INTRODUCTION TO BALITSKIJ - FADIN - KURAEV - LIPATOV EQUATION

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GOAL: REVIEW DERIVATION OF BFKL EQUATION

PLAN:

1. PRELIMINARIES
2. OCTET EXCHANGE IN QCD
3. GLUON EMISSION IN QCD
4. $2 \to 2+n$ GLUON AMPLITUDE
5. BFKL EQUATION
6. SOLUTIONS , PROPERTIES , APPLICATIONS

DGLAP IN DLA LIMIT – – GLUON EVOLUTION

$$\frac{\partial f_i(x,\mu^2)}{\partial \log \mu^2} = \sum_j \int_x^1 \frac{dx'}{x'} P_{ij}(x/x')\, f_j(x',\mu^2)$$

Small-x limit : $P_{gg}(z) = \frac{C_A}{z}$ $\qquad P_{qq} \sim \text{const} \qquad P_{qg} \sim \text{const}$

$$P_{gq} \sim \frac{C_F}{z}$$

For x-moments of f : $\frac{\partial}{\partial \log Q^2} \tilde{f}(j,Q^2) = \frac{\alpha_s(Q^2)}{2\pi} \gamma(j,Q^2) \tilde{f}(j,Q^2)$

Most singular piece for P_{gg} : $\gamma_{gg}(j) = \frac{2C_A}{j-1} + \ldots$

$$g(x,Q^2) \simeq \int_{c_j} \frac{dj}{2\pi i}\, x^{-j}\, \tilde{g}_0(j) \exp\left(\frac{2C_A\alpha_s}{j-1} \frac{1}{2\pi} \overbrace{\log \frac{Q^2}{Q_0^2}}^{t} \right)$$

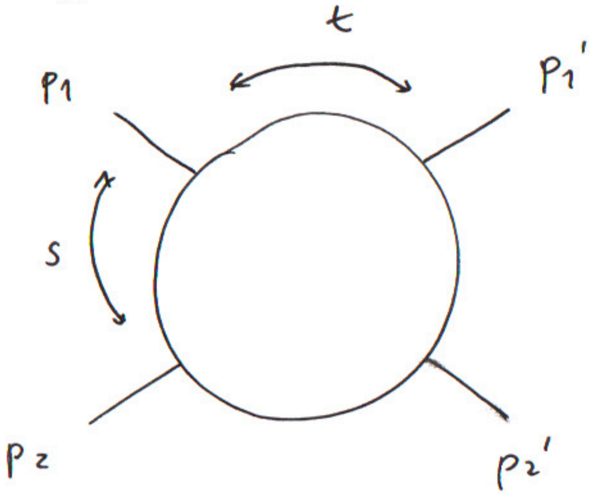
$$= \int \frac{dj}{2\pi i}\, \tilde{g}_0(j) \exp\left(\frac{\bar{\alpha}_s}{j-1} t + jY \right)$$

$t, Y \to \infty \Rightarrow$ saddle point at $j = 1 + \sqrt{\frac{\bar{\alpha}_s t}{Y}} \Rightarrow g(x,Q^2) \sim \frac{1}{x} e^{2\sqrt{\bar{\alpha}_s Y t}}$

→ Emergence of large $\log 1/x$ coefficients in perturbative expansion.

PRELIMINARIES – HIGH ENERGY SCATTERING

$2 \to 2$



$p_i^2 = m_i^2$

Mandelstam : $s = (p_1 + p_2)^2$

$t = (p_1' - p_1)^2$

$u = (p_2' - p_1)^2$

$$s + t + u = \sum_i m_i^2$$

Amplitude : $A(s,t)$

At high energies : $s \gg |t|, m_i^2$

$$u \simeq -s$$

$$A(s,t) \sim s$$

$$\sigma \sim \frac{|A(s,t)|^2}{s^2} \sim \text{const}$$

Optical theorem: $S = \mathbb{1} + iT$

$$S^\dagger S = \mathbb{1} \Rightarrow T - T^\dagger = i T^\dagger T$$

$$\langle f | \hat{T} | i \rangle - \langle f | \hat{T}^\dagger | i \rangle =$$

$$= i \sum_n \int d\phi_n \langle f | T^\dagger | n \rangle \langle n | \hat{T} | i \rangle$$

$$\Rightarrow \sigma \sim \mathrm{Disc}_s A(s,t) =$$

$$= A(s+i\varepsilon, t) - A(s-i\varepsilon, t) \Big|_{\varepsilon \to 0}$$

PRELIMINARIES: REGGE APPROACH

Partial wave expansion: $$A = \sum_{\ell=0}^{\infty} (2\ell+1)\, A_\ell\, P_\ell(\underset{\uparrow\ \cos\vartheta}{z_s})$$

In variables s,t: $$z_s = 1 + \frac{2t}{s-4m^2}$$

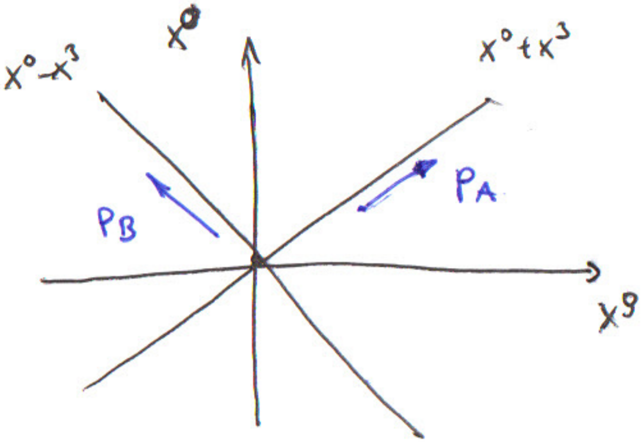
The amplitude $A(s,t)$ is analytic in complexified variable plane except of singularities of physical origin (poles at $s = m_i^2$, multiparticle states $\to$ cuts etc.)

