Angular momentum and decay distributions in high energy physics: an introduction and use cases for the LHC

- Basics of dilepton decay distributions. Examples: quarkonium and vector bosons
- A general demonstration of an old and surprising “perturbative-QCD” relation, using only rotation invariance
- Model-independent spin characterization of the Higgs-like di-photon resonance
Why should we study particle polarizations?

• test of perturbative QCD [$Z$ and $W$ decay distributions]
• constrain universal quantities [$\sin\theta_W$ and/or proton PDFs from $Z/W/\gamma^*$ decays]
• accelerate discovery of new particles or characterize them [Higgs, $Z'$, anomalous $Z+\gamma$, graviton, ...]
• understand the formation of hadrons (non-perturbative QCD)
We want to know the relative contributions of the following processes, differing for how/when the observed $Q$-$Q$bar bound state acquires its quantum numbers.

- **Colour-singlet processes:** quarkonia produced directly as observable *colour-neutral* $Q$-$Q$bar pairs

  purely perturbative

- **Colour-octet processes:** quarkonia are produced through *coloured* $Q$-$Q$bar pairs of *any possible* quantum numbers

  perturbative $\otimes$ non-perturbative

Transition to the observable state.

*Quantum numbers change!* $J$ can change! $\rightarrow$ polarization!
Polarization of vector particles

\( J = 1 \rightarrow \) three \( J_z \) eigenstates \( |1, +1\rangle, |1, 0\rangle, |1, -1\rangle \) wrt a certain \( z \)

Measure polarization = measure (average) angular momentum composition

Method: study the angular distribution of the particle decay in its rest frame

The decay into a fermion-antifermion pair is an especially clean case to be studied

The shape of the observable angular distribution is determined by a few basic principles:

1) “helicity conservation”

2) rotational covariance of angular momentum eigenstates

\[
\frac{1}{2} |1, +1\rangle + \frac{1}{2} |1, -1\rangle - \frac{1}{\sqrt{2}} |1, 0\rangle
\]

3) parity properties
EW and strong forces preserve the *chirality* (L/R) of fermions. In the relativistic (massless) limit, *chirality* = *helicity* = spin-momentum alignment → the *fermion spin* never flips in the coupling to gauge bosons:
example: dilepton decay of $J/\psi$

$J/\psi$ angular momentum component along the polarization axis $z$:

$$M_{J/\psi} = -1, 0, +1$$

determined by production mechanism

The two leptons can only have total angular momentum component $M'_{\ell^+\ell^-} = +1$ or $-1$ along their common direction $z'$

0 is forbidden
2: rotation of angular momentum eigenstates

change of quantization frame:
\[ R(\vartheta, \varphi): \quad z \rightarrow z' \]
\[ y \rightarrow y' \]
\[ x \rightarrow x' \]

\[ | J, M' \rangle = \sum_{M=-J}^{+J} D_{MM'}^{J}(\vartheta, \varphi) | J, M \rangle \]

Wigner D-matrices

Example:
\[ \frac{1}{\sqrt{2}} | 1, +1 \rangle + \frac{1}{\sqrt{2}} | 1, -1 \rangle - \frac{1}{\sqrt{2}} | 1, 0 \rangle \]
example: $M = 0$

\[ J/\psi (M_{J/\psi} = 0) \rightarrow e^+e^- (M'_{e^+e^-} = +1) \]

\[ |1, +1\rangle = D^{1}_{-1, +1}(\vartheta, \phi) |1, -1\rangle + D^{1}_{0, +1}(\vartheta, \phi) |1, 0\rangle + D^{1}_{+1, +1}(\vartheta, \phi) |1, +1\rangle \]

→ the $J_{z'}$ eigenstate $|1, +1\rangle$ “contains” the $J_{z}$ eigenstate $|1, 0\rangle$
with component amplitude $D_{0, +1}^{1}(\vartheta, \phi)$

→ the decay distribution is

\[ |\langle 1, +1 \mid \mathcal{O} \mid 1, 0 \rangle|^2 \propto |D_{0, +1}^{1}(\vartheta, \phi)|^2 = \frac{1}{2} \left( 1 - \cos^2\vartheta \right) \]

\[ e^+e^- \leftarrow J/\psi \]
3: parity

$|1, -1\rangle$ and $|1, +1\rangle$ distributions are mirror reflections of one another

$P(-1) > P(+1)$

$P(-1) = P(+1)$

$P(-1) < P(+1)$

$\frac{dN}{d\Omega} \propto |D_{-1,+1}^{1}(\theta,\phi)|^2 \propto 1 + \cos^2\theta - 2\cos\theta$

$\frac{dN}{d\Omega} \propto |D_{+1,+1}^{1*}(\theta,\phi)|^2 \propto 1 + \cos^2\theta + 2\cos\theta$

Are they equally probable?

