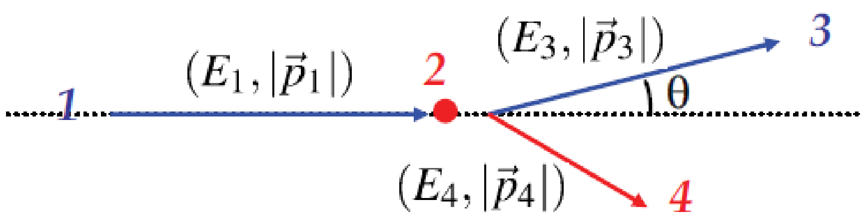
e.g. electron or neutrino scattering

In this limit $$|\vec{p}_i^*|^2 = \frac{(s-m_2)^2}{4s}$$

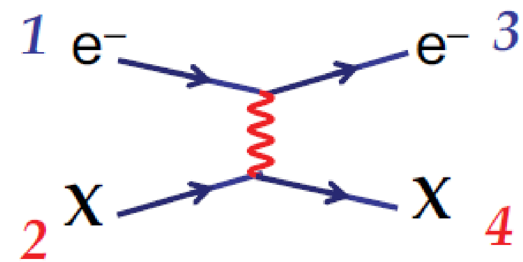
$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi(s-m_2^2)^2}|M_{fi}|^2} \qquad (m_1 = 0)$$

2->2 body scattering in the Lab Frame

- The other commonly occurring case is scattering from a fixed target in the Laboratory Frame (e.g. electron-proton scattering)
- First take the case of elastic scattering at high energy where the mass of the incoming particles can be neglected: $m_1 = m_3 = 0, \quad m_2 = m_4 = M$



e.g.



- Wish to express the cross section in terms of scattering angle of the e⁻

$$d\Omega = 2\pi d(\cos\theta)$$

therefore

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\frac{dt}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt}$$

Integrating over $d\phi$

- The rest is some rather tedious algebra.... start from four-momenta

$$p_1 = (E_1, 0, 0, E_1), \quad p_2 = (M, 0, 0, 0), \quad p_3 = (E_3, E_3\sin\theta, 0, E_3\cos\theta), \quad p_4 = (E_4, \vec{p}_4)$$

so here $\quad t = (p_1 - p_3)^2 = -2p_1.p_3 = -2E_1E_3(1-\cos\theta)$

But from (E,p) conservation $\quad p_1 + p_2 = p_3 + p_4$

and, therefore, can also express t in terms of particles 2 and 4

$$\begin{aligned} t &= (p_2 - p_4)^2 = 2M^2 - 2p_2.p_4 = 2M^2 - 2ME_4 \\ &= 2M^2 - 2M(E_1 + M - E_3) = -2M(E_1 - E_3) \end{aligned}$$

Note E_1 is a constant (the energy of the incoming particle) so

$$\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)} = 2M\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)}$$

- Equating the two expressions for t gives $$E_3 = \frac{E_1 M}{M + E_1 - E_1\cos\theta}$$

so $$\frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} = \frac{E_1 M}{(M + E_1 - E_1\cos\theta)^2} = E_1^2 M\left(\frac{E_3}{E_1 M}\right)^2 = \frac{E_3^2}{M}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2\pi}\frac{\mathrm{d}t}{\mathrm{d}(\cos\theta)}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{2\pi}2M\frac{E_3^2}{M}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi}\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{E_3^2}{\pi}\frac{1}{16\pi(s-M^2)^2}|M_{fi}|^2$$

using $s = (p_1 + p_2)^2 = M^2 + 2p_1.p_2 = M^2 + 2ME_1$

Particle 1 massless ➡ $(p_1^2 = 0)$

gives $(s - M^2) = 2ME_1$

➡ $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2|M_{fi}|^2$$

In limit $m_1 \to 0$

In this equation, E_3 is a function of θ:

$$E_3 = \frac{E_1 M}{M + E_1 - E_1\cos\theta}$$

giving

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2 \qquad (m_1 = 0)$$

General form for 2→2 Body Scattering in Lab. Frame

★ The calculation of the differential cross section for the case where m_1 can not be neglected is longer and contains no more "physics" (see appendix II). It gives:

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\cdot\frac{1}{p_1 m_1}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2$$

Again there is **only one independent variable**, θ, which can be seen from conservation of energy

$$E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{\underbrace{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta}_{\vec{p}_4 = \vec{p}_1 - \vec{p}_3} + m_4^2}$$

i.e. $|\vec{p}_3|$ is a function of θ

Summary

★ Used a Lorentz invariant formulation of Fermi's Golden Rule to derive decay rates and cross-sections in terms of the **Lorentz Invariant Matrix Element** (wave-functions normalised to 2E/Volume)

Main Results:

★ Particle decay:

$$\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2}\int |M_{fi}|^2 \mathrm{d}\Omega$$

Where p^* is a function of particle masses

$$p^* = \frac{1}{2m_i}\sqrt{\left[(m_i^2 - (m_1+m_2)^2\right]\left[m_i^2 - (m_1-m_2)^2\right]}$$

★ Scattering cross section in C.o.M. frame:

$$\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}\int |M_{fi}|^2 \mathrm{d}\Omega^*$$

★ Invariant differential cross section (valid in all frames):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2$$

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$$

Summary cont.