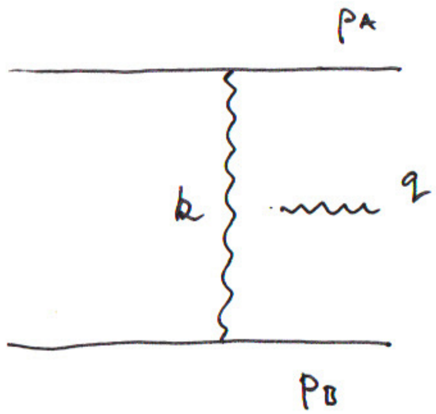
Analiticity + dispersion integrals $\to$ analitic continuation to complex values of angular momentum ℓ

Assuming $A(s,t)$ has poles in complex ℓ plane :

$$A(s,t) \sim \beta_{AB}(t)\, \xi^{t}_{\alpha(t)}\, s^{\alpha(t)} \qquad \beta_{AB}(t) = \beta_A(t)\,\beta_B(t)$$

PRELIMINARIES - HIGH ENERGY KINEMATICS





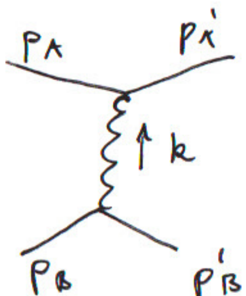
$p_i^{\pm} = p^0 \pm p^3$ $\qquad p_A \parallel n^+ \qquad p_B \parallel n^-$

$$\boxed{k = \alpha p_A \pm \beta p_B + k_T}$$

$$k^2 = 2\alpha\beta\, p_A p_B - \vec{k}_T^{\,2} = s\alpha\beta - \vec{k}^{\,2}$$

$$d^4 k = \frac{s}{2}\, d\alpha\, d\beta\, d^2\vec{k}_T$$

Dominance of the transverse degrees of freedom in exchanged particle propagators:



$\longrightarrow \quad p_A'^{\,2} = (p_A + k)^2 \Rightarrow k^2 + 2 p_A k = 0$

$\alpha\beta s - \vec{k}^{\,2} + \beta s = 0$

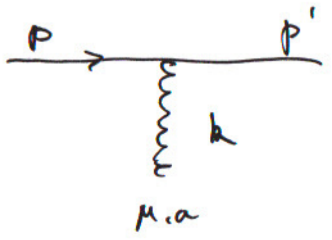
$\longrightarrow \quad \alpha\beta s - \vec{k}^{\,2} + \alpha s = 0$

$\alpha \simeq -\frac{\vec{k}^{\,2}}{s}$

$\beta \simeq \frac{\vec{k}^{\,2}}{s}$

$k^2 \simeq -\vec{k}^{\,2}$

PRELIMINARIES: QCD AT HIGH ENERGIES



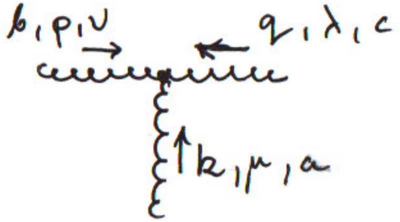
$g T^a \gamma^\mu$

Gordon identity:

$$\bar u(p')\gamma^\mu u(p) = \bar u(p')\left[\frac{(p'+p)^\mu}{2m} + \frac{i\underbrace{\sigma^{\mu\nu}}_{[\gamma^\mu,\gamma^\nu]}\underbrace{(p'_\nu - p_\nu)}_{k_\nu}}{2m}\right]u(p)$$

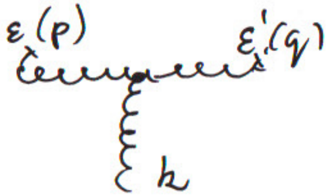
$$\bar u_{\lambda'}(p')\, u_\lambda(p) = 2m\,\delta_{\lambda\lambda'} + O(1/s)$$

$$\Rightarrow T^a\, \bar u_{\lambda'}(p')\,\gamma^\mu\, u_\lambda(p) \simeq 2p^\mu\, \delta_{\lambda\lambda'}\, T^a$$



$$\longrightarrow \quad igf^{abc}\,(2p^\mu)\,\delta_{\lambda\lambda'}$$

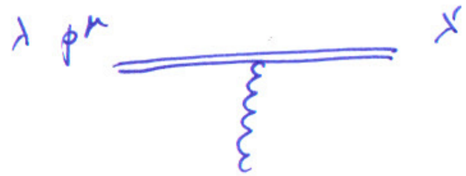
$$igf^{abc}\Big[\underline{(p-q)^\mu g^{\nu\lambda}} + (q-k)^\nu g^{\mu\lambda} + (k-p)^\lambda g^{\mu\nu}\Big]$$



$\varepsilon'_{\lambda'}(q)\cdot q = 0$

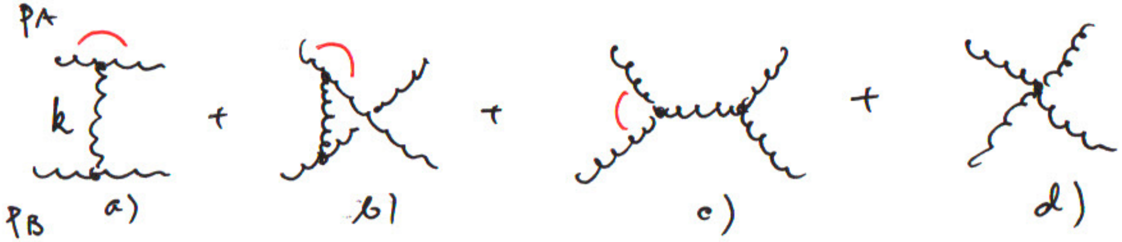
$\varepsilon_\lambda(p)\cdot p = 0$

In general



$\simeq \quad 2p^\mu\,\delta_{\lambda\lambda'}$ [COLOR]

OCTET EXCHANGE – TREE

AMPLITUDE: 