$\frac{dN}{d\Omega} \propto 1 + \cos^2\theta + 2[ P(+1) - P(-1) ] \cos\theta$
3: parity

\[ |1, -1\rangle \] and \[ |1, +1\rangle \] distributions are mirror reflections of one another.

Decay distribution of \[ |1, 0\rangle \] state is always parity-symmetric:

\[
\frac{dN}{d\Omega} \propto |D_{0, +1}^{1*}(\vartheta, \varphi)|^2 \propto 1 - \cos^2 \vartheta
\]
“Transverse” and “longitudinal”

Transverse polarization, like for real photons. The word refers to the alignment of the field vector, not to the spin alignment!

\[ |J/\psi\rangle = |1, +1\rangle \]

or \[ |1, -1\rangle \]

\[ \frac{dN}{d\Omega} \propto 1 + \cos^2 \theta \]

(parity-conserving case)

“Longitudinal” polarization

\[ |J/\psi\rangle = |1, 0\rangle \]

\[ \frac{dN}{d\Omega} \propto 1 - \cos^2 \theta \]
Why “photon-like” polarizations are common

We can apply helicity conservation at the production vertex to predict that all vector states produced in fermion-antifermion annihilations ($q\bar{q}$ or $e^+e^-$) at Born level have transverse polarization.

\[ q\bar{q} \text{ rest frame} = V \text{ rest frame} \]

\[ V = \gamma^*, Z, W \]

\[ |V\rangle = |1, +1\rangle \]

\[ (|1, -1\rangle) \]

The “natural” polarization axis in this case is the relative direction of the colliding fermions (Collins-Soper axis).

Drell-Yan is a paradigmatic case
But not the only one
The most general distribution

\[ \frac{dN}{d\Omega} \propto 1 + \lambda_\theta \cos^2 \theta + \lambda_\phi \sin^2 \theta \cos 2\phi + \lambda_{\theta\phi} \sin 2\theta \cos \phi + 2A_\theta \cos \theta + 2A_\phi \sin \theta \cos \phi \]

average polar anisotropy
average azimuthal anisotropy

chosen polarization axis

production plane
particle rest frame

parity violating

correlation polar - azimuthal
Polarization frames

**Helicity axis (HX):** quarkonium momentum direction

**Gottfried-Jackson axis (GJ):** direction of one or the other beam

**Collins-Soper axis (CS):** average of the two beam directions

**Perpendicular helicity axis (PX):** perpendicular to CS

production plane
Frame dependence

For $|p_L| << p_T$, the CS and HX frames differ by a rotation of $90^\circ$

$\psi = 2 \cos dN \frac{d\theta}{d\Omega} - \propto \Omega$

$dN \frac{d\Omega}{d\Omega} \propto 1 - \cos^2 \theta$

longitudinal

$|\psi\rangle = |0\rangle$

(pure state)

$\psi = 2 \cos dN \frac{d\theta}{d\Omega} - \propto \Omega$

$dN \frac{d\Omega}{d\Omega} \propto 1 + \cos^2 \theta - \sin^2 \theta \cos 2\varphi$

“transverse”

$|\psi\rangle = \frac{1}{\sqrt{2}} |+1\rangle - \frac{1}{\sqrt{2}} |-1\rangle$

(mixed state)
All reference frames are equal…
but some are more equal than others

What do different detectors measure with *arbitrary* frame choices?

Gedankenszenario:
- **dileptons are fully transversely polarized in the CS frame**
- the decay distribution is measured at the $\Upsilon(1S)$ mass by 6 detectors with different **dilepton acceptances**:

| Detector       | $|y| < \text{value}$ |
|----------------|---------------------|
| CDF            | $0.6$               |
| D0             | $1.8$               |
| ATLAS & CMS    | $2.5$               |
| ALICE $e^+e^-$ | $0.9$               |
| ALICE $\mu^+\mu^-$ | $2.5 < y < 4$ |
| LHCb           | $2 < y < 4.5$       |
The lucky frame choice

(CS in this case)

\[ \frac{dN}{d\Omega} \propto 1 + \cos^2 \theta \]

\[ \lambda_{y,\mu} \]

\[ \lambda_{y,\phi} \]

ALICE $\mu^+\mu^-$ / LHCb
ATLAS / CMS
D0
ALICE $e^+e^-$
CDF
Less lucky choice
(HX in this case)

\[ \lambda_\theta = +0.65 \]
\[ \lambda_\theta = -0.10 \]

