Elementary Particle Physics: theory and experiments

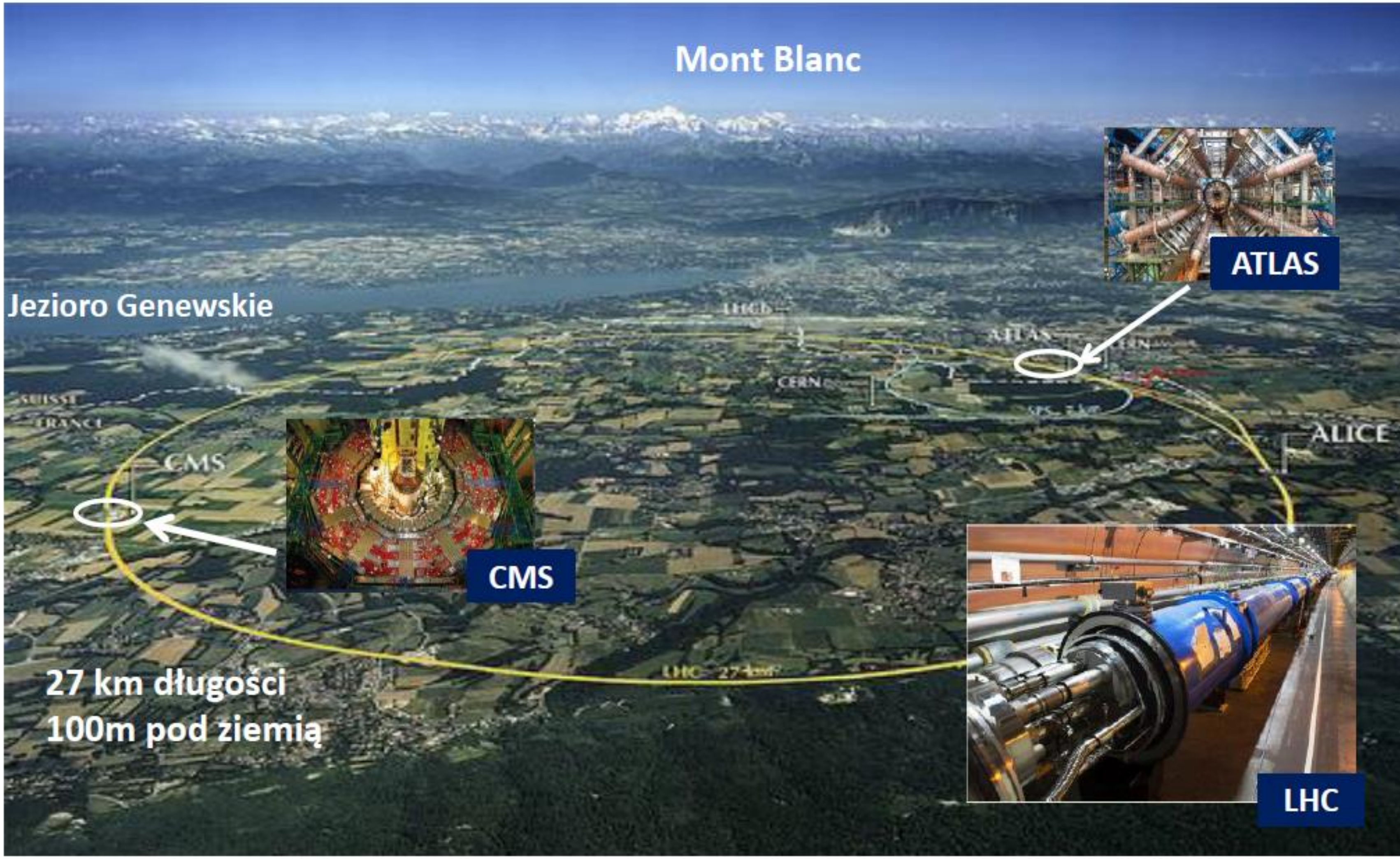
Detectors for HEP: ATLAS at LHC

Calculating cross-sections and decay rates

Some slides taken from M. A. Thomson lectures at Cambridge University in 2011

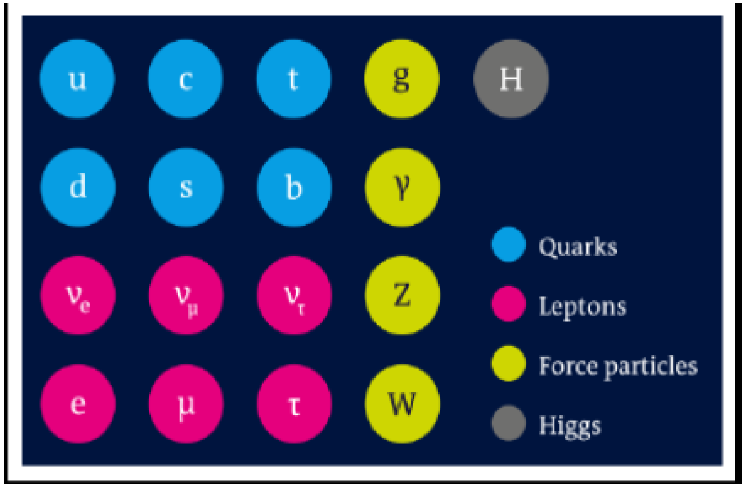
Prof. dr hab. Elżbieta Richter-Wąs

LHC (Large Hadron Collider)

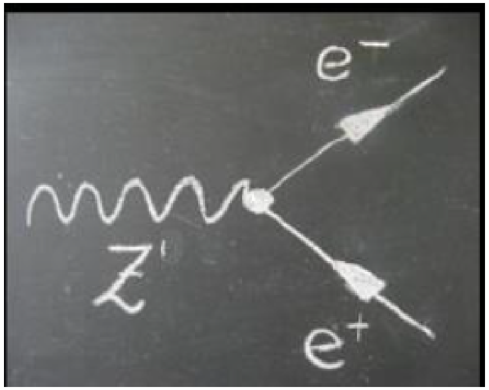


Which particles are detected?

1) **Charged leptons, photons and hadrons: e, μ,γ,π,K,p,n…** (maybe new long-lived particles, i.e. particles which enter detector)
2) B (and D) mesons and τ leptons have cτ ~ 0.09 ..0. x 10^{-3}m large enough for additional vertex reconstruction
3) Neutrinos (maybe also new particles) are reconstructed as mising transverse momentum
4) All other particles which decay or hadronise in primary vertex (top quark decays before hadronises)

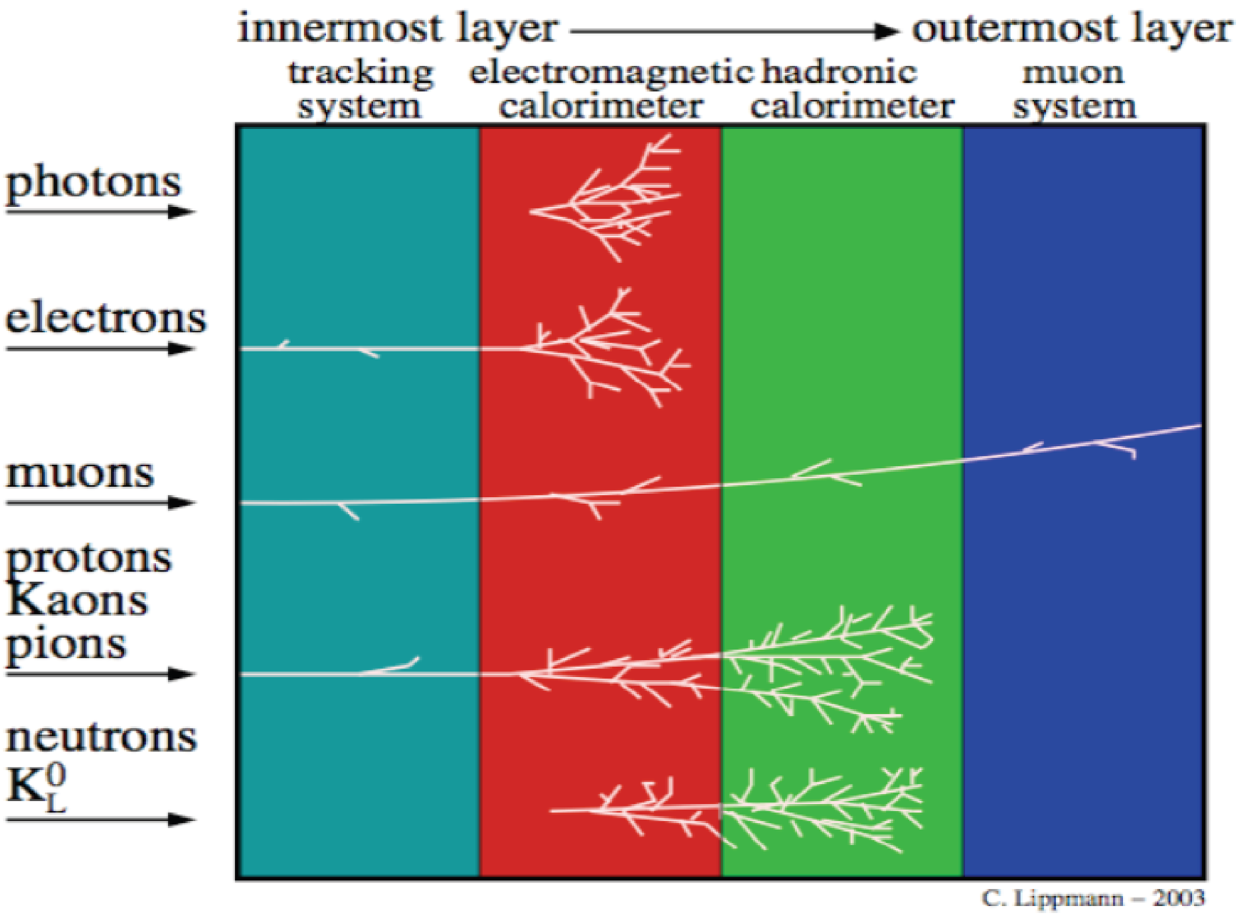


Only e, μ, γ of the fundamental Standard Model Particles are directly detected



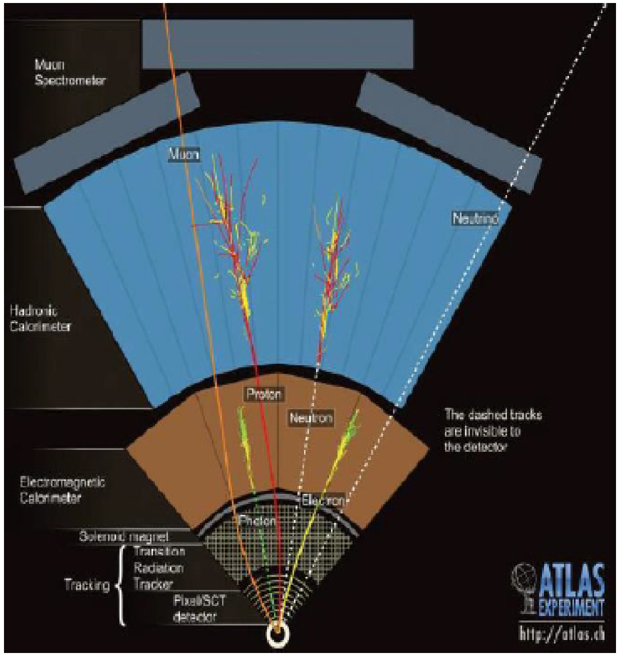
Heavy particles W, Z decay immediatelly

Sketch of particles interaction with detector



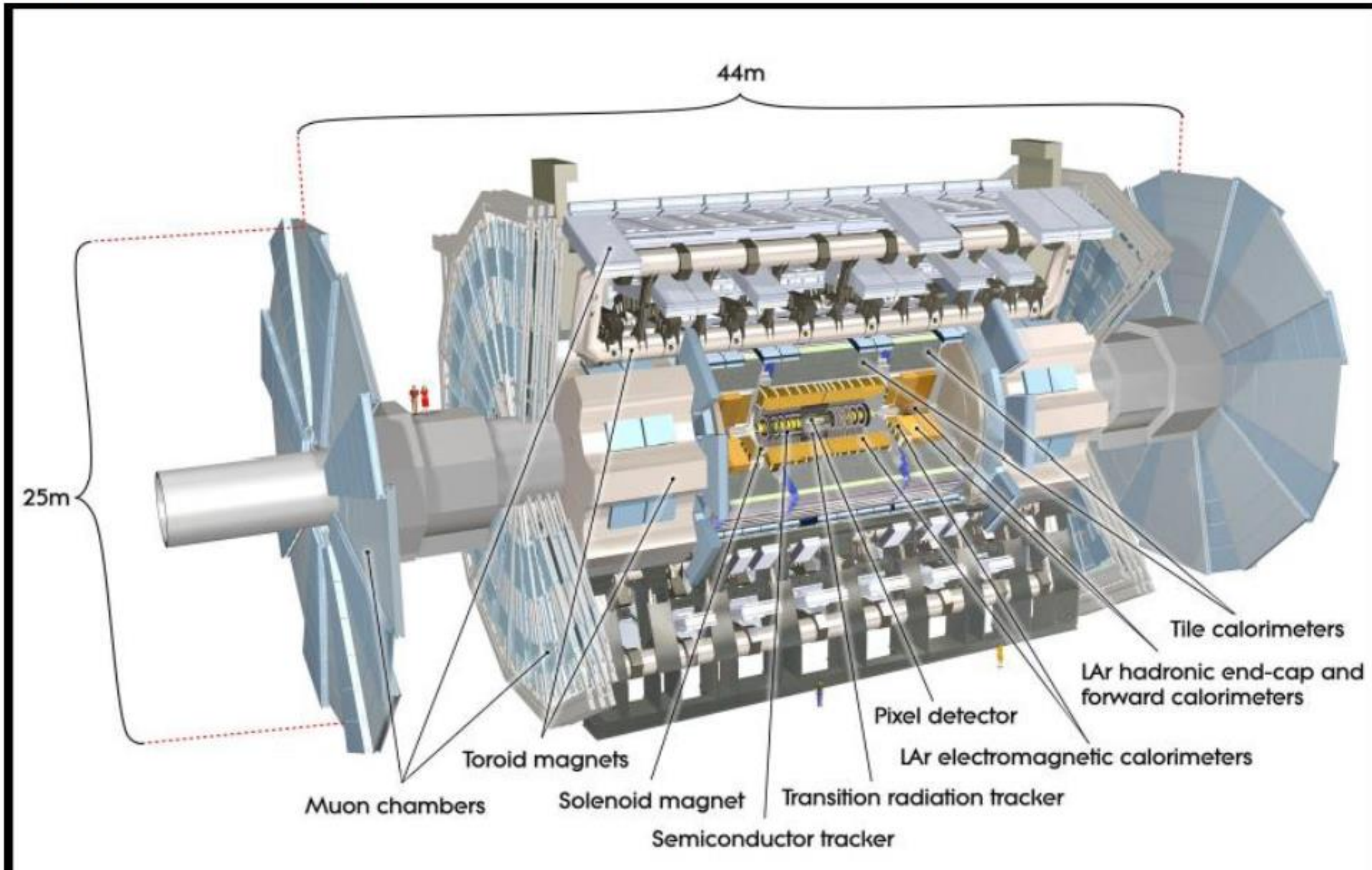
The observables?

1) Photon makes photo-efect, Compton scattering and **pair production**. It has no track but an **electromagnetic cascade** in the calorimeter.
2) Charged particles makes scattering, **ionisation**, excitation and bremsstrahlung, transition and cherenkov radiation. They produce **tracks**.
3) Electrons make **electromagnetic cascades** (clusters) in the calorimeter
4) Hadrons also interact strongly via inelastic interactions, e.g. neutron capture, induced fission, etc. They make **hadronic cascades** (clusters) int he hadronic calorimeter.
5) Only weakly interacting particles (neutrinos) are reconstructed as **missing transverse momentum** („missing energy”).