$= A$

$A_{(0)} = 0$ $\qquad A^{(a)} \sim \dfrac{p_A^\mu \; p_B^\nu}{k^2} g_{\mu\nu} \sim \dfrac{s}{t}$

$A^{(c)}, A^{(a)}, A^{(d)} \sim \mathcal{O}(1), \mathcal{O}(\frac{1}{s})$

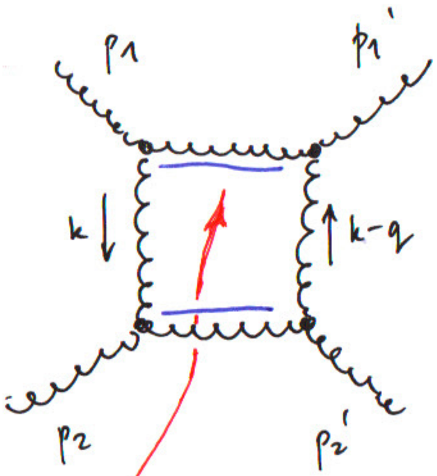
Dominance of t-channel exchange : $A_{(8)} \simeq g^2 \left(-\frac{2s}{t}\right) \delta_{\lambda_1\lambda_1'} \delta_{\lambda_2\lambda_2'}$
$\times$ [COLOR]

Universality : up to color charge the same amplitude of octet exchange for gg, gq, $q\bar{q}$ etc.

Prototype of Regge (high energy) factorisation :

$$A_{(8)} \sim \underbrace{(2p_A^\mu T^a{}_g)}_{\text{vertex } A} \underbrace{\frac{g_{\mu\nu}}{k^2}}_{\text{exchange}} \underbrace{(2p_B^\nu T^a{}_g)}_{\text{vertex } B}$$

OCTET EXCHANGE - ONE LOOP CORRECTION



$$i\mathcal{M}_1 \cong \int \frac{d^4k}{(2\pi)^4} \frac{\underbrace{(2p_{1\alpha}\, g_{\mu_1\nu_1})}\,\underbrace{(2p_{1\beta}\, g_{\nu_1\mu_3})}\,\overbrace{\underbrace{(2p_2)^\alpha}_{}}^{(g_{\nu_2\mu_4} 2p_2^\beta)} g_{\mu_2\nu_2}}{k^2 (p_1-k)^2 (p_2+k)^2 (k-q)^2},$$

$$\times\ \epsilon^{(\lambda_1)}_{\mu_1}\, \epsilon^{(\lambda_2)}_{\mu_2}\, \epsilon^{*(\lambda_3)}_{\mu_3}\, \epsilon^{*(\lambda_4)}_{\mu_4}\, g^4\, [\text{COLOR}]$$

$$= \delta_{\lambda_1\lambda_3}\, \delta_{\lambda_2\lambda_4}\, g^4\, (4 p_1\cdot p_2)^2\, \frac{s}{2} \int \frac{d\alpha\, d\beta\, d^2k_T}{(2\pi)^4} \frac{1}{[\cdots]}$$

$$\times\ [\text{COLOR}]$$

Propagators:

$$i\epsilon + (p_2+k)^2 = \alpha(1-\beta)s - \vec{k}^2 + i\epsilon$$

$$i\epsilon + (p_1-k)^2 = (1-\alpha)\beta s - \vec{k}^2 + i\epsilon$$

Pole in integration over β

Pole contribution: $\beta \approx 0,\quad (p_2+k)^2 \simeq \alpha s - \vec{k}^2 \quad \Rightarrow$

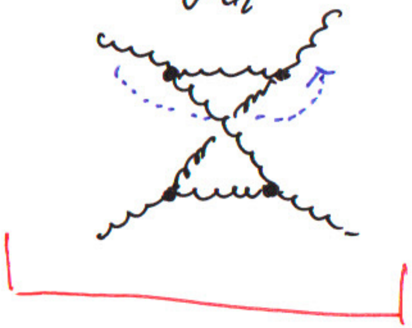
$$\Rightarrow \quad i\mathcal{M}_1 \cong \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\, g^4\, (2s)^2\, \frac{s}{2}\, \frac{(-2\pi i)}{s} \int \frac{\boxed{d\alpha}\, d^2k}{(2\pi)^4} \frac{[\text{COLOR}]}{\vec{k}^2\, \boxed{(\alpha s - \vec{k}^2)}\, (\vec{k}-\vec{q})^2}$$

$$\frac{\vec{k}^2}{s} \ll \alpha \ll 1 \quad \Rightarrow \quad \boxed{\int d\alpha \to \log s = Y} \quad \Rightarrow \quad i\mathcal{M}_1 \simeq \underbrace{\left[-g^2\, \frac{2s}{t}\, \delta_{\lambda_1\lambda_3}\delta_{\lambda_2\lambda_4}\right]}_{\text{tree}} \frac{g^2 t}{(2\pi)^3} \int \frac{d^{2+\varepsilon}k}{\vec{k}^2 (\vec{q}-\vec{k})^2}\, (Y$$

$$\times\ [\text{COLOR}]$$

OCTET EXCHANGE AT ONE LOOP CNTD.

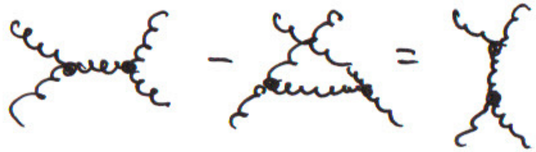
In Feynman gauge

$iM^{(1)} =$ $M_1^{(1)}$  + $M_2^{(1)}$ 

$M_2^{(1)}$ – crossed diagram $M_1^{(1)}$

Kinematical part : $s \to u \simeq -s$ $\quad (u+s+t = \sum_i m_i^2)$

$$\boxed{s \ln s} \to u \ln u = (-s)\ln(-s) \underset{LLA}{=} \boxed{-s \ln s}$$

Color factors: combine using Jacobi identity : 