ALICE $\mu^+\mu^- / LHCb$
ATLAS / CMS
D0
ALICE $e^+e^-$
CDF

artificial (experiment-dependent!)
kinematic behaviour
\[ \rightarrow \text{measure in more than one frame!} \]
Frames for Drell-Yan, Z and W polarizations

- polarization is always fully transverse... $V = \gamma^*, Z, W$

Due to helicity conservation at the $q-\bar{q}-V$ ($q-q^*-V$) vertex, $J_z = \pm 1$ along the $q-\bar{q}$ ($q-q^*$) scattering direction $z$

- ...but with respect to a subprocess-dependent quantization axis

$O(\alpha_s^0)$

Due to helicity conservation at the $q-\bar{q}-V$ ($q-q^*-V$) vertex, $J_z = \pm 1$ along the $q-\bar{q}$ ($q-q^*$) scattering direction $z$

$z = \text{relative dir. of incoming } q \text{ and } q\bar{q}$

($\sim$ Collins-Soper frame)

important only up to $p_T = \mathcal{O}(\text{parton } k_T)$

$O(\alpha_s^1)$

QCD corrections

$z = \text{dir. of one incoming quark}$

($\sim$ Gottfried-Jackson frame)

$z = \text{dir. of outgoing } q$

($= \text{parton-cms-helicity} \approx \text{lab-cms-helicity}$)
“Optimal” frames for Drell-Yan, Z and W polarizations

Different subprocesses have different “natural” quantization axes

For \textbf{\textit{s-channel processes}} the \textbf{natural axis} is
the direction of the outgoing quark
(= direction of dilepton momentum)

→ optimal frame (= maximizing polar anisotropy): HX

(example: Z
\text{y} = +0.5

neglecting parton-parton-cms vs proton-proton-cms difference!)

\text{HX, CS, PX, GJ1 (negative beam), GJ2 (positive beam)}
“Optimal” frames for Drell-Yan, Z and W polarizations

Different subprocesses have different “natural” quantization axes

For \( t\) - and \( u\)-channel processes the natural axis is the direction of either one or the other incoming parton (~ “Gottfried-Jackson” axes)

\[ \rightarrow \text{optimal frame: geometrical average of GJ1 and GJ2 axes} = \text{CS (} p_T < M \text{) and PX (} p_T > M \text{)} \]

\[ \text{example: } Z \]

\[ y = +0.5 \]

\[ HX \]

\[ CS \]

\[ PX \]

\[ GJ1 = GJ2 \]
A complementary approach: frame-independent polarization

The *shape* of the distribution is (obviously) frame-invariant (= invariant by rotation)

→ it can be characterized by frame-independent parameters:

\[
\tilde{\lambda} = \frac{\lambda_{\theta} + 3\lambda_{\phi}}{1 - \lambda_{\phi}}, \quad \lambda^* = \frac{\lambda_{\theta} - 3\lambda^*}{1 + \Lambda^*}, \quad \Lambda^* = \frac{1}{4}\left\{\lambda_{\theta} - \lambda_{\phi} \pm \sqrt{(\lambda_{\theta} - \lambda_{\phi})^2 + 4\lambda_{\theta}\lambda_{\phi}}\right\}
\]

\[
\tilde{A} = \frac{\sqrt{A_{\theta}^2 + A_{\phi}^2}}{3 + \lambda_{\theta}}
\]

rotations in the production plane
Reduces acceptance dependence

Gedankenscenario: vector state produced in this subprocess admixture:
- 60% processes with natural transverse polarization in the CS frame
- 40% processes with natural transverse polarization in the HX frame

\[ M = 10 \text{ GeV/c}^2 \]

- Immune to “extrinsic” kinematic dependencies
  → less acceptance-dependent
  → facilitates comparisons
- useful as closure test

CDF \[ |y| < 0.6 \]
D0 \[ |y| < 1.8 \]
ATLAS/CMS \[ |y| < 2.5 \]
ALICE e^+e^- \[ 2.5 < y < 4 \]
ALICE \( \mu^+\mu^- \) \[ 2 < y < 4.5 \]
Physical meaning: Drell-Yan, Z and W polarizations

- Polarization is always fully transverse...
  