The ATLAS example

Typical 4π cylindrical onion structure



Reconstructed properties

From the hits, tracks, clusters, missing transverse momentum and vertices we reconstruct the particles properties:

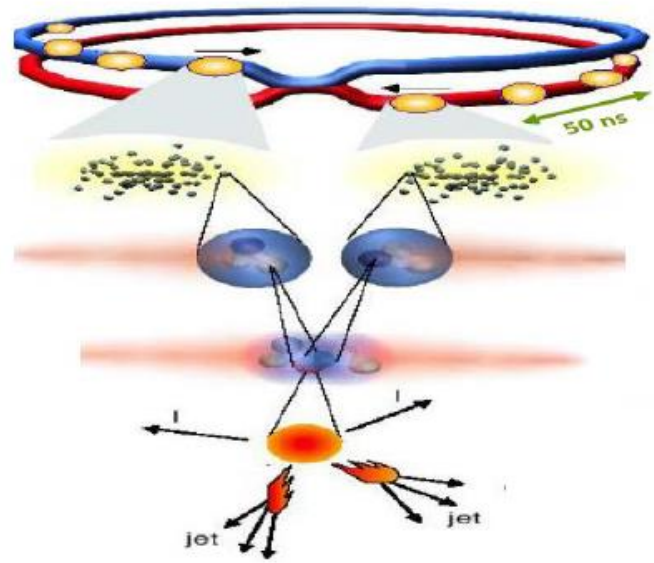
1) Momentum from curved tracks
2) Charge from track curvature
3) Energy from full absorption in calorimeters and curved tracks
4) Spin from angular distributions
5) Mass from invariant mass from decay products
6) Lifetime from time of flight measurement
7) Identity from dE/dx, lifetime or special behaviour (like transition ratiation)

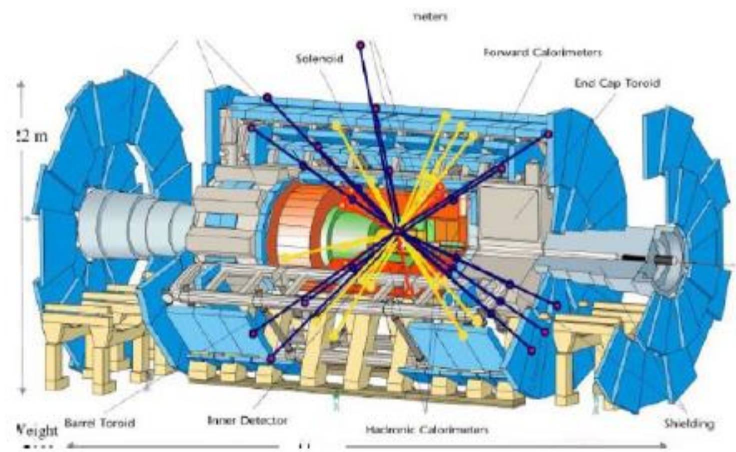
Detector design constraints (I)

- **Constraints from physics:**
  1) High detection efficiency demands minimal cracks and holes, high coverage
  2) High resolution demands little material like support structures, cables, cooling pipes, electronics etc. (avoid multiple scattering)
  3) Irradiation hard active materials to avoid degradation and changes during operation
  4) Low noise
  5) Easy maintenance (materials get radioactive)
  6) …

Detector design constraints (II)

- **Enviromental contraints, i.e. from LHC design parameters:**
  1) Collision events every ~ 25ns
  2) Muons from previous event still in detector when current enters tracker
  3) High occupancy in the inner detector
  4) Pile up (more proton proton collisions in each bunch crosing)
  5) High irradiation
  6) …



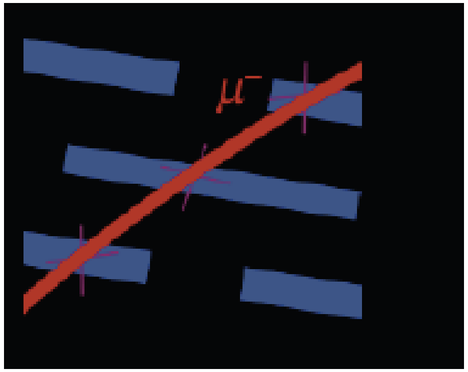


Magnet system

- Use Lorentz force to curve tracks

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

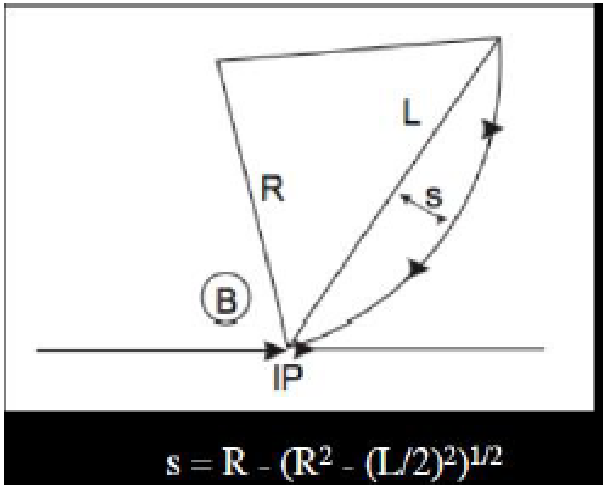
Electric force — Magnetic force



- Max E is about 50MV/m in high vacum, just B field used (5T gives ~10^3 stronger force)
- Curvature or radius: q v B = m v^2/T => p = q B R
- At least three hits needed to reconstruct a unique R of a track
- Remember solenoid resolution:

$(\Delta p_T/p_T)_{solenoid} \sim (\Delta s/L^2 B) p_T$

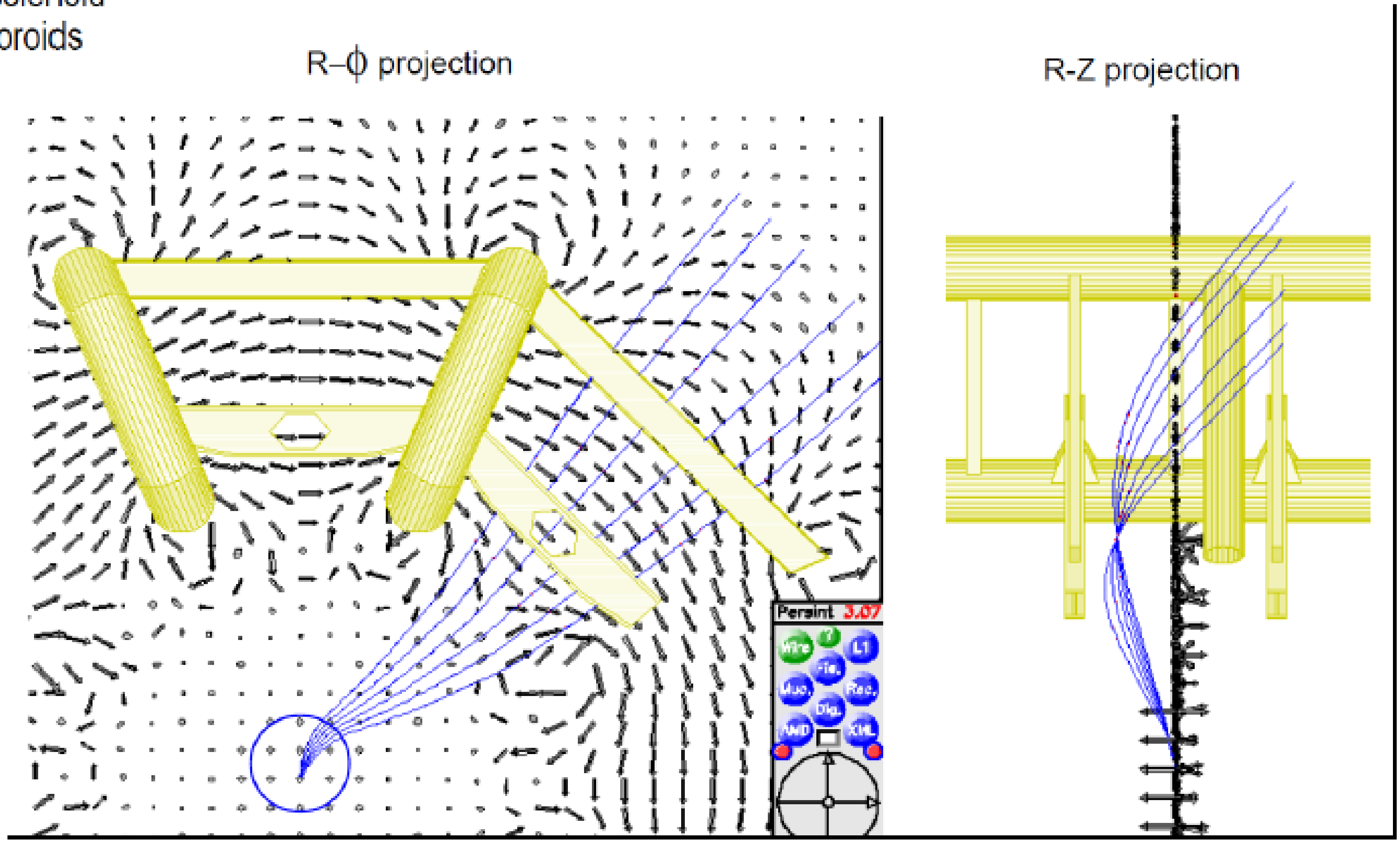
(in GeV with s in μm, L in cm and B in T. Large B is good against high occupancy.



s = sagitta

Charged particles in magnetic field

ATLAS magnetic field
1 solenoid
3 toroids



Size and field examples

ATLAS barrel toroid
20.5 kA, 3.9 T

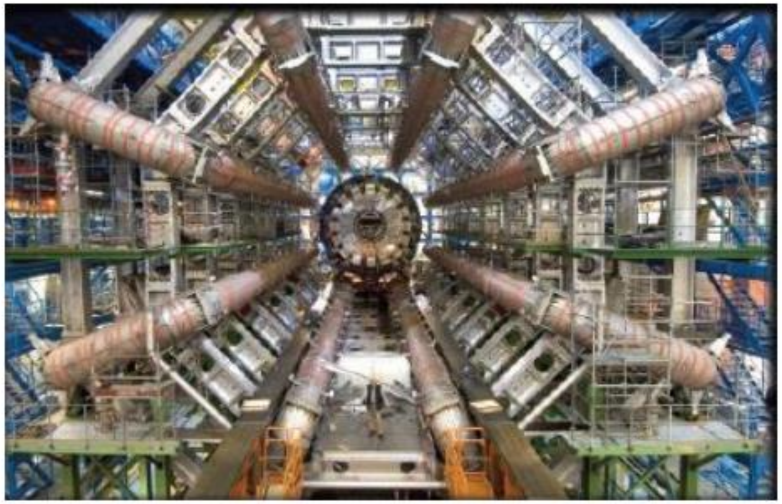
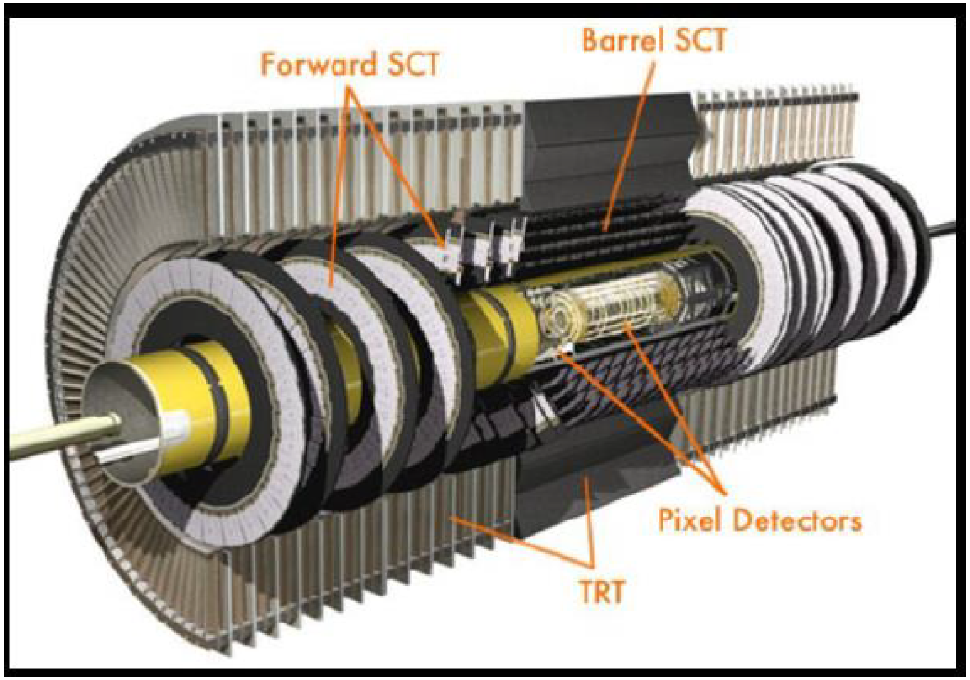


Table 1
Main parameters of some HEP detector magnets (solenoids).

	CDF	CLEO-II	ALEPH	ZEUS	H1	KLOE	BaBar	Atlas	CMS
B (T)	1.5	1.5	1.5	1.8	1.2	0.6	1.5	2.0	4.0
R (m)	1.5	1.55	2.7	1.5	2.8	2.6	1.5	1.25	3.0
L (m)	4.8	3.5	6.3	2.45	5.2	3.9	3.5	3.66	12.5

The magnet layout is a major constraint for the rest of the detector!
See A. Gadi, A magnet system for HEP experiments, NIMA 666 (2012) 10-24

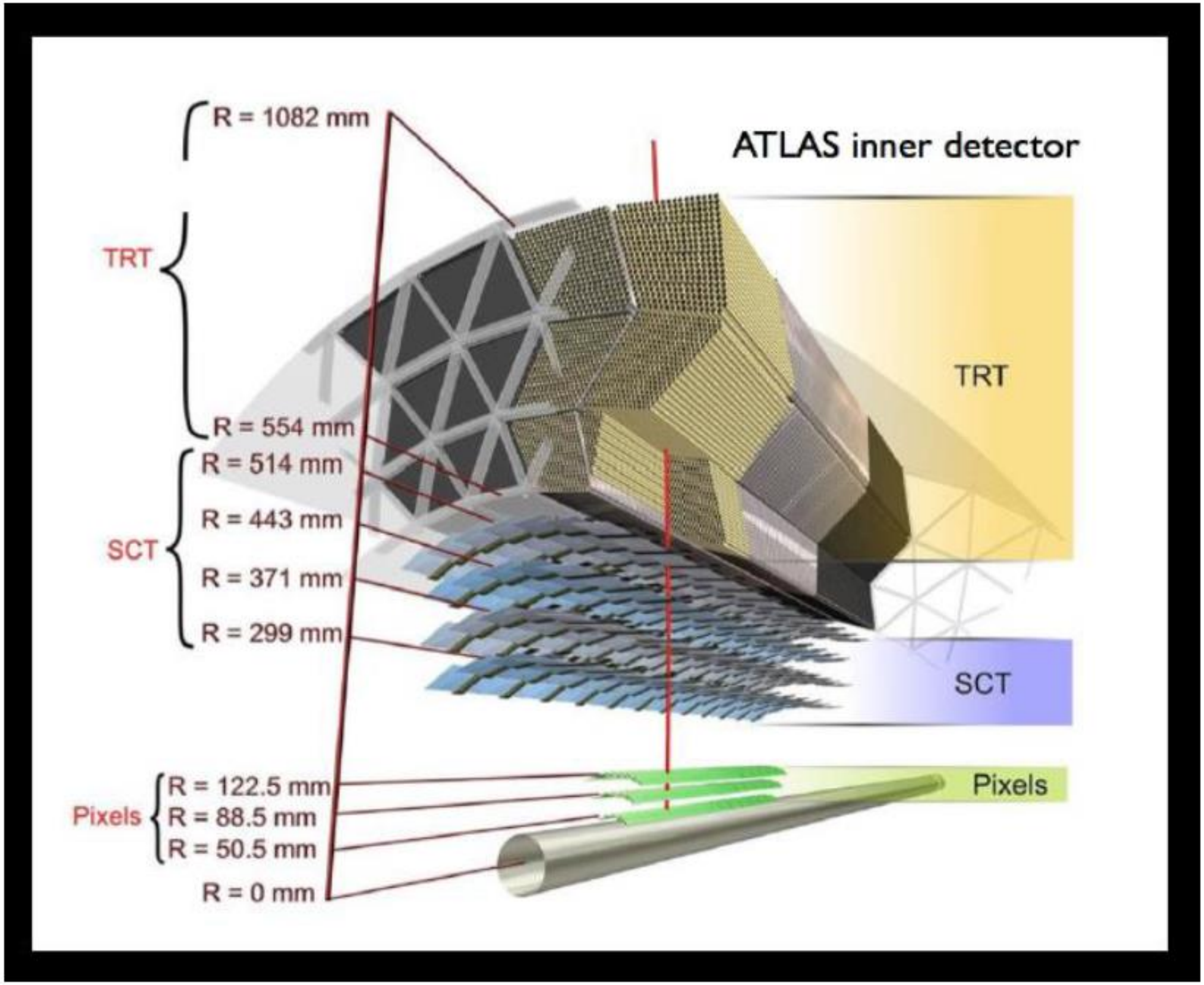
ATLAS Inner Detector



- 3 layers of pixel modules in barrel
- 2x5 disks of forward pixel disks
- 4 layers of strip (SCT) modules in barrel
- 2x9 disks of forward strip modules

Figure : ATLAS Inner detector (ID) in LHC run 1 with pixel and strip (SCT) silicon and transition radiation (TRT) detectors. The length is about 5.5 m.

