DGLAP

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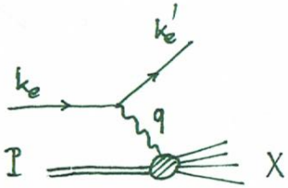
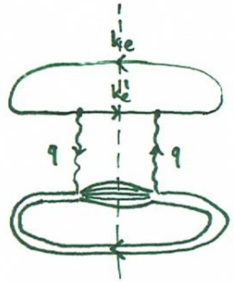


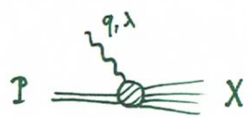
$$d\sigma^{ep\to eX} = \underbrace{\frac{1}{4(k_e\cdot P)}}_{\text{flux}} \; \underbrace{\frac{e^4}{q^4}}_{\text{photon propagator}} \; \overset{\text{lepton}}{L_e^{\mu\nu}} \; \underbrace{4\pi W_{\mu\nu}}_{\text{hadrons}} \; \underbrace{\frac{d^3k_e'}{2E_e'(2\pi)^3}}_{\text{leptonic phase-space}}$$



$$L_e^{\mu\nu} = \frac{1}{2}\mathrm{Tr}\left[\not{k}_e'\gamma^\mu \not{k}_e\gamma^\nu\right]$$

$$e^2\,\frac{g_{\nu\alpha}}{q^2}\,\frac{g_{\mu\rho}}{q^2}\,e^2$$

$$4\pi W^{\alpha\rho} = \frac{1}{2}\sum_\sigma\sum_n\prod_{i=1}^{n}\frac{d^3p_i}{2E_i(2\pi)^3}\langle P,\sigma|J^{\alpha\dagger}|p_i\rangle\langle p_i|J^\rho|P,\sigma\rangle(2\pi)^4\delta\Big(q+P-\sum_i p_i\Big)$$

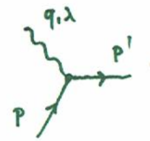


$$\sigma_\lambda^{\gamma^* p\to X} = \underbrace{\frac{1}{4(q\cdot P)}}_{\text{flux (convention)}}\; e^2\,\varepsilon_\lambda^\mu\,\varepsilon_\lambda^{*\nu}\;\underbrace{4\pi W_{\mu\nu}}_{\text{hadrons}}$$

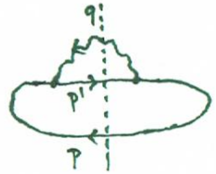
$$W_{\mu\nu} = F_1\left(-g_{\mu\nu}+\frac{q_\mu q_\nu}{q^2}\right) + \frac{F_2}{(P\cdot q)}\left(P_\mu-\frac{P\cdot q}{q^2}q_\mu\right)\left(P_\nu-\frac{P\cdot q}{q^2}q_\nu\right)$$

$q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$, parity inv., $(W^{\mu\nu})^* = W^{\nu\mu}$

Parton model:



$$d\hat{\sigma}_{\lambda}^{\gamma^* q \to q} = \frac{1}{4(p\cdot q)} \overline{\sum_{\sigma,\sigma'}} |\mathcal{M}_\lambda|^2 \underbrace{(2\pi)^4 \delta(p+q-p') \frac{d^4p'}{(2\pi)^4} 2\pi \delta(p'^2)}_{dP_s}$$



$$= e^2 e_i^2 \frac{1}{2} \mathrm{Tr}\left[\not{p}' \not{\varepsilon}_\lambda \not{p} \not{\varepsilon}^*_\lambda\right]$$

$$d\hat{\sigma}_{\lambda,i}^{\gamma^* q \to q} = \frac{1}{4(p\cdot q)} e^2 e_i^2 \, \varepsilon^\mu_\lambda \varepsilon^{*\nu}_\lambda \, \frac{1}{2} \mathrm{Tr}\left[\not{p}' \gamma_\mu \not{p} \gamma_\nu\right] dP_s$$