$[f^{..}f^{..} + f^{..}f^{..} + f^{..}f^{..} = 0]$

Y

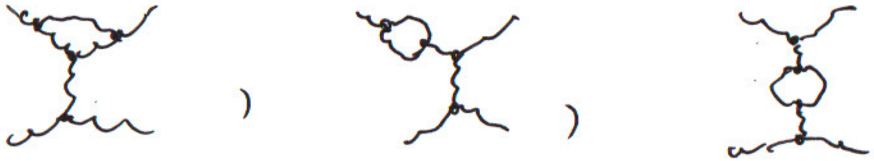
$$A^{NLO}_{(8)} = A^{LO}_{(8)} \left[\frac{g^2 N_c t}{2(2\pi)^3} \int \frac{d^{2+\epsilon}\vec{k}}{\vec{k}^2 (\vec{q}-\vec{k})^2} \right] \log s$$

$A^{NLO}_{(8)} = A^{LO}_{(8)} Y \omega(t)$; $\boxed{\omega(t) = \frac{N\alpha_s}{\pi} \frac{t}{4} \int \frac{d^{2+\epsilon}\vec{k}}{\vec{k}^2 (\vec{q}-\vec{k})^2}}$

Mind IR divergence and UV convergence!

TOWARDS RESUMMATION OF $\sum_n (\alpha_s y)^n$ — GAUGE DEPENDENCE

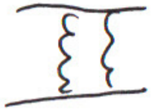
One loop amplitude (virtual NLO) is gauge independent but topologies of diagrams depend on gauge. So far Feynman gauge diagrams



do not contribute at high energy limit

In physical (e.g. Coulomb) gauge $\vec{\nabla}\cdot\vec{A}^a = 0 \iff \partial_\mu A^\mu_a - (N\cdot\partial)(N\cdot$

Propagator: $D^{ab}_{\mu\nu}(k) = \dfrac{\delta^{ab}}{k^2 + i\epsilon}\left[g_{\mu\nu} - \dfrac{(k\cdot N)(k_\mu N_\nu + k_\nu N_\mu) - k_\mu k_\nu}{(kN)^2 - k^2} \right]$

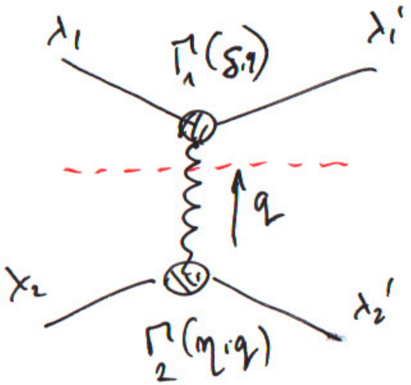
In this gauge the "multiple exchange" diagrams  + ...

do not generate logarithmic enhancement $\sim \alpha_s$ **logs**

Instead vertex corrections and external wave function corrections contribute to NLO octet exchange. Result - gauge independent.

RESUMMATION OF VIRTUAL CORRECTIONS

In Coulomb gauge virtual corrections are associated with vertices



$$A_8^{(LL)} = q^2 \, \frac{-2s}{t} \, \delta_{\lambda_1\lambda_1'} \, \delta_{\lambda_2\lambda_2'} \, \Gamma_1 \Gamma_2$$

$$\Gamma_1 = \Gamma_{1,0} \, \gamma_1(\xi, q) \qquad \xi = \log \frac{p_1^+}{\mu}$$

$$\Gamma_2 = \Gamma_{2,0} \, \gamma_2(\eta, q) \qquad \eta = \log \frac{p_2^-}{\mu}$$

Lorentz invariance $\Rightarrow$ arbitrary rapidity choice for factorisation of Γ_1 and Γ_2 $\Longrightarrow$

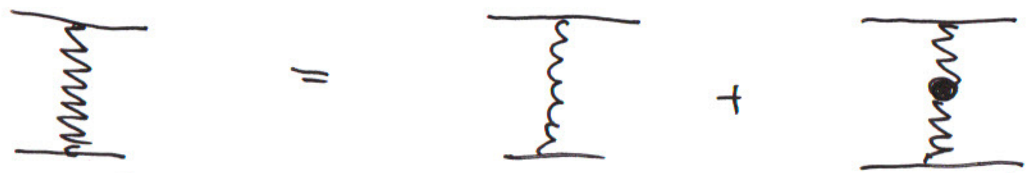
$$\gamma_2(\eta,q)\,\gamma_1(\xi,q) = \gamma(\xi+\eta, q) \Rightarrow \begin{cases} \gamma_1(\xi,q) = A(q)\, e^{\omega(q)\xi} \\ \gamma_2(\eta,q) = B(q)\, e^{\omega(q)\eta} \end{cases}$$

Tree level : $A(q) = B(q) = 1 \Rightarrow$ $$\boxed{A_8^{(LL)} \sim A_8^{(TREE)} \left(\frac{s}{\mu^2}\right)^{\omega(t)}}$$

Reggeized gluon trajectory

MORE ON REGGEIZED GLUON

- Gluon Regge trajectory is a gauge invariant object
- However it is divergent in the infrared
  (IR divergency cancellation is not supposed to work for virtual corrections separately)
- In alternative "bootstrap" picture gluon trajectory obeys a Bethe-Salpeter type equation :