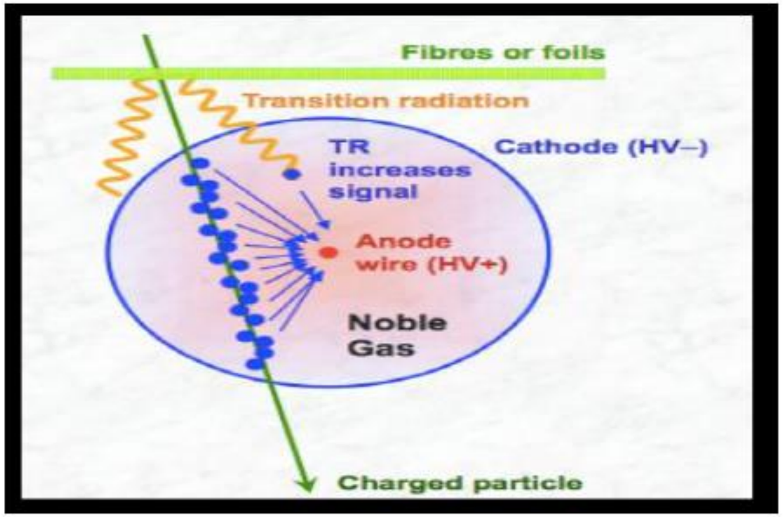
ATLAS Inner Detector

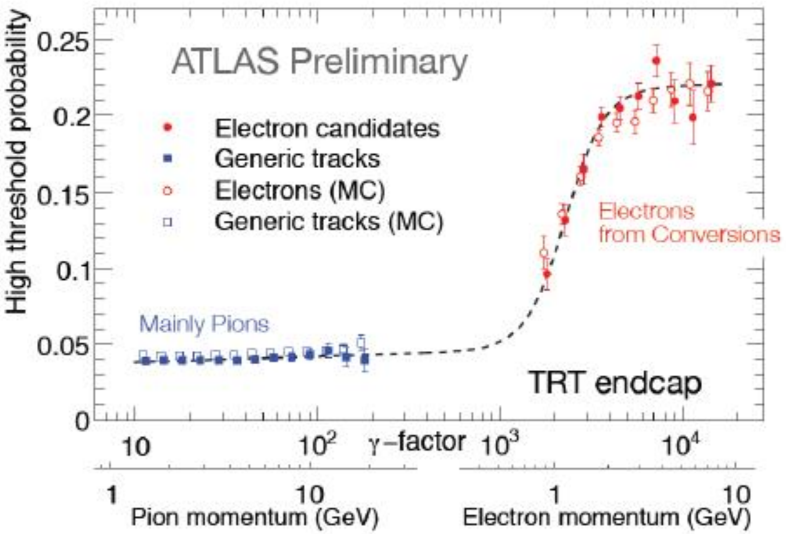


Transition Radiation Tracker

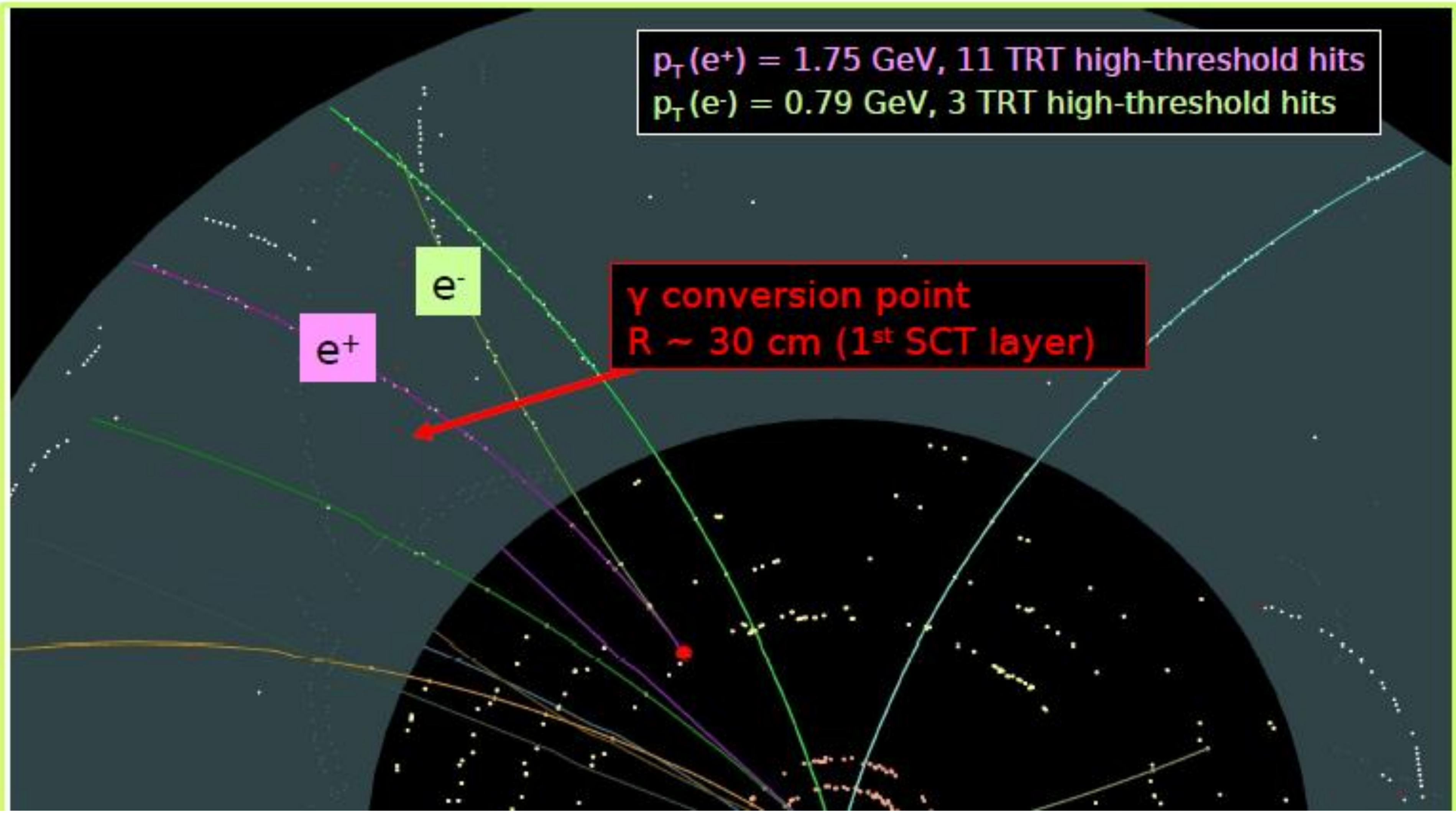
Combine tracking with particle identification (PID)

- Charged particles radiate photons when crossing material borders.
- **$e^{\pm}$** radiate x-rays more than heavier particles.
- Use this particle PID, i.e. distinguis e^{+-} from hadrons.
- ATLAS has a TR detector in the inner detector. It uses gas for detection.

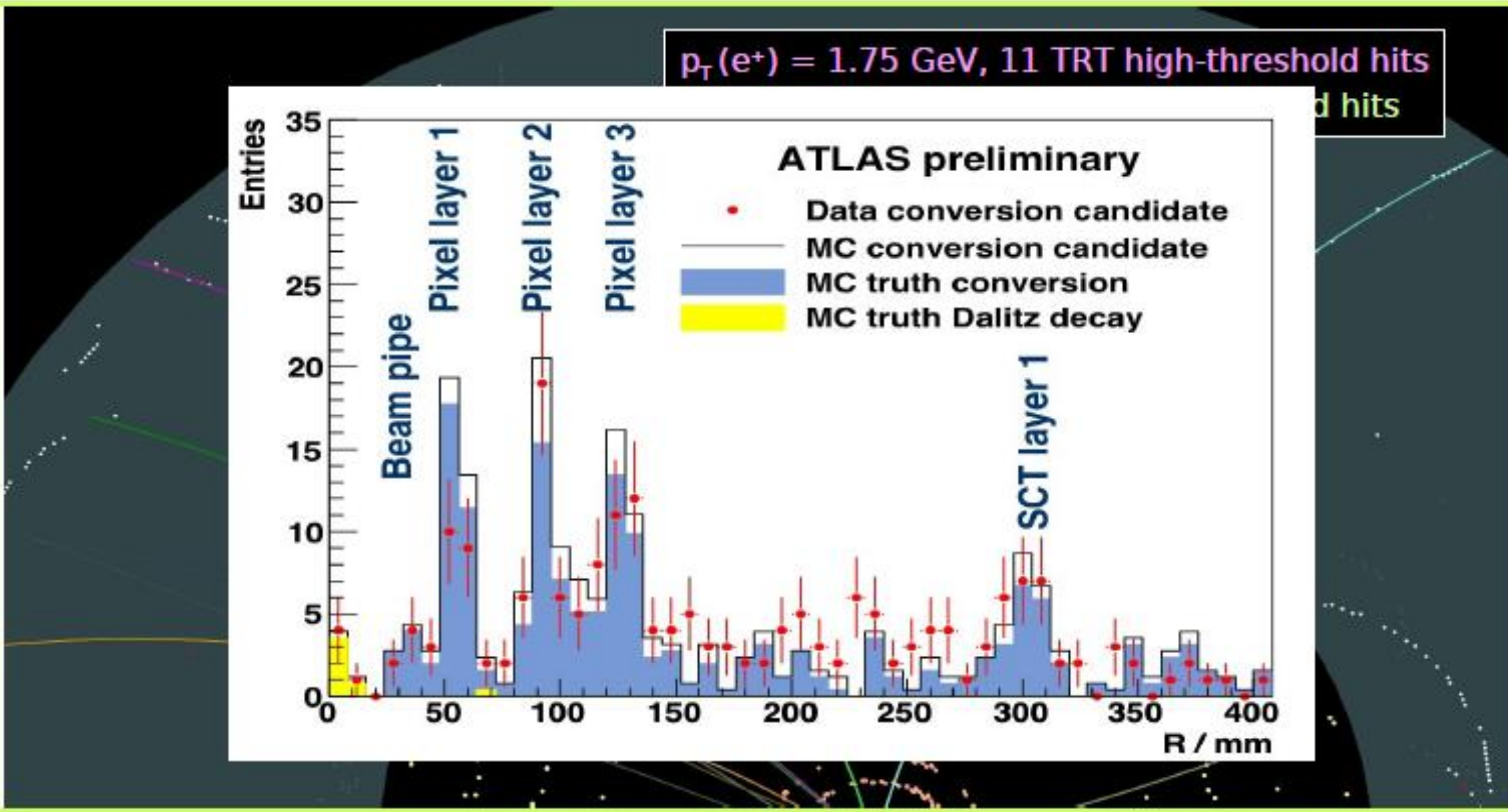




$\gamma \rightarrow e^+e^-$ conversions



γ→ e+e- conversions



ATLAS EM Calorimeter

Accordion Pb/LAr |η|<3.2 ~170k channels

Precision measurement |η|<2.5

3 layers up to |η|=2.5 + presampler |η|<1.8

2 layers 2.5<|η|<3.2

Layer 1 (γ/π^0 rej. + angular meas.)

$\Delta\eta.\Delta\varphi$ = 0.003 x 0.1

Layer 2 (shower max)

$\Delta\eta.\Delta\varphi$ = 0.025 x 0.0.25

Layer 3 (Hadronic leakage)

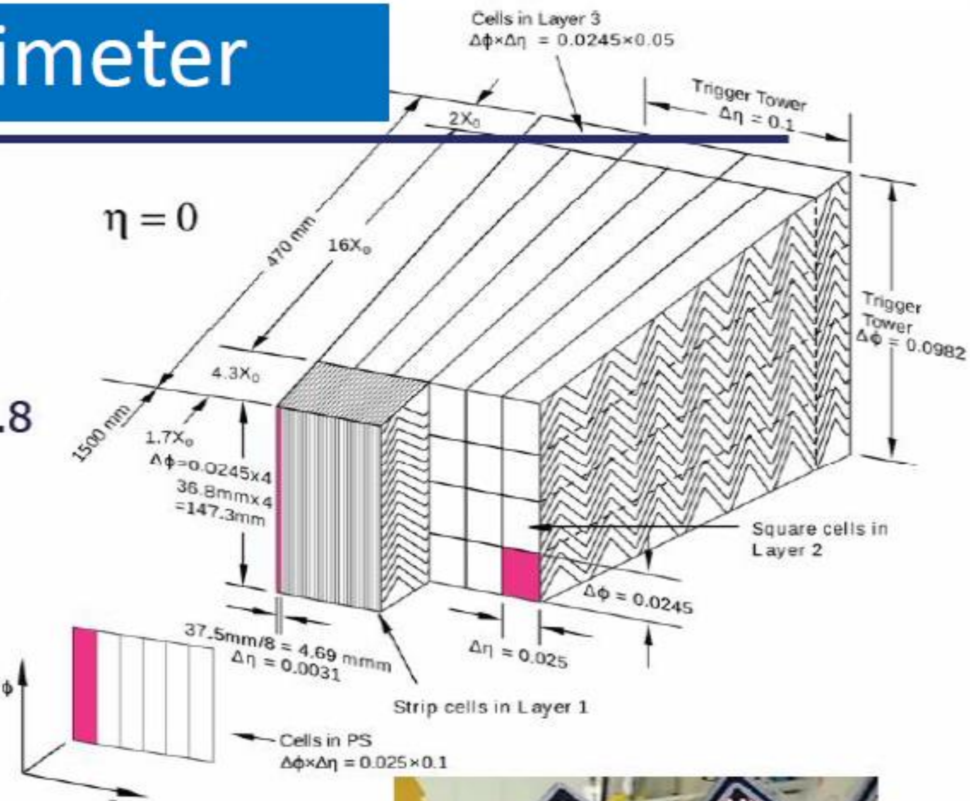
$\Delta\eta.\Delta\varphi$ = 0.05 x 0.0.025