Factorization theorem: $d\sigma_\lambda^{\gamma^* p \to X} = \sum_i \int_0^1 d\xi \, f_i(\xi) \, d\hat{\sigma}_{\lambda,i}^{\gamma^* q \to q}$ $\qquad \xi = \frac{p\cdot q}{P\cdot q}$

$$d\sigma_\lambda^{\gamma^* p \to X} = \frac{1}{4(P\cdot q)} e^2 \varepsilon^\mu_\lambda \varepsilon^{*\nu}_\lambda \underbrace{\sum_i e_i^2 \int_0^1 \frac{d\xi}{\xi} f_i(\xi) \frac{1}{2} \mathrm{Tr}\left[\not{p}' \gamma_\mu \not{p} \gamma_\nu\right] dP_s}_{4\pi W_{\mu\nu}}$$

Parton model: $$4\pi W_{\mu\nu} = \sum_i e_i^2 \int_0^1 \frac{d\xi}{\xi} f_i(\xi) \frac{1}{2} \mathrm{Tr}\left[(\not p + \not q)\gamma_\mu \not p \gamma_\nu\right] dP_s$$

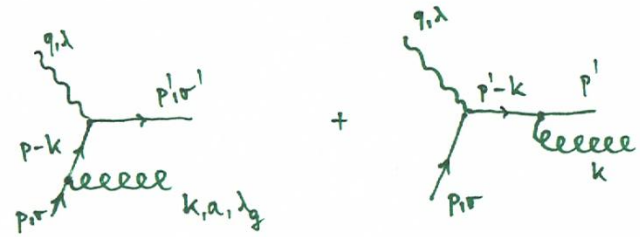
$p^2 \approx 0$ $$p^\mu p^\nu W_{\mu\nu} = 0 \quad \Rightarrow \quad F_2(x) = 2x F_1(x)$$ Callan - Gross relation

$$g^{\mu\nu} W_{\mu\nu} = -\frac{F_2}{x}$$

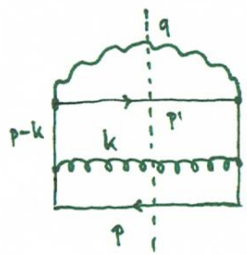
$$F_2(x) = \sum_i e_i^2 x f_i(x)$$

Longitudinel structure function: $F_L = F_2 - 2x F_1 = 0$

Real emission:



$$d\hat{\sigma}_{\lambda,i}^{\gamma^* q\to q} = \frac{1}{4(p\cdot q)}\,\overline{|\mathcal{M}_\lambda|^2}\,\underbrace{(2\pi)^4\delta(p+q-p'-k)\,\frac{d^4p'}{(2\pi)^4}\,2\pi\delta(p'^2)\,\frac{d^4k}{(2\pi)^4}\,2\pi\delta(k^2)}_{dPS_2}$$



$$= g^2e^2\,\frac{e_i^2 C_F}{(p-k)^4}\,\frac{1}{2}\sum_{\lambda_g}\mathrm{Tr}\left[\not p'\not\varepsilon\,(\not p-\not k)\not\varepsilon_g^*\,\not p\,\not\varepsilon_g(\not p-\not k)\not\varepsilon^*\right]$$

colour factor: $\displaystyle \frac{1}{N_c}\sum_{\alpha,\beta=1}^{N_c}\sum_{a=1}^{N_c^2-1}(t^a)_{\alpha\beta}(t^a)_{\beta\alpha} = \frac{1}{N_c}\sum_{a=1}^{N_c^2-1}\mathrm{Tr}(t^a t^a) = \frac{N_c^2-1}{2N_c} = C_F$

Divergences: $(p-k)^2 = -2p\cdot k = -2p_0k_0(1-\cos\theta)$

soft divergence $k_0\to 0$

colinear divergence $\theta\to 0$

Sudakov parametrization : $p, n,\quad p^2=n^2=0,\quad p\cdot n=1 \qquad p=(p,\vec{0}_\perp,p),\quad n=\left(\frac{1}{2p},\vec{0}_\perp,-\frac{1}{2p}\right)$

$$k=(1-z)p+\beta n+k_\perp\,,\qquad pk_\perp = nk_\perp = 0$$

$$k^2 = 2\beta(1-z)+k_\perp^2 = 0 \;\Rightarrow\; \underbrace{\beta=\frac{\vec{k}_\perp^2}{2(1-z)}}\,,\qquad (p-k)^2 = -\frac{\vec{k}_\perp^2}{1-z}$$