  Due to helicity conservation at the $q\bar{q}-V$ ($q-q^*-V$) vertex, $J_z = \pm 1$ along the $q\bar{q}$ ($q-q^*$) scattering direction $z$

- ...but with respect to a subprocess-dependent quantization axis

In all these cases the $q-q-V$ lines are in the production plane (planar processes); The CS, GJ, pp-HX and qg-HX axes only differ by a rotation in the production plane
Example: $Z/\gamma^*/W$ polarization (CS frame) as a function of contribution of LO QCD corrections:

**Case 1:** dominating $q\bar{q}$ QCD corrections

- $\lambda_\theta$ vs. $\tilde{\lambda}$
- $\lambda_\theta$ is constant, maximal and independent of process admixture.

**Case 2:** dominating $q-g$ QCD corrections

- $\lambda_\theta$ vs. $\tilde{\lambda}$
- $\lambda_\theta$ is far from being maximal
- $\lambda_\theta$ depends on process admixture
  - need pQCD and PDFs

$\lambda_\theta$ is constant, maximal and independent of process admixture.

**W by CDF&D0**

- "unpolarized"?
- No, $\tilde{\lambda} = +1$!

**Mass dependent!**
Example: $Z/\gamma^*W$ polarization (CS frame) as a function of contribution of LO QCD corrections:

Case 1: dominating $q\bar{q}$ QCD corrections

- \( \tilde{\lambda} = +1 \)
- \( f_{QCD} \)
- \( M = 150 \text{ GeV}/c^2 \)
- \( p_T = 50 \text{ GeV}/c \)
- \( p_T = 200 \text{ GeV}/c \)
- \( \lambda_{cs} \)
- \( \lambda_{cs} \) vs \( f_{QCD} \)

Case 2: dominating $qg$ QCD corrections

- \( \tilde{\lambda} = +1 \)
- \( f_{QCD} \)
- \( M = 80 \text{ GeV}/c^2 \)
- \( p_T = 50 \text{ GeV}/c \)
- \( p_T = 200 \text{ GeV}/c \)
- \( \lambda_{cs} \)
- \( \lambda_{cs} \) vs \( f_{QCD} \)

On the other hand, \( \tilde{\lambda} \) forgets about the direction of the quantization axis. This information is crucial if we want to disentangle the $qg$ contribution, the only one resulting in a rapidity-dependent \( \lambda_\theta \).

Measuring \( \lambda_\theta \)(CS) as a function of rapidity gives information on the gluon content of the proton.
The Lam-Tung relation

A fundamental result of the theory of vector-boson polarizations (Drell-Yan, directly produced $Z$ and $W$) is that, at leading order in perturbative QCD,

$$\lambda_g + 4\lambda_\phi = 1$$

independently of the polarization frame

Lam-Tung relation, PRD 18, 2447 (1978)

This identity was considered as a surprising result of cancellations in the calculations.

Today we know that it is only a special case of general frame-independent polarization relations, corresponding to a transverse intrinsic polarization:

$$\tilde{\lambda} = \frac{\lambda_g + 3\lambda_\phi}{1 - \lambda_\phi} = +1 \quad \Rightarrow \quad \lambda_g + 4\lambda_\phi = 1$$

It is, therefore, simply a consequence of
1) rotational invariance
2) properties of the quark-photon/$Z$/W coupling

Experimental tests of the LT relation are not tests of QCD!
Beyond the Lam-Tung relation

Even when the Lam-Tung relation is violated, \( \tilde{\lambda} \) can always be defined and is always frame-independent.

\[ \tilde{\lambda} = +1 \rightarrow \text{Lam-Tung. New interpretation: only vector boson – quark – quark couplings (in planar processes) → automatically verified in DY at QED & LO QCD levels and in several higher-order QCD contributions} \]

\[ \tilde{\lambda} = +1 - \mathcal{O}(0.1) \rightarrow \text{vector-boson – quark – quark couplings in non-planar processes (higher-order contributions)} \]

\[ \tilde{\lambda} \ll +1 \rightarrow \text{contribution of different/new couplings or processes} \]
\[ \tilde{\lambda} > +1 \rightarrow \text{(e.g.: Z from Higgs, W from top, triple ZZ\gamma coupling, higher-twist effects in DY production, etc...)} \]
Spin characterization of the Higgs-like di-photon resonance

Usual approach to “determine” the J of T: comparison between \( J=0 \) hypothesis and \textbf{ONE} alternative hypothesis. Example:

- SM Higgs boson
  - \( J=0 \)
  - \( M_{gg} = 0 \)
  - \( M_{\gamma\gamma} = 0 \)

- graviton with minimal-couplings to SM bosons (≈ “boson helicity conservation”)
  - \( J=2 \)
  - \( M_{gg} = \pm 1, \pm 2 \)
  - \( M_{\gamma\gamma} = \pm 1, \pm 2 \)