Energy Resolution: design for η~0

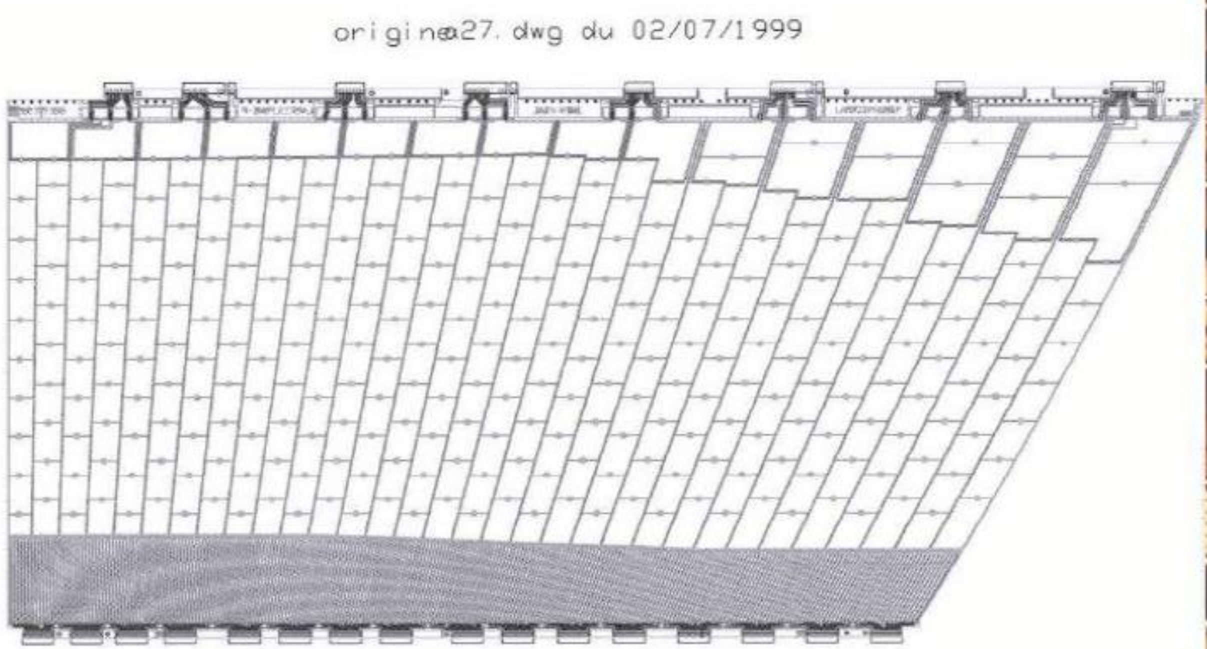
$\Delta E/E \sim 10\%/\sqrt{E} \oplus 150\ \mathrm{MeV}/E \oplus 0.7\%$

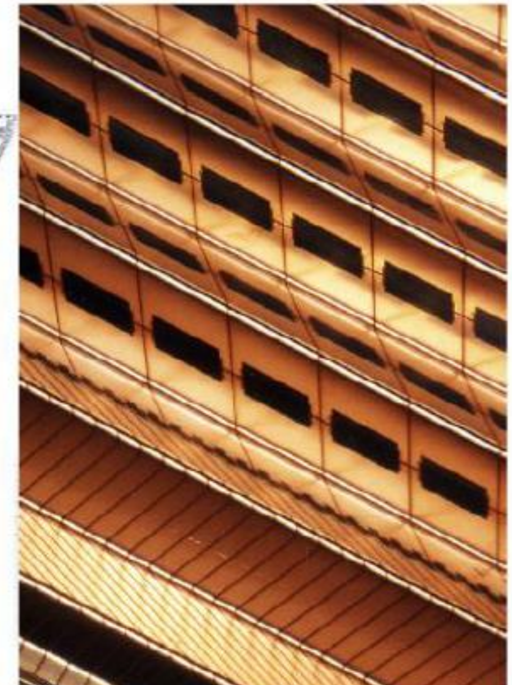
Angular Resolution

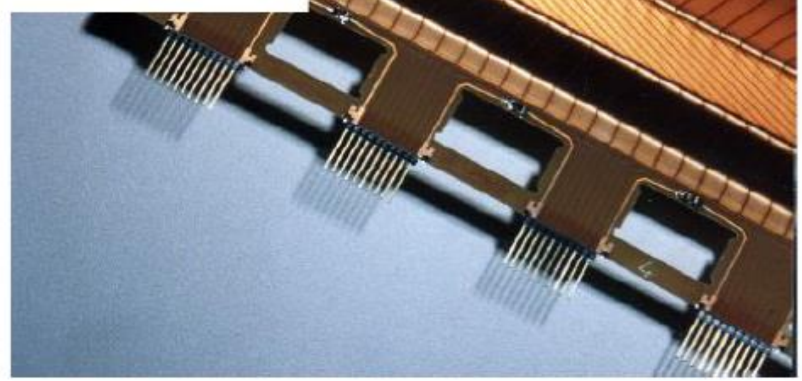
50mrad/√E(GeV)

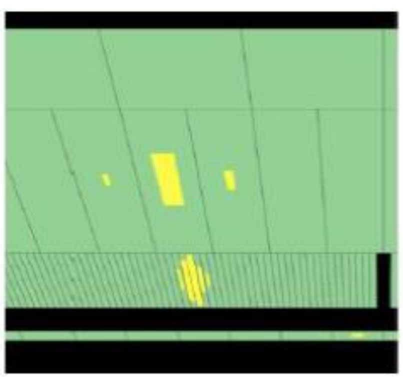


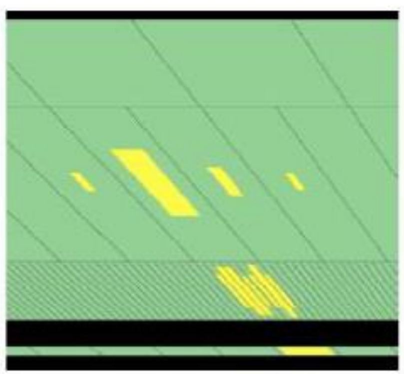
The segmentation



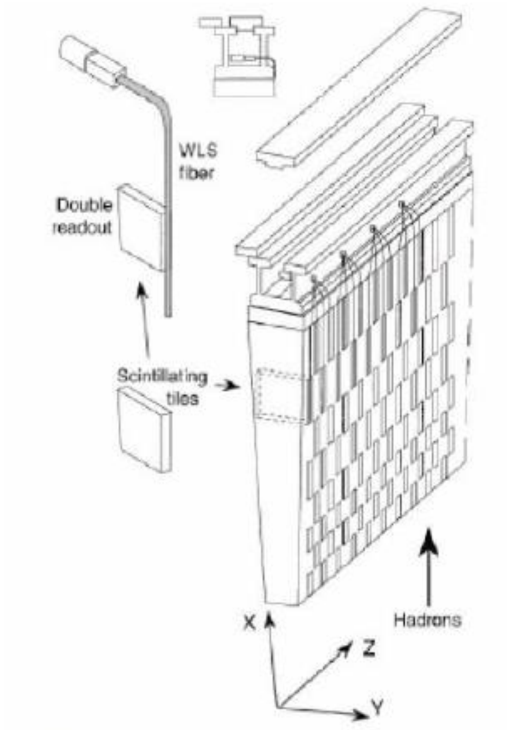








ATLAS Hadronic Calorimeter (Tile)



Fe/Scint with WLS fiber Readout via PMT

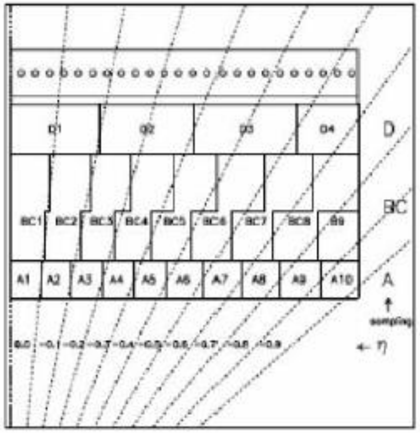


Figure 5-15 Cell geometry of half of a barrel module. The fibres of each cell are routed to one PMT.

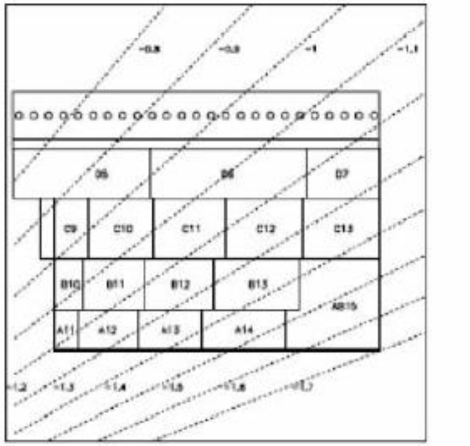
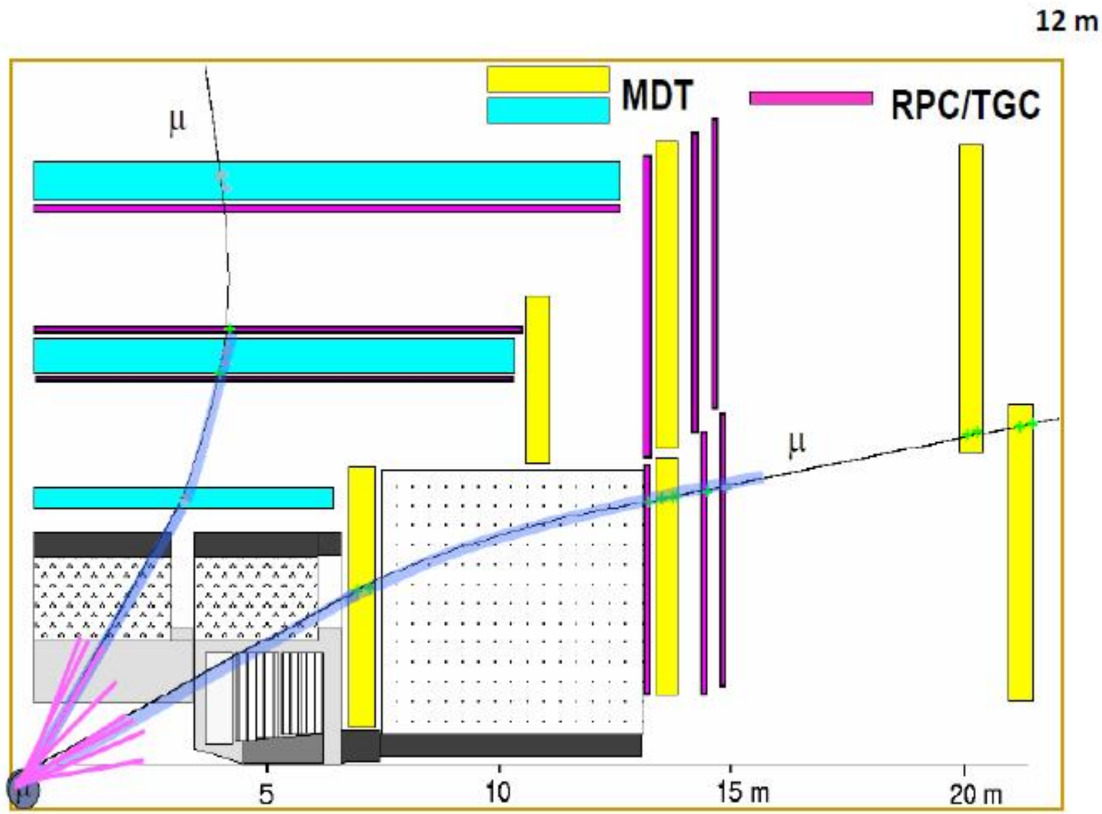


Figure 5-16 Proposed cell geometry for the extended barrel modules (version "a la barrel").

Muon system in ATLAS

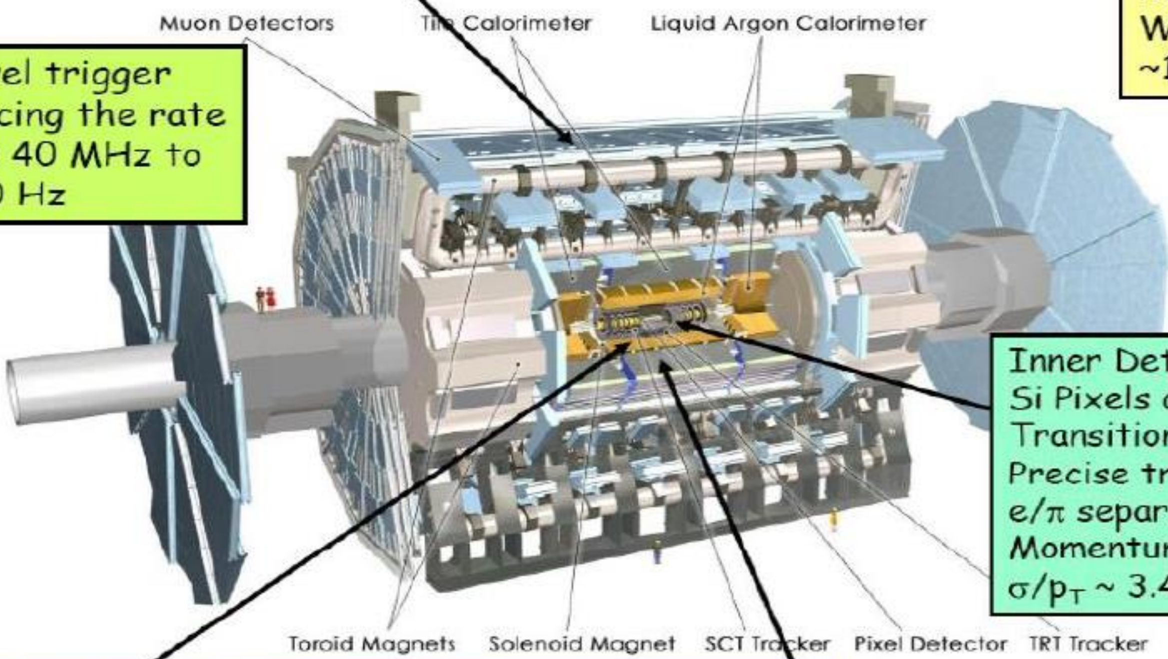


The ATLAS detector

Muon Spectrometer ($|\eta|<2.7$) : air-core toroids with gas-based chambers
Muon trigger and measurement with momentum resolution < 10% up to E_μ ~ TeV

Length : ~ 46 m
Radius : ~ 12 m
Weight : ~ 7000 tons
~10^8 electronic channels

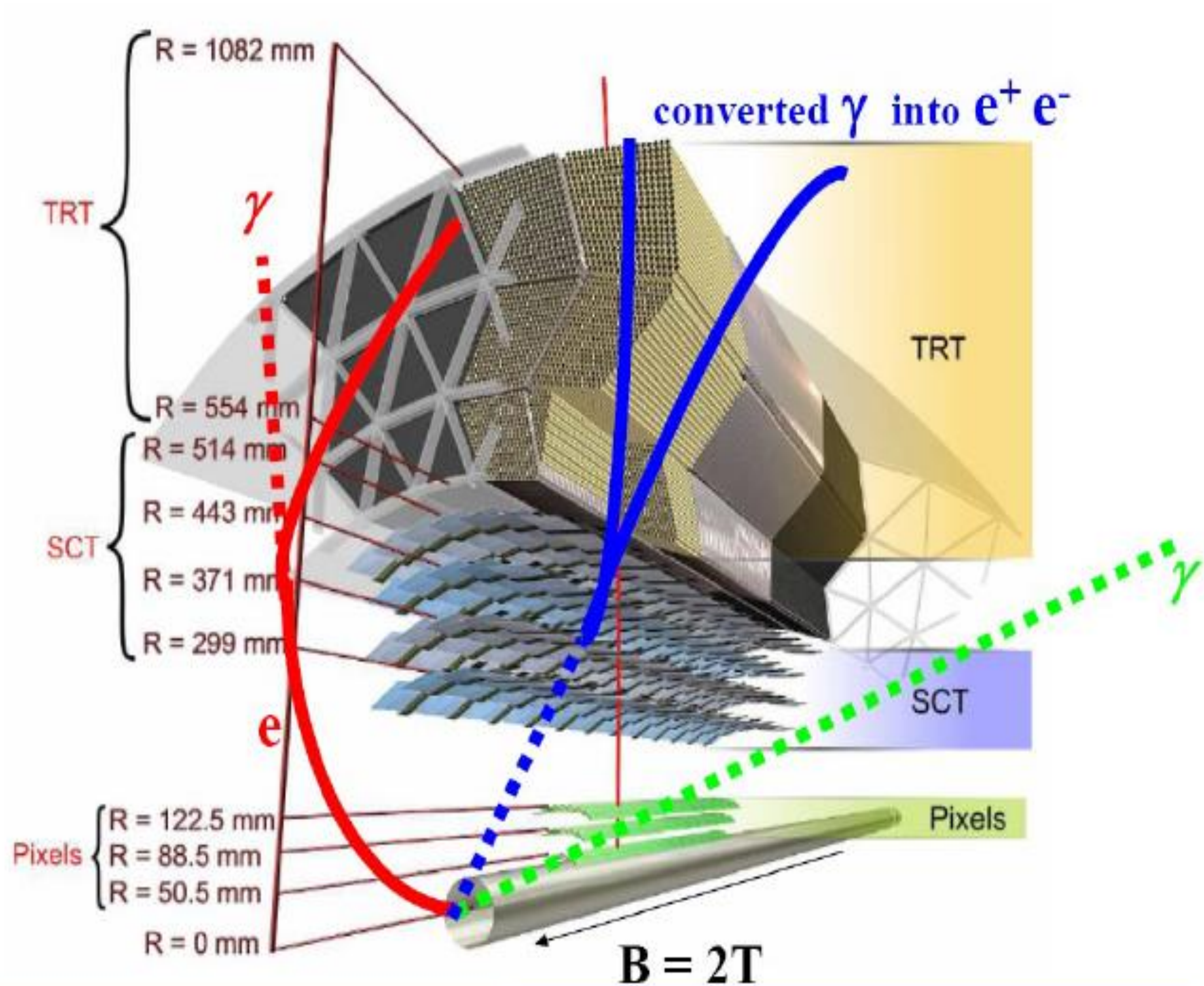
3-level trigger reducing the rate from 40 MHz to ~200 Hz