colinear divergence $k_\perp \to 0$

soft divergence $z\to 1$

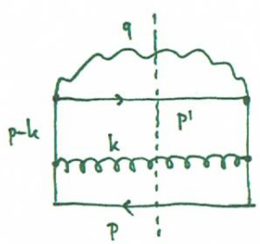
Gauge: $\sum_{\lambda_g}\varepsilon_\mu\varepsilon_\nu^* = -g_{\mu\nu}+\frac{n_\mu k_\nu + n_\nu k_\mu}{(n\cdot k)}$

$$\sum_{\lambda_g} Tr\left[\not p'\not\varepsilon(\not p-\not k)\underbrace{\not\varepsilon_g^*\not p\not\varepsilon_g}(\not p-\not k)\not\varepsilon^*\right]=\cdot\cdot$$

$$\sum_{\lambda_g}\not\varepsilon_g^*\not p\not\varepsilon_g = -\gamma_\mu\not p\gamma^\mu+\frac{1}{1-z}(\not n\not p\not k+\not k\not p\not n) = 2\not p+\frac{1}{1-z}\left[2(n\cdot p)\not k-2(n\cdot k)\not p+2(p\cdot k)\not n\right]=\frac{2}{1-z}\left[\not k+\beta\not n\right]$$

$$(\not p-\not k)(\not k+\beta\not n)(\not p-\not k) = 2\beta\left[(1+z^2)\not p+(1-z)\beta\not n - z\not k_\perp\right]$$

$$\cdot\cdot \approx \frac{4(1+z^2)}{1-z}\beta\, Tr\left[(\not p+\not q)\not\varepsilon\not p\not\varepsilon^*\right] + \text{subleading terms}$$



$$= g^2e^2e_i^2\, C_F\,\frac{1-z}{\vec{k}_\perp^2}\left(\frac{1+z^2}{1-z}\right) Tr\left[(\not p+\not q)\not\varepsilon\not p\not\varepsilon^*\right]$$

Phase space: $dPS_2 = (2\pi)^4\,\delta(p+q-p'-k)\,\frac{d^4p'}{(2\pi)^4}\,2\pi\,\delta(p'^2)\,\frac{d^4k}{(2\pi)^4}\,2\pi\,\delta(k^2) = 2\pi\,\delta((p+q-k)^2)\,\frac{d^4k}{(2\pi)^3}\,\delta(k^2) = \dots$

$$(p+q-k)^2 = (zp+q-\beta p'-k_\perp)^2 \approx (zp+q)^2 = 2z\,p\cdot q - Q^2 = 2z\xi\,P\cdot q - Q^2 = 2P\cdot q\,z\left(\xi-\frac{x}{z}\right)$$

$$\dots = \frac{2\pi}{2(P\cdot q)z}\,\delta\left(\xi-\frac{x}{z}\right)\frac{1}{16\pi^2}\,\frac{dz}{1-z}\,dk_\perp^2$$

Correction to cross-section: $d\sigma^{(1)}_\lambda = \frac{1}{4(q\cdot P)}\,e^2\,\varepsilon^\mu_\lambda\,\varepsilon^{*\nu}_\lambda\,\sum_i e_i^2 \int_0^1 \frac{d\xi}{\xi}\,f_i(\xi)\,\frac{\alpha_s}{2\pi}\,\underbrace{C_F\,\frac{1+z^2}{1-z}}_{P_{qq}(z)}\,\frac{1}{2}\mathrm{Tr}\left[(\not p+\not q)\gamma_\mu\not p\gamma_\nu\right]\frac{\pi}{(P\cdot q)}\,\delta\left(\xi-\frac{x}{z}\right)\frac{dz}{z}\,\frac{dk_\perp^2}{k_\perp^2}$