★ Differential cross section in the lab. frame ($m_1 = 0$)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2 \quad \Leftrightarrow \quad \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2$$

★ Differential cross section in the lab. frame ($m_1 \neq 0$)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{|\vec{p}_1| m_1} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \cdot |M_{fi}|^2$$

with $E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta + m_4^2}$

<u>**Summary of the summary:**</u>

★ Have now dealt with kinematics of particle decays and cross sections
★ The fundamental particle physics is in the matrix element
★ The above equations are the basis for all calculations that follow

Appendix I: Lorentz Invariant Flux

- **Collinear collision:** $a \longrightarrow \quad \longleftarrow b$

$$v_a, \vec{p}_a \qquad\qquad v_b, \vec{p}_b$$

$$\begin{aligned} F = 2E_a 2E_b (v_a + v_b) &= 4E_a E_b \left(\frac{|\vec{p}_a|}{E_a} + \frac{|\vec{p}_b|}{E_b} \right) \\ &= 4(|\vec{p}_a| E_b + |\vec{p}_b| E_a) \end{aligned}$$

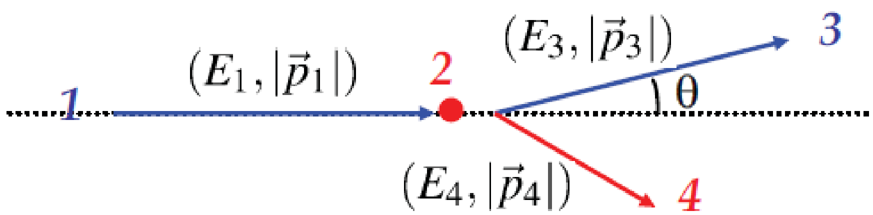
To show this is Lorentz invariant, first consider

$$p_a . p_b = p_a^\mu p_{b\mu} = E_a E_b - \vec{p}_a . \vec{p}_b = E_a E_b + |\vec{p}_a||\vec{p}_b|$$

Giving

$$\begin{aligned} F^2/16 - (p_a^\mu p_{b\mu})^2 &= (|\vec{p}_a| E_b + |\vec{p}_b| E_a)^2 - (E_a E_b + |\vec{p}_a||\vec{p}_b|)^2 \\ &= |\vec{p}_a|^2 (E_b^2 - |\vec{p}_b|^2) + E_a^2 (|\vec{p}_b|^2 - E_b^2) \\ &= |\vec{p}_a|^2 m_b^2 - E_a^2 m_b^2 \\ &= -m_a^2 m_b^2 \\ F &= 4\left[(p_a^\mu p_{b\mu})^2 - m_a^2 m_b^2\right]^{1/2} \end{aligned}$$

Appendix II: general 2->2 scattering in lab frame



$$p_1 = (E_1, 0, 0, |\vec{p}_1|), \ \ p_2 = (M, 0, 0, 0), \ \ p_3 = (E_3, E_3 \sin\theta, 0, E_3 \cos\theta), \ \ p_4 = (E_4, \vec{p}_4)$$

again

$$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\frac{dt}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt}$$

But now the invariant quantity t:

$$\begin{aligned} t &= (p_2 - p_4)^2 = m_2^2 + m_4^2 - 2p_2.p_4 = m_2^2 + m_4^2 - 2m_2 E_4 \\ &= m_2^2 + m_4^2 - 2m_2(E_1 + m_2 - E_3) \end{aligned}$$

$$\Rightarrow \quad \frac{dt}{d(\cos\theta)} = 2m_2 \frac{dE_3}{d(\cos\theta)}$$

Which gives $$\frac{d\sigma}{d\Omega} = \frac{m_2}{\pi} \frac{dE_3}{d(\cos\theta)} \frac{d\sigma}{dt}$$

To determine $dE_3/d(\cos\theta)$, first differentiate $E_3^2 - |\vec{p}_3|^2 = m_3^2$

$$2E_3 \frac{dE_3}{d(\cos\theta)} = 2|\vec{p}_3| \frac{d|\vec{p}_3|}{d(\cos\theta)} \qquad \text{(AII.1)}$$

Then equate $t = (p_1 - p_3)^2 = (p_4 - p_2)^2$ to give

$$m_1^2 + m_3^2 - 2(E_1 E_3 - |\vec{p}_1||\vec{p}_3|\cos\theta) = m_4^2 + m_2^2 - 2m_2(E_1 + m_2 - E_3)$$

Differentiate wrt. $\cos\theta$

$$(E_1 + m_2)\frac{dE_3}{d\cos\theta} - |\vec{p}_1|\cos\theta \frac{d|\vec{p}_3|}{d\cos\theta} = |\vec{p}_1||\vec{p}_3|$$

Using (1) ➡

$$\frac{dE_3}{d(\cos\theta)} = \frac{|\vec{p}_1||\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \qquad \text{(AII.2)}$$

$$\frac{d\sigma}{d\Omega} = \frac{m_2}{\pi} \frac{dE_3}{d(\cos\theta)} \frac{d\sigma}{dt} = \frac{m_2}{\pi} \frac{dE_3}{d(\cos\theta)} \frac{1}{64\pi s|\vec{p}_i^*|^2} |M_{fi}|^2$$

It is easy to show $|\vec{p}_i^*|\sqrt{s} = m_2|\vec{p}_1|$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}E_3}{\mathrm{d}(\cos\theta)} \frac{m_2}{64\pi^2 m_2^2 |\vec{p}_1|^2} |M_{fi}|^2$$

and using (AII.2) obtain

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{p_1 m_1} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \cdot |M_{fi}|^2$$