→ Reggeized gluon is a composite

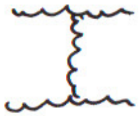
Reggeized = Tree + Reggeized at NLO (but $\mathcal{O}(\alpha_s)$)

with :
$$F(\omega, t) = \frac{1}{\mu^2} \int_{\mu^2}^{\infty} \frac{ds'}{s'} \left(\frac{s'}{\mu^2}\right)^{-\omega} A_8^{(LL)}(s', t)$$

$$F(\omega, t) = \frac{F_0(t)}{\omega - 1 - \omega(t)}$$ → simple Regge pole

REAL GLUON EMISSION: 2 → 3 PROCESS AND EFFECTIVE LIPATOV VERTEX

REAL GLUON EMISSIONS FROM THE DOMINANT DIAGRAM



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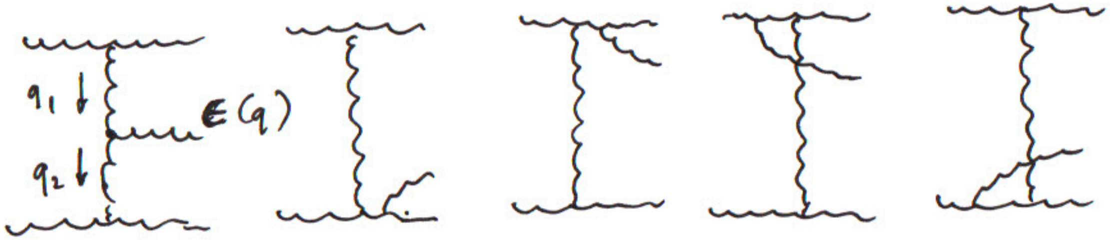
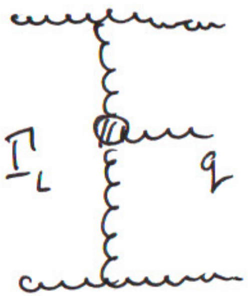


A lowest order diagram set for 2→3 with on-shell external particles: a gauge invariant amplitude

Result:

$$A_{2\to3} = -2s\, g\,\delta_{\lambda_1\lambda_1'}\,\frac{1}{q_1^2}\,\Gamma_L^{\mu}\,\varepsilon_\mu(q)\,\frac{1}{q_2^2}\,g\,\delta_{\lambda_2\lambda_2'}$$



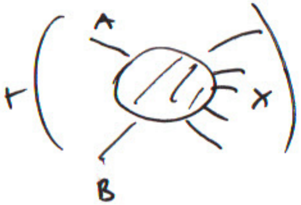
Effective Lipatov vertex

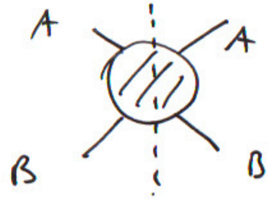
$$\Gamma_L^{\mu}(q_1,q_2) = \left(\alpha_1 + \frac{2q_1^2}{\beta_2 s}\right)p_1^{\mu} - \left(\beta_2 + \frac{2q_2^2}{\alpha_1 s}\right)p_2^{\mu} - (q_{1T}+q_{2T})^{\mu}$$

In high energy limit, multi-Regge kinematics

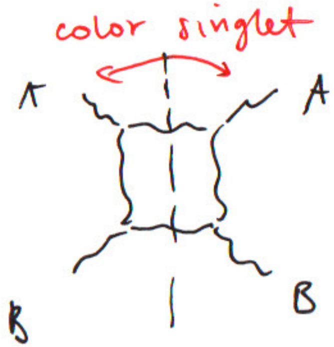
$q_\mu \Gamma_L^{\mu} = 0 \Rightarrow$ gauge invariance of the vertex

TOWARDS DESCRIPTION OF THE TOTAL CROSS SECTION

Optical theorem : σ_{tot} 

$\sim$ 2 Im 

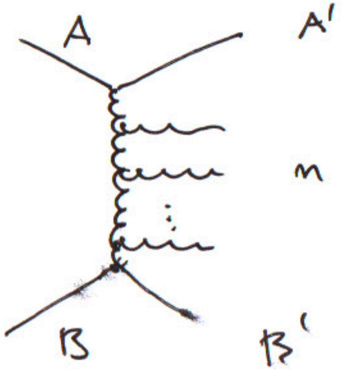
Lowest order in QCD at high energies



Virtual corrections : resummed in Regge trajectory of gluons

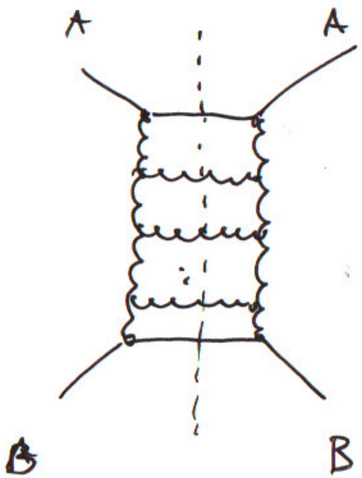
To be computed : real corrections, corresponding to gluon emissions :

Amplitude $2 \to 2+n$

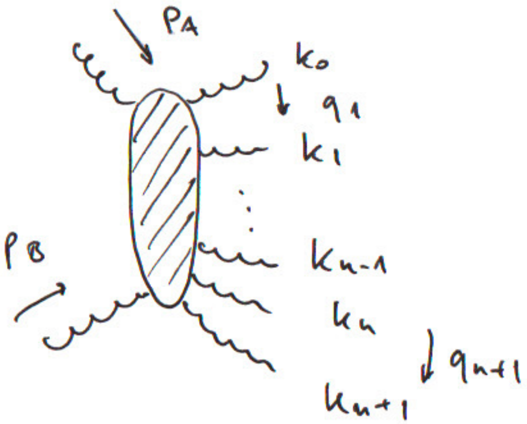


n gluons

and $\sigma_{tot} \sim \sum_n dPS_{2+n}$



PHASE SPACE FACTORISATION FOR $2 \to 2+n$



$$dPS_{n+2} = \prod_{i=0}^{n+1} \frac{d^4 k_i}{(2\pi)^3} \delta(k_i^2)\, \theta(k_i^0)\; \delta\left(\sum_{i=0}^{n+1} k_i - p_A - p_B\right)$$

$$q_i = \tilde{\alpha}_i p_A - \tilde{\beta}_i p_B + q_{iT}$$

Longitudinal part : $\frac{s}{2} \int d\alpha_i \, d\beta_i \; \delta\left((\alpha_i - \alpha_{i+1})(\beta_{i+1} - \beta_i)s - \vec{k}_i^2\right)$