Inner Detector ($|\eta|<2.5$, B=2T):
Si Pixels and strips (SCT) +
Transition Radiation straws
Precise tracking and vertexing,
e/π separation (TRT).
Momentum resolution:
$\sigma/p_T \sim 3.4\times10^{-4}\, p_T(\text{GeV}) \oplus 0.015$

EM calorimeter: Pb-LAr Accordion
e/γ trigger, identification and measurement
E-resolution: ~ 1% at 100 GeV, 0.5% at 1 TeV

HAD calorimetry ($|\eta|<5$): segmentation, hermeticity
Tilecal Fe/scintillator (central), Cu/W-LAr (fwd)
Trigger and measurement of jets and missing E_T
E-resolution: $\sigma/E \sim 50\%/\sqrt{E} \oplus 0.03$

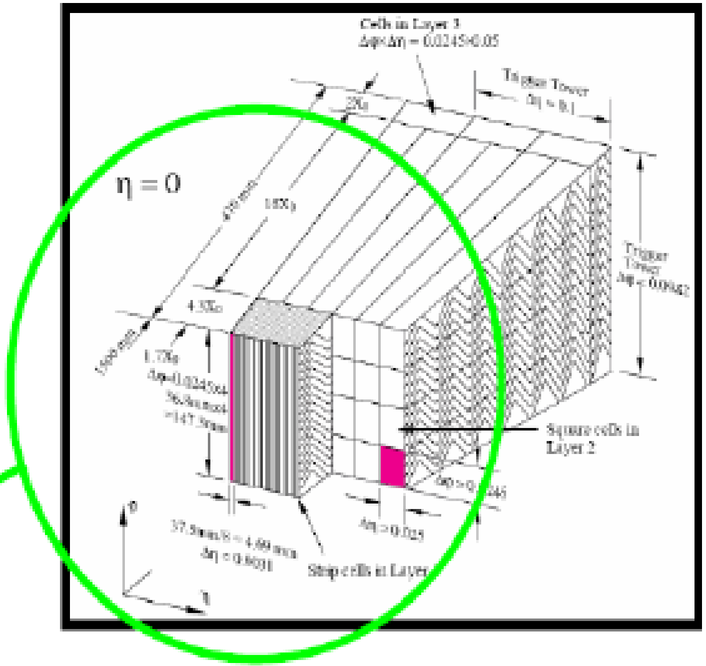


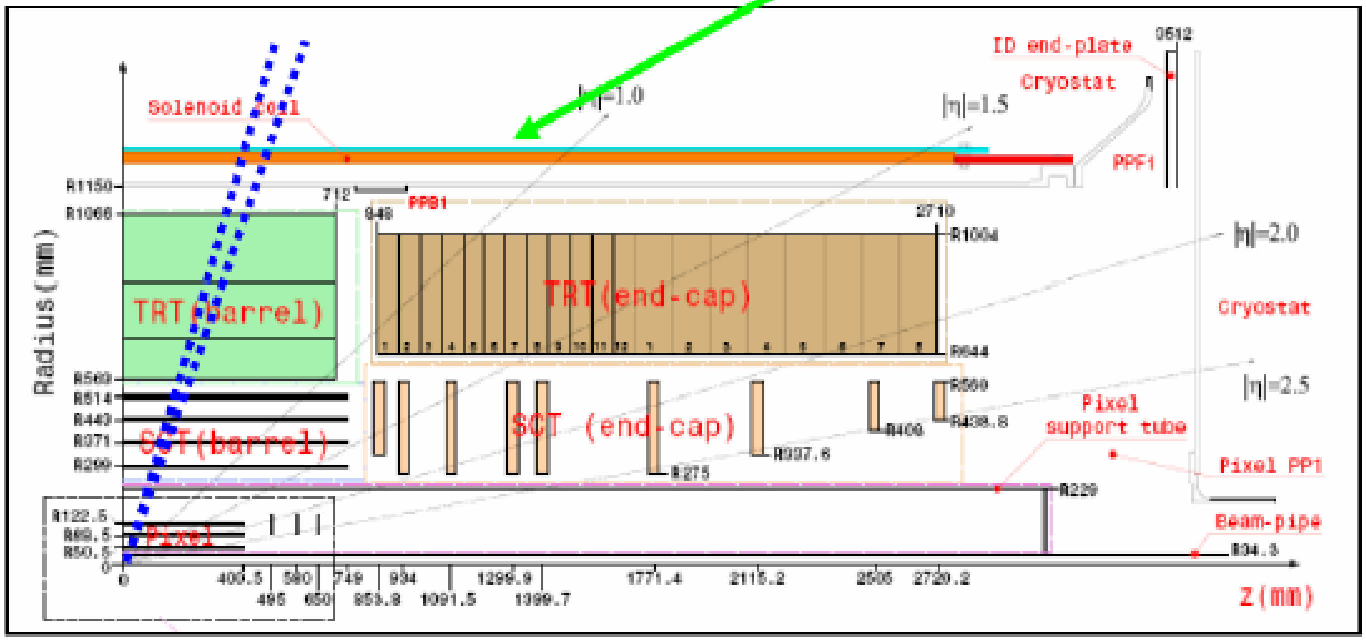
R = 1082 mm
converted γ into e+ e-
TRT
γ
TRT
R = 554 mm
R = 514 mm
R = 443 mm
SCT
R = 371 mm
R = 299 mm
γ
SCT
e
Pixels
R = 122.5 mm
R = 88.5 mm
R = 50.5 mm
R = 0 mm
Pixels
B = 2T

This will be very useful to reject the background from π^0

opening of photons coming from a π^0 (p_T=40 GeV)

$\Delta R > .007$





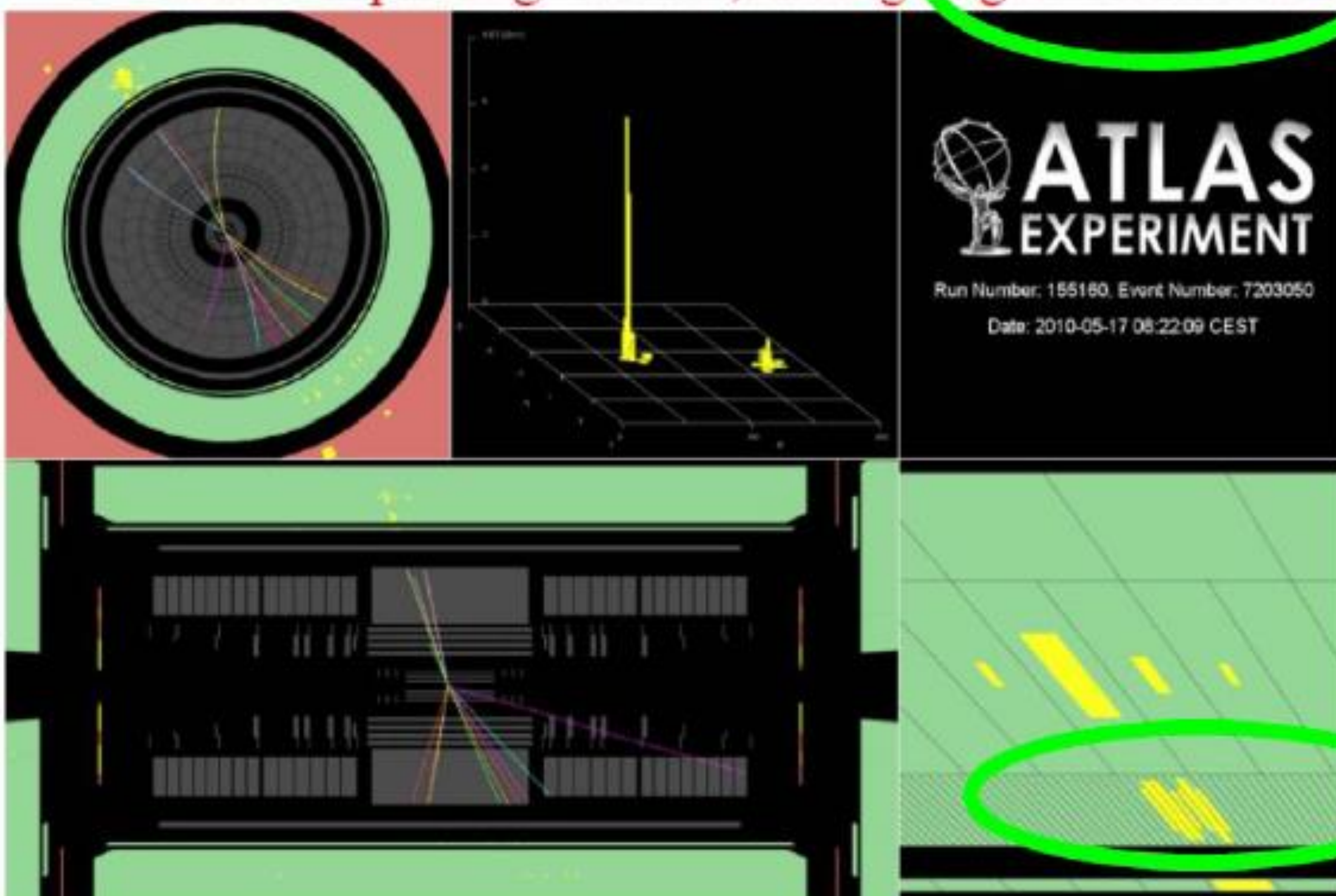
granularity of 1st sampling of calorimeter

$\Delta\eta \sim .003$

Photon identification with shower shapes

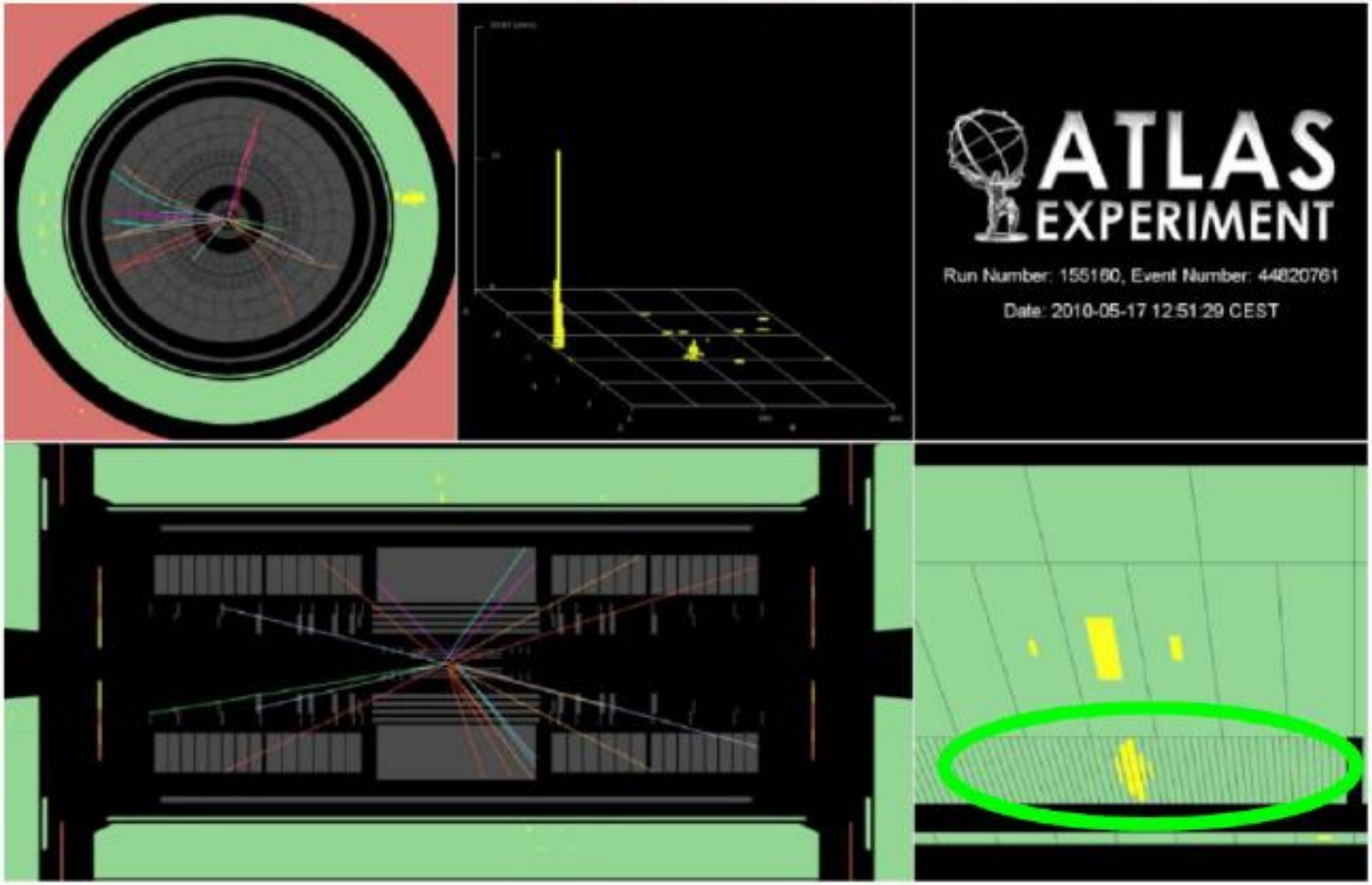
reminder: opening angle between the two photons of a π^0 of p_T = 40 GeV ***is > 0.007 to be compared with size of strip calo 1st sampling ~0.003***