Decay distribution calculated case-by-case
Spin characterization of the Higgs-like di-photon resonance

Usual approach to “determine” the J of T:
comparison between J=0 hypothesis and ONE alternative hypothesis. Example:

SM Higgs boson

graviton with minimal-couplings to SM bosons (~ “boson helicity conservation”)
Likelihood Ratio Approach

• **Method:**
  - measure distribution of the likelihood ratio between hypothesis A and hypothesis B
  \[ \mathcal{L}[B] / \mathcal{L}[A] \quad \mathcal{L} \propto \text{decay angular distribution} \]
  - here \( A = \text{SM Higgs} (J_A = 0) \), \( B = \text{a new-physics hypothesis} (J_B) \)

• **Ingredients** (for each set of A and B hypotheses):
  - the angular momentum quantum numbers \( J_A \) and \( J_B \)
  - the coupling properties of A and B to initial and final particles (gluons and photons)
  - calculations of the helicity amplitudes for the production and decay processes

• **Question** addressed:
  - is the observed resonance more likely to be particle A or particle B?

• The **answer**
  - may be given unhesitatingly, i.e. \( \mathcal{L}[A] \gg \mathcal{L}[B] \), even when neither A nor B coincide with the correct hypothesis
  - is never conclusive until the whole set of possible models for A and B is explored.

Do we know this set of models in a totally model-independent way? As a matter of fact, a very restricted set of “B” models is currently considered...
MPC approach

- **Method:**
  - measure the angular distribution

\[
\frac{dN}{d\Omega} \propto 1 + \lambda_2 \cos^2 \Theta + \lambda_4 \cos^4 \Theta + \lambda_6 \cos^6 \Theta + \ldots + \lambda_N \cos^N \Theta
\]

- **Ingredients:**
  - angular momentum conservation
  - initial gluons and final photons are transversely polarized
  - no hypothesis on J nor on couplings, no explicit calculations of helicity amplitudes

[J=1 hypothesis forbidden by Landau-Yang theorem]

The general physical parameter domains of the J=2, 3 and 4 cases are mutually exclusive!
**MPC approach**

**Method:**
- measure the angular distribution

\[
\frac{dN}{d\Omega} \propto 1 + \lambda_2 \cos^2 \vartheta + \lambda_4 \cos^4 \vartheta + \lambda_6 \cos^6 \vartheta + \ldots + \lambda_N \cos^N \vartheta
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[J=1 hypothesis forbidden by Landau-Yang theorem]

The general physical parameter domains of the J=2, 3 and 4 cases are mutually exclusive!

And do not include the origin (J=0)!
MPC approach

Method:

- measure the angular distribution

\[ \frac{dN}{d\Omega} \propto 1 + \lambda_2 \cos^2 \theta + \lambda_4 \cos^4 \theta + \lambda_6 \cos^6 \theta + \ldots + \lambda_N \cos^N \theta \]

Ingredients:

- angular momentum conservation
- Initial gluons and final photons are transversely polarized
- no hypothesis on J nor on couplings, no explicit calculations of helicity amplitudes

The \( \cos \theta \) distribution discriminates the spin univocally:

\( J = 0 \)  \hspace{1cm} \( J = 2 \)  \hspace{1cm} \( J = 3 \)  \hspace{1cm} \( J = 4 \)
MPC approach

• **Method:**
  - measure the angular distribution

\[
\frac{dN}{d\Omega} \propto 1 + \lambda_2 \cos^2 \theta + \lambda_4 \cos^4 \theta + \lambda_6 \cos^6 \theta + \ldots + \lambda_N \cos^N \theta
\]

• **Ingredients:**
  - angular momentum conservation
  - Initial gluons and final photons are transversely polarized
  - no hypothesis on J nor on couplings, no explicit calculations of helicity amplitudes

• This method directly addresses the **question:**
  - how much is J?

• The **answer**
  - is model-independent and can be compared to any theory
  - is always conclusive, if the measurement is sufficiently precise
LR vs MPC

The binary strategy of the LR approach aims at discriminating between two hypotheses:

- $J = 0$ (SM Higgs)
- $J = 2$ minimally-coupling graviton
The binary strategy of the LR approach aims at discriminating between two hypotheses:

From this point of view, this measurement would correspond to a J=0 characterization.
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From this point of view, this measurement would correspond to a J=0 characterization.

In the MPC approach it would exclude all models lying outside the ellipse, but it would not exclude J=2, nor J=3!
The binary strategy of the LR approach aims at discriminating between two hypotheses:

In the MPC approach, this measurement would represent an unequivocal spin-0 characterization.
Further reading


