



Jurgen Honig
Józef Spałek

**A Primer _____
to the Theory
of Critical _____
— Phenomena**

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Preface

This book is intended to be an introduction to the basic theoretical background for the general area of critical phenomena taking place near a continuous phase transition. These phenomena encompass a singular behavior of physical properties, that is, the behavior of measurable physical quantities that may take infinite values at the critical point (e.g., at the critical temperature). Although many books and review articles already deal with this topic, they are often written at an advanced level of mathematical sophistication. We therefore attempted to provide the necessary background information readers will need to study the advanced presentations. Particular emphasis is placed on developing the concept of the order parameter and on a systematic approach starting from Landau mean-field theory.

We generally have tried to keep the discussion at a reasonably elementary level. Some familiarity with the elements of quantum mechanics, Fourier series and transforms, and complex variables is assumed. However, we have not hesitated to include some specialty topics relevant to our aim; this more advanced material, marked with (*), can be omitted on a first reading. Also, to render the individual chapters more self-contained, we have repeated some presentations in different chapters. Although we have tried to be careful, nothing can ever be expected to be error-free. We will therefore appreciate being informed where we have been remiss.

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