π^0 candidate passing "loose", failing "tight" selection



tight selection uses mainly calo 1st sampling

Photon candidate passing “tight” selection

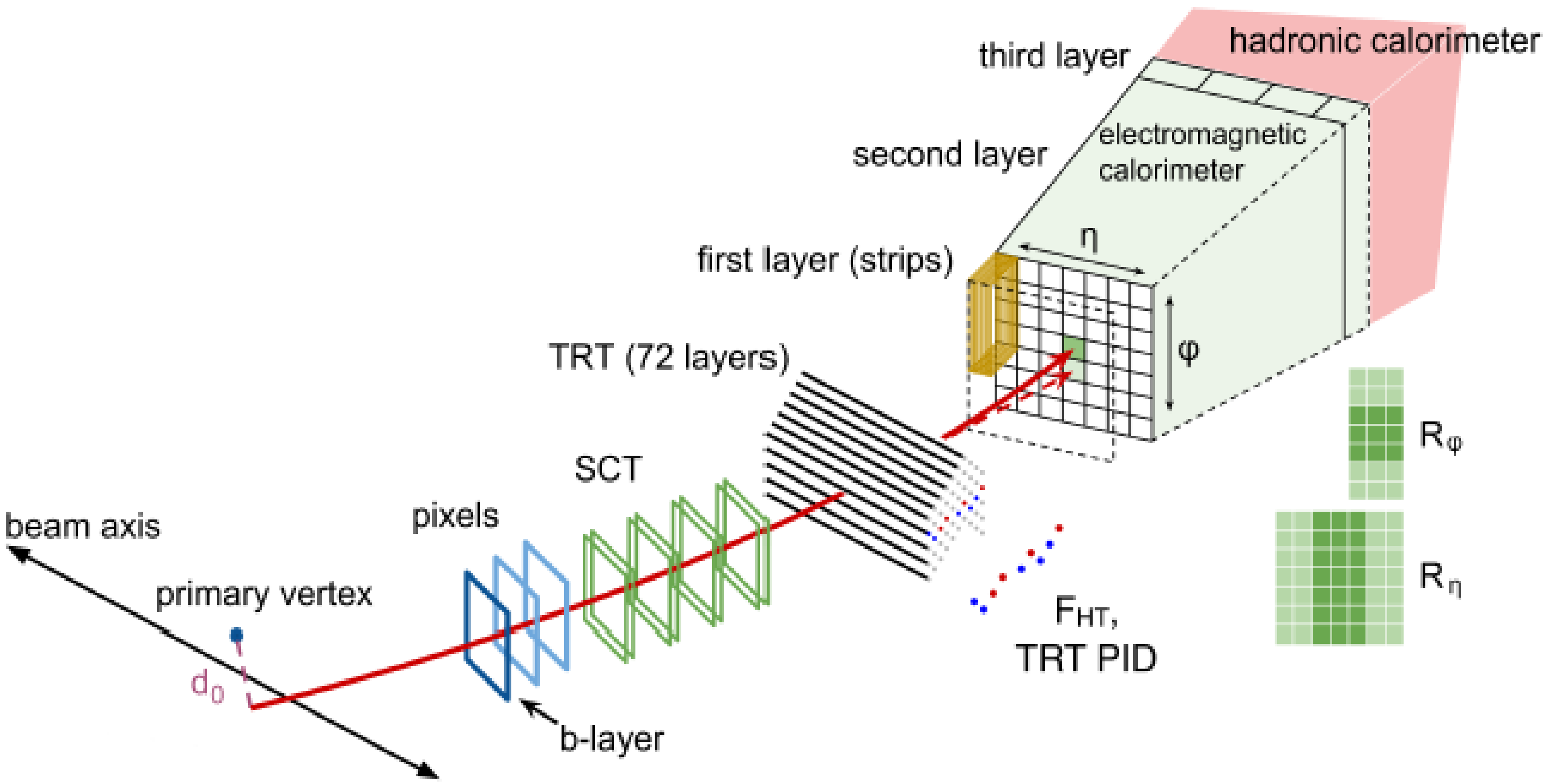


Nice shape in first sampling of EM calormeter

Electron identification

ElectronID attempts to make optimal use of the tracking and calorimetric detectors. This includes shower widths, ratios of various energy deposits, tracking hits, track-cluster matching, etc. (IBL not shown below!)

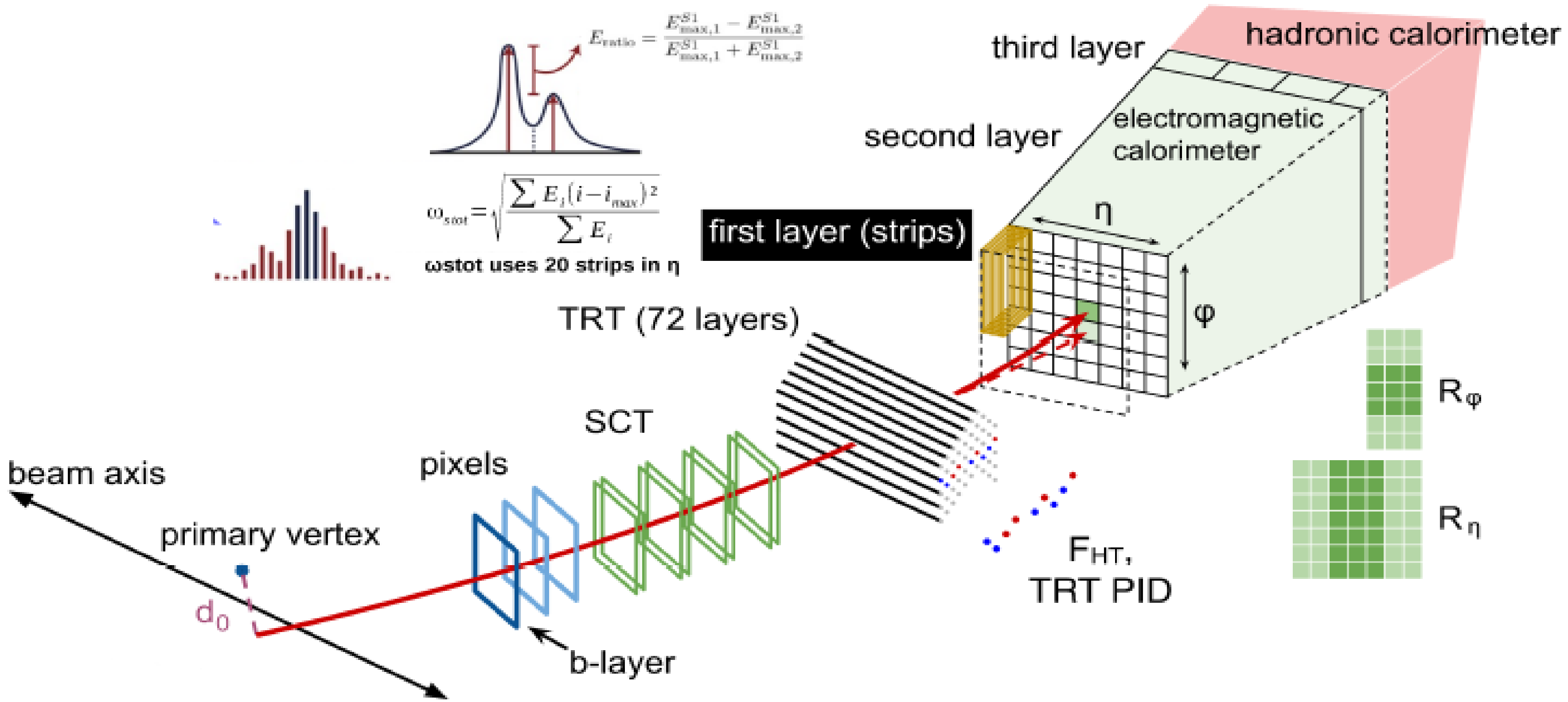
Thanks to Joey and Kurt for the image!



Electron identification

First layer of the EMCAL

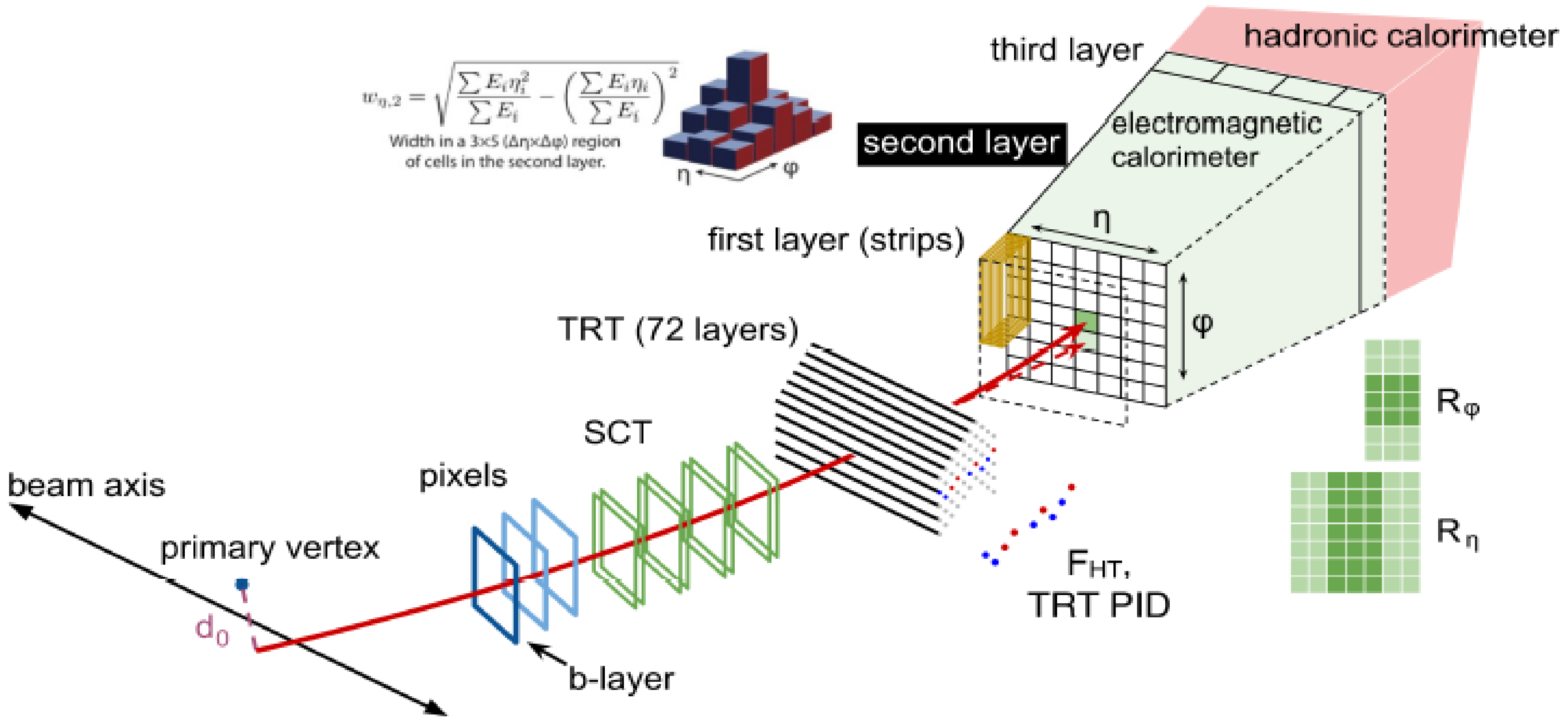
$$E_{Ratio} = \frac{E_{max,1} - E_{max,2}}{E_{max,1} + E_{max,2}}; \; \omega_{stot} = \sqrt{\frac{\sum E_i (i - i_{max})^2}{\sum E_i}}$$



Electron identification

Second layer of the EMCAL

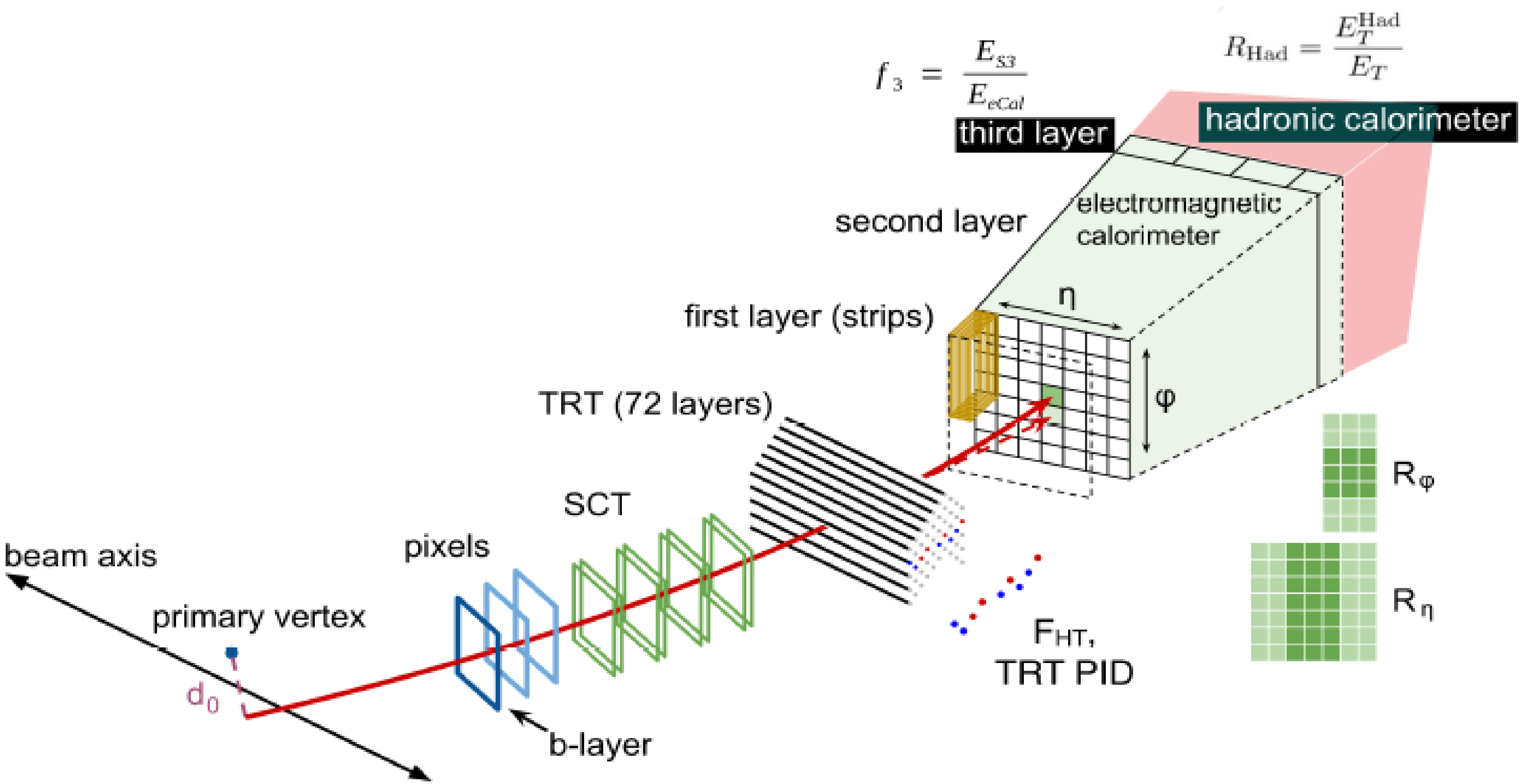
$$W_{\eta_2} = \sqrt{\frac{\sum E_i \eta_i^2}{\sum E_i} - \left(\frac{\sum E_i \eta_i}{\sum E_i}\right)}; R_\eta = \frac{E^{S2}_{3\times 7}}{E^{S2}_{7\times 7}}; R_\phi = \frac{E^{S2}_{3\times 3}}{E^{S2}_{3\times 7}}$$



Electron identification

Third layer of the EMCAL and HCAL

$$f_3 = \frac{E_{S3}}{E_{EMCAL}};\ R_{Had} = \frac{E_T^{Had}}{E_T}$$



Nuclear Instruments & Methods in Physics Research

topical issue

Instrumentation and detector technologies for frontier high energy physics

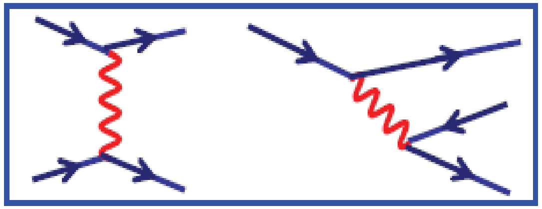
Volume 666, pages 1 - 222 (21 February 2012)

Edited by:
Archana Sharma (CERN)

Technological advances in radiation detection have been pioneered and led by particle physics. The ever increasing complexity of the experiments in high energy physics has driven the need for developments in high performance silicon and gaseous tracking detectors, electromagnetic and hadron calorimetry, transition radiation detectors and novel particle identification techniques. Magnet systems have evolved with superconducting magnets being used in present and, are being designed for use in, future experiments. The alignment system, being critical for the overall detector performance, has become one of the essential design aspects of large experiments. The electronic developments go hand in hand to enable the exploitation of these detectors designed to operate in the hostile conditions of radiation, high rate and luminosity. This volume provides a panorama of the state-of-the-art in the field of radiation detection and instrumentation for large experiments at the present and future particle accelerators.