Correction to structure function: $F_2(x) = F_2^{(0)} + F_2^{(1)} = \sum_i e_i^2\, x\left[f_i(x) + \frac{\alpha_s}{2\pi}\int_{\mu_0^2}^{Q^2}\frac{dk_\perp^2}{k_\perp^2}\int_x^1\frac{dz}{z}\,P_{qq}(z)\,f_i\left(\frac{x}{z}\right)\right]$

$$\left.\begin{aligned} f_i(x,Q^2) &= f_i(x) + \frac{\alpha_s}{2\pi}\ln\frac{Q^2}{\mu_0^2}\int_x^1\frac{dz}{z}\,P_{qq}(z)\,f_i\left(\frac{x}{z}\right)\\ f_i(x,\mu^2) &= f_i(x) + \frac{\alpha_s}{2\pi}\ln\frac{\mu^2}{\mu_0^2}\int_x^1\frac{dz}{z}\,P_{qq}(z)\,f_i\left(\frac{x}{z}\right)\end{aligned}\right\}\quad f_i(x,Q^2) = f_i(x,\mu^2) + \frac{\alpha_s}{2\pi}\ln\frac{Q^2}{\mu^2}\int_x^1\frac{dz}{z}\,P_{qq}(z)\,f_i\left(\frac{x}{z}\right)$$

factorization scale

absorption of IR divergency into redefinition of f.

Evolution equation:

$$\frac{\partial f_i(x,Q^2)}{\partial \ln\mu^2} = 0 = \frac{\partial f_i(x,\mu^2)}{\partial \ln\mu^2} - \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z} P_{qq}(z)\, f_i\left(\frac{x}{z}\right)$$

$$\mu^2 \frac{\partial f_i(x,\mu^2)}{\partial \mu^2} = \frac{\alpha_s}{2\pi}\int_x^1 \frac{dz}{z} P_{qq}(z)\, f_i\left(\frac{x}{z},\mu^2\right) \qquad \text{DGLAP}$$

Dokshitzer - Gribov - Lipatov - Altarelli - Parisi

Integral form:

$$f_i(x,Q^2) = f_i(x,\mu^2) + \frac{\alpha_s}{2\pi}\int_{\mu^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2}\int_x^1 \frac{dz}{z} P_{qq}(z)\, f_i\left(\frac{x}{z},k_\perp^2\right)$$

Virtual corrections:

$$P_{qq}(z) \longrightarrow P_{qq}(z) = C_F\left[\frac{1+z^2}{1-z} - \delta(1-z)\int_0^1 dy\,\frac{1+y^2}{1-y}\right] \equiv C_F\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right]$$

$$\int_x^1 dz\,\frac{1}{(1-z)_+} f(z) = \int_x^1 dz\,\frac{f(z)-f(1)}{1-z} + f(1)\ln(1-x)$$

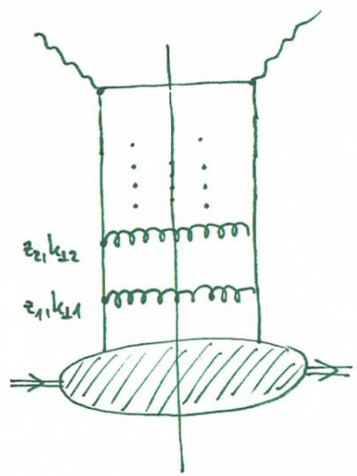
DGLAP is Fredholm type integral equation:

$$f(x,Q^2) = f_0(x,\mu^2) + \frac{\alpha_s}{2\pi}\int_{\mu^2}^{Q^2}\frac{dk_\perp^2}{k_\perp^2}\int_x^1\frac{dz}{z}P\left(\frac{x}{z}\right)f_0(z,k_\perp^2) + \frac{\alpha_s}{2\pi}\int_{\mu^2}^{Q^2}\frac{dk_\perp^2}{k_\perp^2}\int_x^1\frac{dz}{z}P\left(\frac{x}{z}\right)\frac{\alpha_s}{2\pi}\int_{\mu^2}^{k_\perp^2}\frac{dk_\perp'^2}{k_\perp'^2}\int_z^1\frac{dz'}{z'}P\left(\frac{x}{z'}\right)f_0(z',k_\perp'^2) + \dots$$

Structure of infinite series:

$$f(x,Q^2) = f_0(x) + \frac{\alpha_s}{2\pi}\ln\frac{Q^2}{\mu^2}\int_x^1\frac{dz}{z}P\left(\frac{x}{z}\right)f_0(z) + \frac{1}{2}\left(\frac{\alpha_s}{2\pi}\ln\frac{Q^2}{\mu^2}\right)^2\int_x^1\frac{dz}{z}P\left(\frac{x}{z}\right)\int_z^1\frac{dz'}{z'}P\left(\frac{x}{z'}\right)f_0(z') + \dots$$