To avoid multiple counting of identical particle configurations

ordering : $\alpha_0 > \alpha_1 > \ldots > \alpha_n > \alpha_{n+1}$

Then: $\int d\beta_i \;\; \delta(\;) \longrightarrow \int_{\alpha_{i-1}}^{\alpha_{i+1}} \frac{d\alpha_i}{\alpha_i} = \log \frac{\alpha_{i+1}}{\alpha_{i-1}}$

At LL accuracy $y_i = \log\left(\frac{1}{\alpha_i}\right)$ is rapidity of emitted gluon

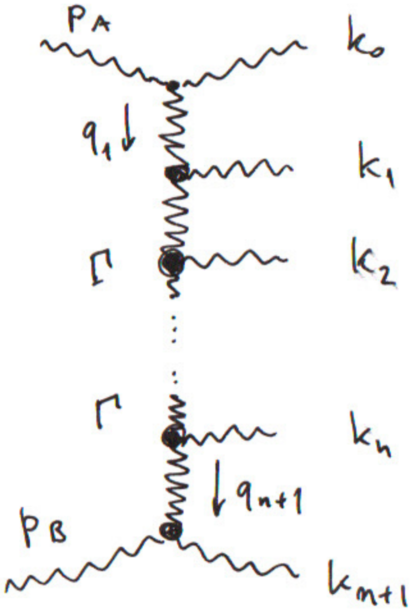
Maximal power of y-logarithms comes from multi-Regge kinematics

$$y_{n+1} \gg y_n \gg \ldots \gg y_2 \gg y_1$$

$$\boxed{dPS_{n+2} = \mathcal{P}_y \prod_i \frac{d^2 k_{Ti} \, dy_i}{2\,(2\pi)^3}}$$

TREE LEVEL AMPLITUDE $2 \to 2+n$

At LL accuracy a simple structure emerges:



$$\mathcal{A} = (-2s)\, g\, \delta_{\lambda_A \lambda_0}$$

$$\frac{1}{q_1^2}\, g\, \Gamma_L^{\mu_1}(q_1, q_2)\, \epsilon^{(\lambda_1)*}_{\mu_1}$$

$$\cdots$$

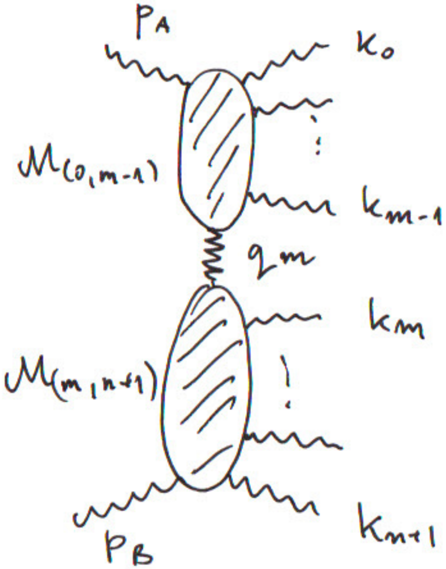
$$\frac{1}{q_n^2}\, g\, \Gamma_L^{\mu_n}(q_n, q_{n+1})\, \epsilon^{(\lambda_n)*}_{\mu_n}$$

$$\frac{1}{q_{n+1}^2}\, g\, \delta_{\lambda_{n+1} \lambda_B}$$

Why?

- There must be a pole in at least one q_m^2 – otherwise scaling requires $\frac{1}{s_{ij}}$ – supressed in multi-Regge kinematics
- Graphs are grouped into sets that have pole in q_i^2, the full amplitude is a sum over these groups
- Factorisation:

$$\mathcal{M}_{(0 \to n+1)} \sim \mathcal{M}^{\mu}_{(0,m-1)} \frac{D_{\mu\nu}}{q_m^2} \mathcal{M}^{\nu}_{(m,n+1)}$$



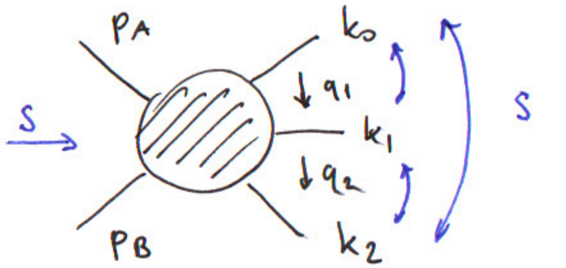
Then iteration of this procedure for new pieces of the amplitude

$\to$ till planar tree level diagram is reached

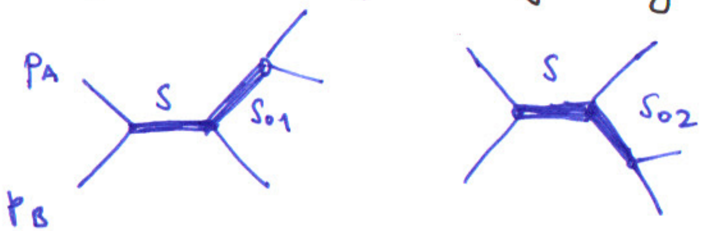
LOOP CORRECTIONS TO TREE $2\to 2+n$

Sketch of the idea:

- One employs analytic properties of multi-channel amplitudes and Regge trajectory of gluon
- Example $2\to 3$