Cross-section and decay rates

- In particle physics we are mainly concerned with particle interactions and decays, i.e. transitions between states



  - these are the experimental observables of particle physics
- Calculate transition rates from Fermi's Golden Rule

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

Γ_{fi} is number of transitions per unit time from initial state $|i\rangle$ to final state $\langle f|$ – not Lorentz Invariant !

T_{fi} is Transition Matrix Element

$$T_{fi} = \langle f|\hat{H}|i\rangle + \sum_{j\neq i} \frac{\langle f|\hat{H}|j\rangle\langle j|\hat{H}|i\rangle}{E_i - E_j} + \ldots$$

$\hat{H}$ is the perturbing Hamiltonian

$\rho(E_f)$ is density of final states

★ Rates depend on MATRIX ELEMENT and DENSITY OF STATES

the ME contains the fundamental particle physics

just kinematics

from M. A. Thomson lectures at Cambridge University in 2011

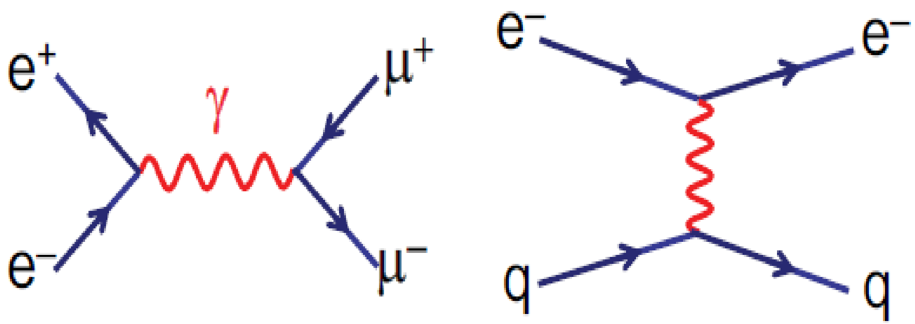
Cross-sections

★ Aiming towards a proper calculation of decay and scattering processes

Will concentrate on:

- $e^+e^- \to \mu^+\mu^-$
- $e^-q \to e^-q$

($e^-q \to e^-q$ to probe proton structure)



⅄ Need relativistic calculations of particle decay rates and cross sections:

$$\sigma = \frac{|M_{fi}|^2}{\text{flux}} \times (\text{phase space})$$

⅄ Need relativistic treatment of spin-half particles:

Dirac Equation

⅄ Need relativistic calculation of interaction Matrix Element:

Interaction by particle exchange and Feynman rules

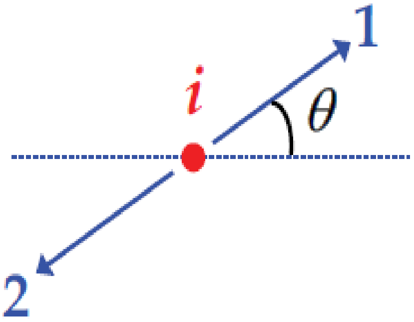
\+ and a few mathematical tricks along, e.g. the Dirac Delta Function

from M. A. Thomson lectures at Cambridge University in 2011

Particle decay rates

- Consider the two-body decay

$$i \to 1 + 2$$



- Want to calculate the decay rate in first order perturbation theory using plane-wave descriptions of the particles (Born approximation):

$$\psi_1 = N e^{i(\vec{p}.\vec{r} - Et)} \qquad (\vec{k}.\vec{r} = \vec{p}.\vec{r} \quad \text{as} \quad \hbar = 1)$$
$$= N e^{-ip.x}$$

where N is the normalisation and $p.x = p^\mu x_\mu$

<u>For decay rate calculation need to know:</u>

- Wave-function normalisation
- Transition matrix element from perturbation theory
- Expression for the density of states

} All in a Lorentz Invariant form

★First consider wave-function normalisation

- Non-relativistic: normalised to one particle in a cube of side a

$$\int \psi\psi^* \mathrm{d}V = N^2 a^3 = 1 \quad \Rightarrow \quad N^2 = 1/a^3$$

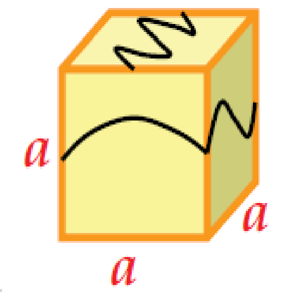
from M. A. Thomson lectures at Cambridge University in 2011

Non-relativistic phase space

- Apply boundary conditions ($\vec{p} = \hbar\vec{k}$):
- Wave-function vanishing at box boundaries

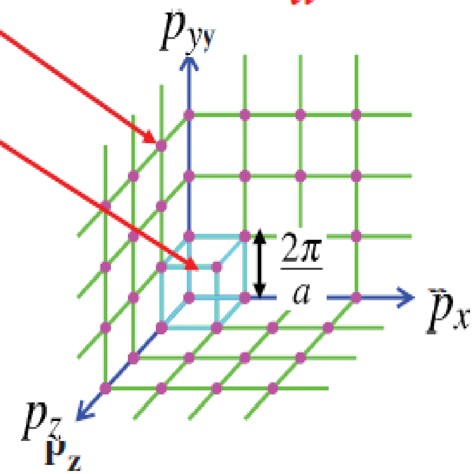
  ➡ quantised particle momenta:

$$p_x = \frac{2\pi n_x}{a};\ p_y = \frac{2\pi n_y}{a};\ p_z = \frac{2\pi n_z}{a}$$



- Volume of single state in momentum space:

$$\left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$



- Normalising to one particle/unit volume gives number of states in element: $d^3\vec{p} = dp_x dp_y dp_z$

$$dn = \frac{d^3\vec{p}}{\frac{(2\pi)^3}{V}} \times \frac{1}{V} = \frac{d^3\vec{p}}{(2\pi)^3}$$

- Therefore density of states in Golden rule:

$$\rho(E_f) = \left|\frac{dn}{dE}\right|_{E_f} = \left|\frac{dn}{d|\vec{p}|}\frac{d|\vec{p}|}{dE}\right|_{E_f}$$

with $p = \beta E$

- Integrating over an elemental shell in momentum-space gives

  $(d^3\vec{p} = 4\pi p^2 dp)$

$$\rho(E_f) = \frac{4\pi p^2}{(2\pi)^3} \times \beta$$

from M. A. Thomson lectures at Cambridge University in 2011

Golden rule revisited

$$\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)$$

- Rewrite the expression for density of states using a delta-function

$$\rho(E_f) = \left|\frac{\mathrm{d}n}{\mathrm{d}E}\right|_{E_f} = \int \frac{\mathrm{d}n}{\mathrm{d}E}\delta(E - E_i)\mathrm{d}E \qquad \text{since } E_f = E_i$$

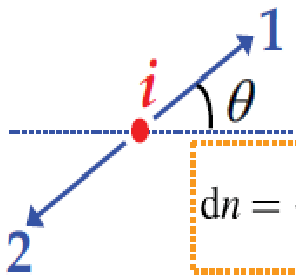
Note : integrating over all final state energies but energy conservation now taken into account explicitly by delta function

- Hence the golden rule becomes: $\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E)\mathrm{d}n$

the integral is over all "allowed" final states of any energy

- For $\mathrm{d}n$ in a two-body decay, only need to consider one particle : mom. conservation fixes the other

$$\Gamma_{fi} = 2\pi \int |T_{fi}|^2 \delta(E_i - E_1 - E_2)\frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3}$$



$$\mathrm{d}n = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}$$

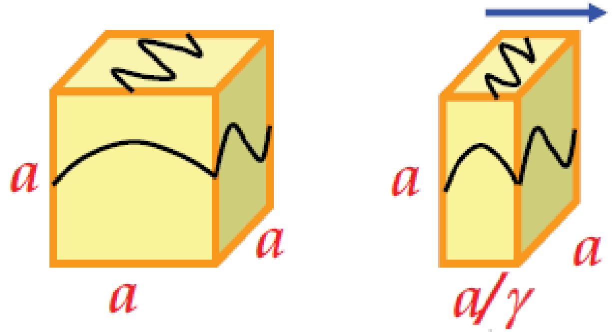
- However, can include momentum conservation explicitly by integrating over the momenta of both particles and using another δ-fn

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \underbrace{\delta(E_i - E_1 - E_2)}_{\text{Energy cons.}}\underbrace{\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)}_{\text{Mom. cons.}}\underbrace{\frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3}}_{\text{Density of states}}$$

from M. A. Thomson lectures at Cambridge University in 2011

Lorentz invariant phase-space

- In non-relativistic QM normalise to one particle/unit volume: $\int \psi^* \psi \mathrm{d}V = 1$
- When considering relativistic effects, volume contracts by $\gamma = E/m$



- Particle density therefore increases by $\gamma = E/m$
  - ★ Conclude that a relativistic invariant wave-function normalisation needs to be **proportional to E particles per unit volume**
- Usual convention: **Normalise to 2E particles/unit volume** $\int \psi'^* \psi' \mathrm{d}V = 2E$
- Previously used ψ normalised to 1 particle per unit volume $\int \psi^* \psi \mathrm{d}V = 1$
- Hence $\psi' = (2E)^{1/2}\psi$ is normalised to $2E$ per unit volume
- <u>Define</u> Lorentz Invariant Matrix Element, M_{fi} , in terms of the wave-functions normalised to $2E$ particles per unit volume

$$M_{fi} = \langle \psi'_1 . \psi'_2 ... | \hat{H} | ... \psi'_{n-1} \psi'_n \rangle = (2E_1 . 2E_2 . 2E_3 .... 2E_n)^{1/2} T_{fi}$$

from M. A. Thomson lectures at Cambridge University in 2011

Golden rule revisited

- For the two body decay

$$i \rightarrow 1 + 2$$

$$\begin{aligned} M_{fi} &= \langle \psi_1' \psi_2' | \hat{H}' | \psi_i' \rangle \\ &= (2E_i.2E_1.2E_2)^{1/2} \langle \psi_1 \psi_2 | \hat{H}' | \psi_i \rangle \\ &= (2E_i.2E_1.2E_2)^{1/2} T_{fi} \end{aligned}$$

★ Now expressing T_{fi} in terms of M_{fi} gives

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2) \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}$$

Note:

- M_{fi} uses relativistically normalised wave-functions. It is Lorentz Invariant
- $\frac{d^3\vec{p}}{(2\pi)^3 2E}$ is the Lorentz Invariant Phase Space for each final state particle the factor of $2E$ arises from the wave-function normalisation
- This form of Γ_{fi} is simply a rearrangement of the original equation **but** the integral is now frame independent (i.e. L.I.)
- Γ_{fi} is inversely proportional to E_i, the energy of the decaying particle. This is exactly what one would expect from time dilation ($E_i = \gamma m$).
- Energy and momentum conservation in the delta functions

from M. A. Thomson lectures at Cambridge University in 2011

Decay rate calculations

$$\Gamma_{fi} = \frac{(2\pi)^4}{2E_i} \int |M_{fi}|^2 \delta(E_i - E_1 - E_2)\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2) \frac{d^3\vec{p}_1}{(2\pi)^3 2E_1} \frac{d^3\vec{p}_2}{(2\pi)^3 2E_2}$$

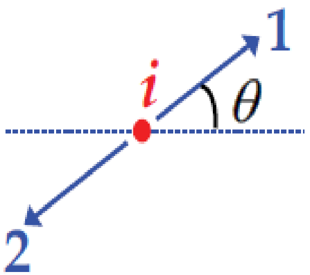
★ Because the integral is Lorentz invariant (i.e. frame independent) it can be evaluated in any frame we choose. The C.o.M. frame is most convenient

- In the C.o.M. frame $E_i = m_i$ and $\vec{p}_i = 0$ $\Rightarrow$

$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2)\delta^3(\vec{p}_1 + \vec{p}_2) \frac{d^3\vec{p}_1}{2E_1} \frac{d^3\vec{p}_2}{2E_2}$$

- Integrating over $\vec{p}_2$ using the δ-function:

$$\Rightarrow \Gamma_{fi} = \frac{1}{8\pi^2 E_i} \int |M_{fi}|^2 \delta(m_i - E_1 - E_2) \frac{d^3\vec{p}_1}{4E_1 E_2}$$



now $E_2^2 = (m_2^2 + |\vec{p}_1|^2)$ since the δ-function imposes $\vec{p}_2 = -\vec{p}_1$

- Writing $d^3\vec{p}_1 = p_1^2 dp_1 \sin\theta d\theta d\phi = p_1^2 dp_1 d\Omega$

For convenience, here $|\vec{p}_1|$ is written as p_1

$$\Rightarrow \quad \Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \delta\left(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right) \frac{p_1^2 dp_1 d\Omega}{E_1 E_2}$$

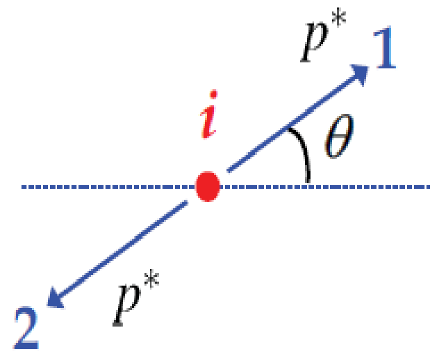
from M. A. Thomson lectures at Cambridge University in 2011

Decay rate calculations

- Which can be written in the form $$\Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 g(p_1)\delta(f(p_1))\mathrm{d}p_1\mathrm{d}\Omega \qquad (2)$$

where $g(p_1) = p_1^2/(E_1E_2) = p_1^2(m_1^2+p_1^2)^{-1/2}(m_2^2+p_1^2)^{-1/2}$

and $f(p_1) = m_i - (m_1^2+p_1^2)^{1/2} - (m_2^2+p_1^2)^{1/2}$



Note:
- $\delta(f(p_1))$ imposes energy conservation.
- $f(p_1) = 0$ determines the C.o.M momenta of the two decay products

  i.e. $f(p_1) = 0$ for $p_1 = p^*$

★ Eq. (2) can be integrated using the property of δ– function derived earlier (eq. (1))

$$\int g(p_1)\delta(f(p_1))\mathrm{d}p_1 = \frac{1}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}}\int g(p_1)\delta(p_1-p^*)\mathrm{d}p_1 = \frac{g(p^*)}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}}$$

where p^* is the value for which $f(p^*) = 0$

- All that remains is to evaluate $\mathrm{d}f/\mathrm{d}p_1$

$$\frac{\mathrm{d}f}{\mathrm{d}p_1} = -\frac{p_1}{(m_1^2+p_1^2)^{1/2}} - \frac{p_1}{(m_2^2+p_1^2)^{1/2}} = -\frac{p_1}{E_1} - \frac{p_1}{E_2} = -p_1\frac{E_1+E_2}{E_1E_2}$$

from M. A. Thomson lectures at Cambridge University in 2011

Decay rate calculations

giving:
$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \left|\frac{E_1E_2}{p_1(E_1+E_2)}\frac{p_1^2}{E_1E_2}\right|_{p_1=p^*} \mathrm{d}\Omega$$
$$= \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \left|\frac{p_1}{E_1+E_2}\right|_{p_1=p^*} \mathrm{d}\Omega$$

- But from $f(p_1)=0$, i.e. energy conservation: $E_1+E_2=m_i$

$$\Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 E_i m_i}\int |M_{fi}|^2 \mathrm{d}\Omega$$

In the particle's rest frame $E_i = m_i$

$$\frac{1}{\tau} = \Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2}\int |M_{fi}|^2 \mathrm{d}\Omega \qquad (3)$$

VALID FOR ALL TWO-BODY DECAYS !