Only non-overlaping singularities allowed by analyticity



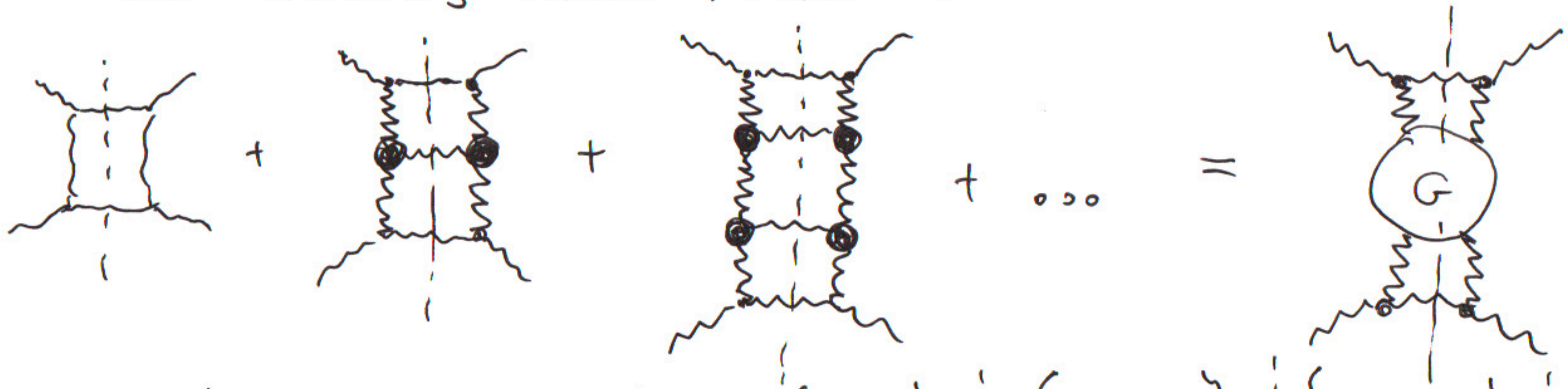
- The singularities in s_{ij} correspond to Reggeized gluon exchanges
- Analiticity $\to$ "signature factors" connected with Reggeons
- LL expansion of signature factors

$$A_{2\to 3} = -2s\,\Gamma_A(q_1)\,\frac{s_{01}^{\omega(q_1)}}{q_1^2}\,\Gamma_L(q_1,q_2)\,\frac{s_{12}^{\omega(q_2)}}{q_2^2}\,\Gamma_B(q_2)$$

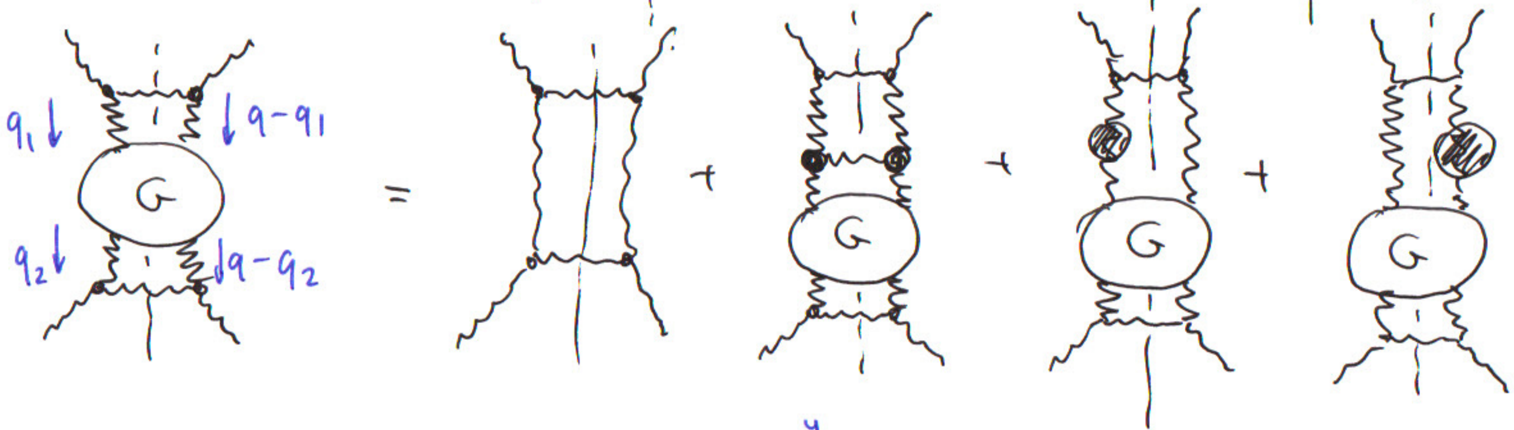
Intermediate gluon propagators "dress up" into Reggeons preserving the topology

THE BALITSKIJ - FADIN - KURAEV - LIPATOV EQUATION

$i\,\mathrm{Disc}_s A =$



Bethe - Salpeter type of resum- - mation



AT LL: $G(\vec{q}_1,\vec{q}_2;\vec{q},y) = G_0(\vec{q}_1,\vec{q}_2;\vec{q}) + \int_0^y dy' \int d^2\vec{q}\,' \, K(\cdots)\, G(\cdots)$

OR: $\dfrac{\partial G}{\partial y} = K \otimes G\,, \quad G\big|_{y=0} = G_0$

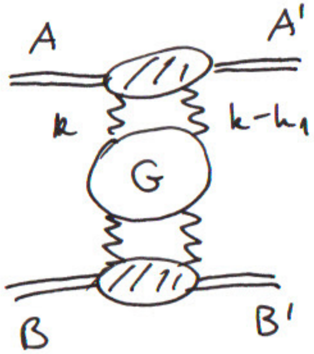
real emission term

$$K\otimes f = \frac{\alpha_s N_c}{\pi}\int \frac{d^2 q'}{2\pi}\left\{\left[\frac{q_2^2}{q'^2 (\vec{q}_2-\vec{q}\,')^2} + \frac{(\vec{q}-\vec{q}_2)^2}{(\vec{q}\,'-\vec{q})^2(\vec{q}_2-\vec{q}\,')^2} - \frac{q^2}{q'^2(\vec{q}\,'-\vec{q})^2}\right] f(\vec{q}\,',\vec{q},y) - \left[\frac{q_2^2}{q'^2(\vec{q}_2-\vec{q}\,')^2} + \frac{(\vec{q}_2-\vec{q}\,)^2}{q'^2+(\vec{q}_2-\vec{q}-\vec{q}\,')^2}\right] f(\vec{q}_2,\vec{q},y)\right.$$

virtual corrections

APPLICATION OF BFKL FORMALISM TO HIGH ENERGY SCATTERING

impact factors

$\mathcal{A} =$ 

$$= c\, \phi^{(0)}_{AA'} \otimes \underbrace{G(Y, \dots) \otimes \phi^{(0)}_{BB'}}_{=}$$

$$= \tilde{C} \int \frac{d^2 k_1}{k_1^2 (\bar{k}-\bar{k}_1)^2}\, \underbrace{\phi^{(0)}_{AA'}(\bar{k}_1, \bar{k})}_{\text{LO}}\, \underbrace{\Phi_{BB'}(\bar{k}_1, \bar{k}; Y)}_{\text{evolved}}$$

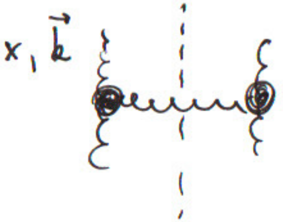
- For singlet-singlet scattering the amplitude is infra-red convergent. Gauge invariance ensures $\phi^{(0)}_{AA'}(\bar{k}_1, \bar{k}_2) \to 0$ for $k_1 \to 0$ or $k_2 \to 0$, also $\mathbf{K} \otimes \phi_{BB'} \to 0$ for $k_i \to 0$

- Integral kernel $\mathbf{K}$ is linear and symmetric $\to$ real eigenvalues

- Dominant behavior of amplitudes at large rapidities $\mathcal{A} \sim s^{1+\omega_0}$ $\Rightarrow$ cross sections enhanced at large energies

SOLUTIONS OF THE BFKL EQUATION (LL)

FORWARD CASE



$$f(x,\vec{k}) = f_0(x,\vec{k}) + \frac{N_c \alpha_s}{\pi} \int_x^1 \frac{dx'}{x'} \int_{k'=0}^{\infty} \frac{d^2k'}{(\vec{k}-\vec{k}')^2} \left[f(x',\vec{k}') \frac{\vec{k}^2}{\vec{k}'^2} - f(x',\vec{k}) \frac{\vec{k}^2}{\vec{k}'^2 + (\vec{k}-\vec{k}')^2} \right]$$

$$\int d\varphi' \rightarrow \qquad f(x,k^2) = f_0(x,k^2) + \bar{\alpha}_s \int_x^1 \frac{dx'}{x'} \int \frac{dk'^2}{k'^2} k^2 \left[\frac{f(x',k'^2) - f(x',k^2)}{|k'^2 - k^2|} + \frac{f(x',k^2)}{\sqrt{4k'^4 + k^4}} \right]$$

Scale invariance $\longrightarrow$ Eigen functions $(k'^2)^\gamma$

$$\mathbf{K} \otimes (k'^2)^\gamma = \chi(\gamma)(k^2)^\gamma$$

$$\bar{\alpha}_s \chi(\gamma) = \bar{\alpha}_s \left[2\psi(1) - \psi(\gamma) - \psi(1-\gamma) \right], \qquad \psi(\gamma) = \frac{\Gamma'(\gamma)}{\Gamma(\gamma)}$$

Double Mellin transform: $\tilde{F}(\gamma,\omega) = \int_0^\infty \frac{dk^2}{k^2} k^{2\gamma} \int_0^1 \frac{dx}{x} x^\omega f(x,k^2)$

In DLA: $\tilde{F}(\gamma,\omega) = \tilde{F}_0(\gamma,\omega) + \frac{\alpha_s}{\gamma\omega} \tilde{F}(\gamma,\omega)$

In general case $\quad \dots \quad \frac{\alpha_s}{\omega} \chi(\gamma) \tilde{F}(\gamma,\omega)$

CNTD.: SOLUTIONS OF THE BFKL EQUATIONS

$$F(\gamma, \omega) = \frac{\tilde{F}_0(\gamma, \omega)}{1 - \frac{\bar{\alpha}_s}{\omega} \chi(\gamma)} \quad ; \quad y = \log 1/x$$

$$f(x, k^2) = \int_{C_\gamma} \frac{d\gamma}{2\pi i} \int_{C_\omega} \frac{d\omega}{2\pi i} (k^2)^\gamma x^{-\omega} \frac{\tilde{F}_0(\gamma, \omega)}{1 - \frac{\bar{\alpha}_s}{\omega} \chi(\gamma)} \sim$$

$$\sim \int \frac{d\gamma}{2\pi i} (k^2)^\gamma e^{\bar{\alpha}_s \chi(\gamma) y} \hat{f}_0(\gamma)$$

DLA limit: $k^2 \to \infty$, $x \to 0$ $\quad \int \frac{d\gamma}{2\pi i} \exp\left[\gamma \log k^2 + \frac{\bar{\alpha}_s y}{\gamma}\right]$

saddle point: $\gamma_s = \sqrt{\bar{\alpha}_s \log 1/x / \log k^2} \Rightarrow \boxed{f(x, k^2) \sim e^{2\sqrt{\bar{\alpha}_s y \log k^2}}}$

Diffusion limit: k^2 - fixed, $x \to 0$

Saddle point at $\gamma = 1/2$, $\chi(1/2) = 4 \ln 2$ and

$$\boxed{f(x, k^2) \propto (k^2)^{1/2} x^{-\omega_0} \exp\left[-\frac{\log^2(k^2/k_0^2)}{2D \log 1/x}\right]}$$

BFKL vs DGLAP

DGLAP

Basis: hard factorisation, renormalisation group

Resummation: $\alpha_s^p \log^q \mu^2$

Parton: collinear: $g(x, \mu^2)$

Kinematics: need higher orders for accuracy

Higher orders: NNLO

Limitations: few, may be inaccurate in certain phase space regions

Theoretical problems - ?

standard tool

BFKL

- Regge factorisation, unitarity, analyticity
- $\alpha_s^p \log^q(1/x)$
- k_T- dependent $f(x, k_T^2)$
- keeps information on parton k_T at LL
- NLL
- Limitations: low-x region, gluon evolution
- Problems: diffusion into infra-red
  large higher order corrections
  unitarity
  multiple scattering ...

make it interesting!