- p^* can be obtained from $f(p_1)=0$

$$(m_1^2+p^{*2})^{1/2} + (m_2^2+p^{*2})^{1/2} = m_i$$

$$\Rightarrow p^* = \frac{1}{2m_i}\sqrt{\left[(m_i^2-(m_1+m_2)^2\right]\left[m_i^2-(m_1-m_2)^2\right]}$$

from M. A. Thomson lectures at Cambridge University in 2011

Cross-section definition

$$\sigma = \frac{\text{no of interactions per unit time per target}}{\text{incident flux}}$$

Flux = number of incident particles/ unit area/unit time

- The "cross section", σ, can be thought of as the effective cross-sectional area of the target particles for the interaction to occur.
- In general this has nothing to do with the physical size of the target although there are exceptions, e.g. neutron absorption

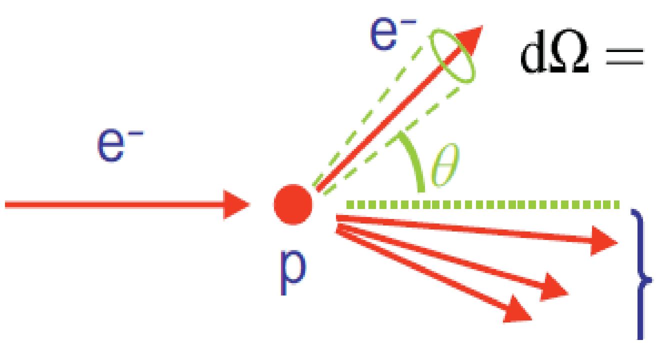
here σ is the projective area of nucleus

Differential Cross section

$$\frac{d\sigma}{d\Omega} = \frac{\text{no of particles per sec/per target into } d\Omega}{\text{incident flux}}$$

or generally

$$\frac{d\sigma}{d\ldots}$$



$$d\Omega = d(\cos\theta)d\phi$$

integrate over all other particles

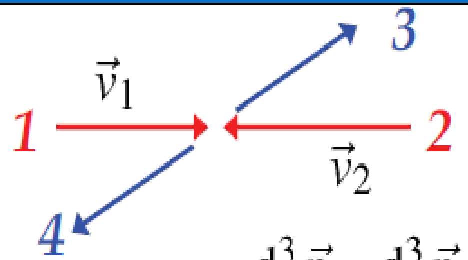
with $$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega$$

from M. A. Thomson lectures at Cambridge University in 2011

Cross-section calculations

- Consider scattering process

$$1+2\rightarrow 3+4$$



- Start from Fermi's Golden Rule:

$$\Gamma_{fi}=(2\pi)^4\int|T_{fi}|^2\delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3}$$

where T_{fi} is the transition matrix for a normalisation of 1/unit volume

- Now

$$\begin{aligned}\text{Rate/Volume} &= (\text{flux of 1})\times(\text{number density of 2})\times\sigma\\ &= n_1(v_1+v_2)\times n_2\times\sigma\end{aligned}$$

- For 1 target particle per unit volume $\text{Rate}=(v_1+v_2)\sigma$

$$\sigma=\frac{\Gamma_{fi}}{(v_1+v_2)}$$

$$\sigma=\frac{(2\pi)^4}{v_1+v_2}\int|T_{fi}|^2\delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3}$$

the parts are not Lorentz Invariant

from M. A. Thomson lectures at Cambridge University in 2011

Cross-section calculations

- To obtain a Lorentz Invariant form use wave-functions normalised to $2E$ particles per unit volume

$$\psi' = (2E)^{1/2}\psi$$

- Again define L.I. Matrix element $M_{fi} = (2E_1\, 2E_2\, 2E_3\, 2E_4)^{1/2} T_{fi}$

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1+v_2)} \int |M_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{2E_3}\frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

- The integral is now written in a Lorentz invariant form
- The quantity $F = 2E_1 2E_2(v_1+v_1)$ can be written in terms of a four-vector scalar product and is therefore also Lorentz Invariant (the Lorentz Inv. Flux)

$$F = 4\left[(p_1^{\mu} p_{2\mu})^2 - m_1^2 m_2^2\right]^{1/2}$$

- Consequently cross section is a Lorentz Invariant quantity

<u>Two special cases of Lorentz Invariant Flux:</u>

- Centre-of-Mass Frame

$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1E_2(|\vec{p}^*|/E_1+|\vec{p}^*|/E_2) \\ &= 4|\vec{p}^*|(E_1+E_2) \\ &= 4|\vec{p}^*|\sqrt{s} \end{aligned}$$

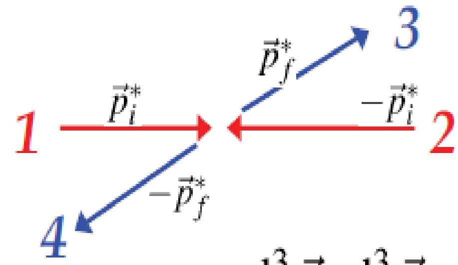
- Target (particle 2) at rest

$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1m_2v_1 \\ &= 4E_1m_2(|\vec{p}_1|/E_1) \\ &= 4m_2|\vec{p}_1| \end{aligned}$$

from M. A. Thomson lectures at Cambridge University in 2011

Cross-section calculations

- We will now apply above Lorentz Invariant formula for the interaction cross section to the most common cases used in the course. First consider 2→2 scattering in C.o.M. frame



- Start from

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2 (v_1 + v_2)} \int |M_{fi}|^2 \delta(E_1 + E_2 - E_3 - E_4) \delta^3(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4) \frac{\mathrm{d}^3\vec{p}_3}{2E_3} \frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

- Here $\vec{p}_1 + \vec{p}_2 = 0 \quad \text{and} \quad E_1 + E_2 = \sqrt{s}$

$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \int |M_{fi}|^2 \delta(\sqrt{s} - E_3 - E_4) \delta^3(\vec{p}_3 + \vec{p}_4) \frac{\mathrm{d}^3\vec{p}_3}{2E_3} \frac{\mathrm{d}^3\vec{p}_4}{2E_4}$$

★ The integral is exactly the same integral that appeared in the particle decay calculation but with m_a replaced by $\sqrt{s}$

$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \frac{|\vec{p}_f^*|}{4\sqrt{s}} \int |M_{fi}|^2 \mathrm{d}\Omega^*$$

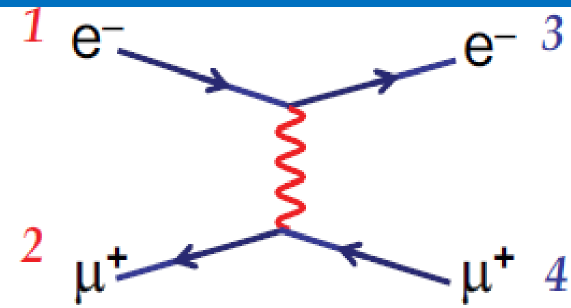
$$\boxed{\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} \int |M_{fi}|^2 \mathrm{d}\Omega^*}$$

from M. A. Thomson lectures at Cambridge University in 2011

Cross-section calculations

- In the case of elastic scattering $|\vec{p}_i^*| = |\vec{p}_f^*|$

$$\sigma_{\text{elastic}} = \frac{1}{64\pi^2 s} \int |M_{fi}|^2 \mathrm{d}\Omega^*$$



- For calculating the total cross-section (which is Lorentz Invariant) the result on the previous page (eq. (4)) is sufficient. However, it is not so useful for calculating the differential cross section in a rest frame other than the C.o.M:

$$\mathrm{d}\sigma = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} |M_{fi}|^2 \mathrm{d}\Omega^*$$

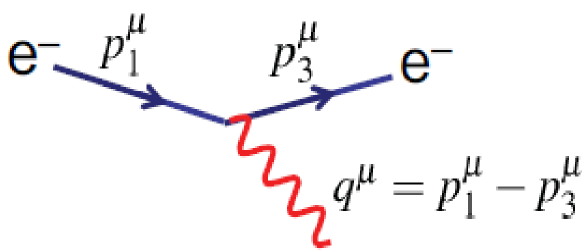
because the angles in $\mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^*$ refer to the C.o.M frame

- For the last calculation in this section, we need to find a L.I. expression fo $\mathrm{d}\sigma$

★ Start by expressing $\mathrm{d}\Omega^*$ in terms of Mandelstam *t* i.e. the square of the four-momentum transfer

$$t = q^2 = (p_1 - p_3)^2$$

Product of four-vectors therefore L.I.



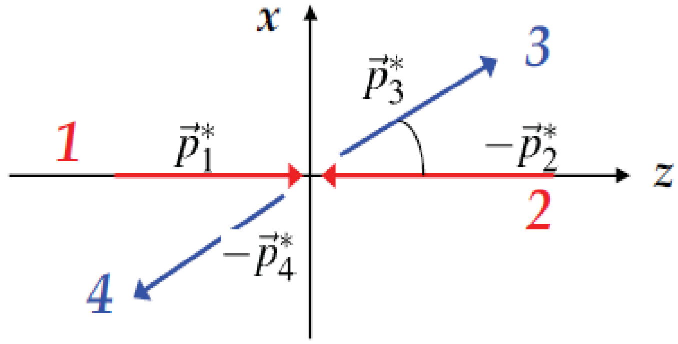
Cross-section calculations

- Want to express $\mathrm{d}\Omega^*$ in terms of Lorentz Invariant $\mathrm{d}t$

  where $t \equiv (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1.p_3 = m_1^2 + m_3^2 - 2p_1.p_3$

- In C.o.M. frame:

$$
\begin{aligned}
p_1^{*\mu} &= (E_1^*, 0, 0, |\vec{p}_1^*|) \\
p_3^{*\mu} &= (E_3^*, |\vec{p}_3^*|\sin\theta^*, 0, |\vec{p}_3^*|\cos\theta^*) \\
p_1^\mu p_{3\mu} &= E_1^* E_3^* - |\vec{p}_1^*||\vec{p}_3^*|\cos\theta^* \\
t &= m_1^2 + m_3^3 - E_1^* E_3^* + 2|\vec{p}_1^*||\vec{p}_3^*|\cos\theta^*
\end{aligned}
$$



giving $$\mathrm{d}t = 2|\vec{p}_1^*||\vec{p}_3^*|\mathrm{d}(\cos\theta^*)$$

therefore $$\mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^* = \frac{\mathrm{d}t\,\mathrm{d}\phi^*}{2|\vec{p}_1^*||\vec{p}_3^*|}$$

hence $$\mathrm{d}\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_3^*|}{|\vec{p}_1^*|}|M_{fi}|^2\mathrm{d}\Omega^* = \frac{1}{2\cdot 64\pi^2 s|\vec{p}_1^*|^2}|M_{fi}|^2\mathrm{d}\phi^*\mathrm{d}t$$

- Finally, integrating over $\mathrm{d}\phi^*$ (assuming no ϕ^* dependence of $|M_{fi}|^2$) gives:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2$$

from M. A. Thomson lectures at Cambridge University in 2011

Lorentz invariant differential cross-section

- All quantities in the expression for $\mathrm{d}\sigma/\mathrm{d}t$ are Lorentz Invariant and therefore, it applies to any rest frame. It should be noted that $|\vec{p}_i^*|^2$ is a constant, fixed by energy/momentum conservation

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$$

- As an example of how to use the invariant expression $\mathrm{d}\sigma/\mathrm{d}t$ we will consider elastic scattering in the laboratory frame in the limit where we can neglect the mass of the incoming particle $E_1 \gg m_1$

e.g. electron or neutrino scattering

In this limit $$|\vec{p}_i^*|^2 = \frac{(s-m_2)^2}{4s}$$

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{16\pi(s-m_2^2)^2}|M_{fi}|^2} \qquad (m_1 = 0)$$

from M. A. Thomson lectures at Cambridge University in 2011

2-> 2 body scattering in the LAB frame

$$t = (p_2 - p_4)^2 = 2M^2 - 2p_2.p_4 = 2M^2 - 2ME_4$$
$$= 2M^2 - 2M(E_1 + M - E_3) = -2M(E_1 - E_3)$$

Note E_1 is a constant (the energy of the incoming particle) so

$$\frac{dt}{d(\cos\theta)} = 2M\frac{dE_3}{d(\cos\theta)}$$

- Equating the two expressions for t gives $$E_3 = \frac{E_1 M}{M + E_1 - E_1\cos\theta}$$

so $$\frac{dE_3}{d(\cos\theta)} = \frac{E_1 M}{(M + E_1 - E_1\cos\theta)^2} = E_1^2 M\left(\frac{E_3}{E_1 M}\right)^2 = \frac{E_3^2}{M}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt} = \frac{1}{2\pi}2M\frac{E_3^2}{M}\frac{d\sigma}{dt} = \frac{E_3^2}{\pi}\frac{d\sigma}{dt} = \frac{E_3^2}{\pi}\frac{1}{16\pi(s-M^2)^2}|M_{fi}|^2$$

using $$s = (p_1 + p_2)^2 = M^2 + 2p_1.p_2 = M^2 + 2ME_1$$

Particle 1 massless ➡ $(p_1^2 = 0)$

gives $$(s - M^2) = 2ME_1$$

➡ $$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2$$

In limit $m_1 \rightarrow 0$

from M. A. Thomson lectures at Cambridge University in 2011

Lorentz invariant differential cross-section

$$t = (p_2 - p_4)^2 = 2M^2 - 2p_2.p_4 = 2M^2 - 2ME_4$$
$$= 2M^2 - 2M(E_1 + M - E_3) = -2M(E_1 - E_3)$$

Note E_1 is a constant (the energy of the incoming particle) so

$$\frac{dt}{d(\cos\theta)} = 2M\frac{dE_3}{d(\cos\theta)}$$

- Equating the two expressions for t gives

$$E_3 = \frac{E_1 M}{M + E_1 - E_1\cos\theta}$$

so
$$\frac{dE_3}{d(\cos\theta)} = \frac{E_1 M}{(M + E_1 - E_1\cos\theta)^2} = E_1^2 M\left(\frac{E_3}{E_1 M}\right)^2 = \frac{E_3^2}{M}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt} = \frac{1}{2\pi}2M\frac{E_3^2}{M}\frac{d\sigma}{dt} = \frac{E_3^2}{\pi}\frac{d\sigma}{dt} = \frac{E_3^2}{\pi}\frac{1}{16\pi(s-M^2)^2}|M_{fi}|^2$$

using $s = (p_1 + p_2)^2 = M^2 + 2p_1.p_2 = M^2 + 2ME_1$

Particle 1 massless ➡ $(p_1^2 = 0)$

gives $(s - M^2) = 2ME_1$

➡
$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2$$

In limit $m_1 \rightarrow 0$

Lorentz invariant differential cross-section

In this equation, E_3 is a function of θ:

$$E_3 = \frac{E_1 M}{M + E_1 - E_1 \cos\theta}$$

giving

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2 \qquad (m_1 = 0)$$

General form for 2→2 Body Scattering in Lab. Frame

★The calculation of the differential cross section for the case where m_1 can not be neglected is longer and contains no more "physics" (see appendix II). It gives:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{p_1 m_1} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \cdot |M_{fi}|^2$$

Again there is only one independent variable, θ, which can be seen from conservation of energy

$$E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{\underbrace{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta}_{\vec{p}_4 = \vec{p}_1 - \vec{p}_3} + m_4^2}$$

i.e. $|\vec{p}_3|$ is a function of θ

from M. A. Thomson lectures at Cambridge University in 2011

Summary

★ Used a Lorentz invariant formulation of Fermi's Golden Rule to derive decay rates and cross-sections in terms of the **Lorentz Invariant Matrix Element** (wave-functions normalised to 2E/Volume)

Main Results:

★ Particle decay:

$$\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 \mathrm{d}\Omega$$

Where p^* is a function of particle masses

$$p^* = \frac{1}{2m_i}\sqrt{\left[(m_i^2 - (m_1+m_2)^2\right]\left[m_i^2 - (m_1-m_2)^2\right]}$$

★ Scattering cross section in C.o.M. frame:

$$\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}\int |M_{fi}|^2 \mathrm{d}\Omega^*$$

★ Invariant differential cross section (valid in all frames):

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2$$

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$$

from M. A. Thomson lectures at Cambridge University in 2011

Summary

★Differential cross section in the lab. frame (m_1=0)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2 \iff \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M+E_1-E_1\cos\theta}\right)^2 |M_{fi}|^2$$

★Differential cross section in the lab. frame ($m_1 \neq 0$)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{|\vec{p}_1| m_1} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1+m_2) - E_3|\vec{p}_1|\cos\theta} \cdot |M_{fi}|^2$$

with $E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta + m_4^2}$

Summary of the summary:

★Have now dealt with kinematics of particle decays and cross sections
★The fundamental particle physics is in the matrix element
★The above equations are the basis for all calculations that follow

from M. A. Thomson lectures at Cambridge University in 2011