LL approximation:



$$k_{\perp n}^2 > k_{\perp n-1}^2 > \dots > k_{\perp 1}^2$$

$$z_1 > z_2 > \dots > z_n$$

Splitting functions:

$$P_{qq} = C_F\left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z)\right]$$

$$P_{qG} = \cancel{N_f}\left[z^2+(1-z)^2\right]\frac{1}{2}$$

$$P_{Gq} = C_F\,\frac{1+(1-z)^2}{z}$$

$$P_{GG} = 2N_c\left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z)\right] + \frac{11N_c-2N_f}{6}\,\delta(1-z)$$

DGLAP:

$$\begin{cases} Q^2\dfrac{\partial}{\partial Q^2} f_i(x,Q^2) = \dfrac{\alpha_s}{2\pi}\displaystyle\int_x^1 \frac{dz}{z}\left[P_{qq}\left(\frac{x}{z}\right) f_i(z,Q^2) + P_{qG}\, g(z,Q^2)\right] \\[2ex] Q^2\dfrac{\partial}{\partial Q^2} g(x,Q^2) = \dfrac{\alpha_s}{2\pi}\displaystyle\int_x^1 \frac{dz}{z}\left[\sum_{i,\bar{i}} P_{Gq}\left(\frac{x}{z}\right)\left(f_i + f_{\bar{i}}\right) + P_{GG}\left(\frac{x}{z}\right) g(z,Q^2)\right] \end{cases}$$

Flavour non-singlet distribution function: $V_i = f_i - f_{\bar{i}}$

Flavour singlet distrib. function: $\Sigma = \sum_i (f_i + f_{\bar{i}})$

$$\begin{cases} Q^2 \frac{\partial}{\partial Q^2} V_i(x,Q^2) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} P_{qq}\left(\frac{x}{z}\right) V_i(z,Q^2) \\[2ex] Q^2 \frac{\partial}{\partial Q^2} \begin{pmatrix} \Sigma(x,Q^2) \\ g(x,Q^2) \end{pmatrix} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} \begin{pmatrix} P_{qq}\left(\frac{x}{z}\right) & 2N_f P_{qG}\left(\frac{x}{z}\right) \\ P_{Gq}\left(\frac{x}{z}\right) & P_{GG}\left(\frac{x}{z}\right) \end{pmatrix} \begin{pmatrix} \Sigma(z,Q^2) \\ g(z,Q^2) \end{pmatrix} \end{cases}$$

Solutions: $\tilde{f}(j) = \int_0^1 dx\, x^{j-1} f(x)$, $f(x) = \int_{c-i\infty}^{c+i\infty} \frac{dj}{2\pi i} x^{-j} \tilde{f}(j)$ Mellin transform

$$\int_0^1 dx\, x^{j-1} \int_x^1 \frac{dz}{z} P\left(\frac{x}{z}\right) f(z) = \int_0^1 \frac{dz}{z} f(z) \int_0^z dx\, x^{j-1} P\left(\frac{x}{z}\right) \underset{\frac{x}{z}=v}{\overset{\uparrow}{=}} \int_0^1 dz\, z^{j-1} f(z) \int_0^1 dv\, v^{j-1} P(v) = \tilde{f}(j)\tilde{P}(j)$$

$$\begin{cases} Q^2 \frac{\partial}{\partial Q^2} \tilde{V}_i(j,Q^2) = \frac{\alpha_s}{2\pi} \gamma_{qq}(j) \tilde{V}_i(j,Q^2) \\[2ex] Q^2 \frac{\partial}{\partial Q^2} \begin{pmatrix} \tilde{\Sigma}(j,Q^2) \\ \tilde{g}(j,Q^2) \end{pmatrix} = \frac{\alpha_s}{2\pi} \begin{pmatrix} \gamma_{qq} & \gamma_{qG} \\ \gamma_{Gq} & \gamma_{GG} \end{pmatrix} \begin{pmatrix} \tilde{\Sigma}(j,Q^2) \\ \tilde{g}(j,Q^2) \end{pmatrix} \end{cases}$$

γ's – anomalous dimensions

Anomalous dimensions: $\gamma_{qq}(j) = \int_0^1 dx\, x^{j-1} P_{qq}(x) = C_F\left[\frac{3}{2} + \int_0^1 dx\, \frac{x^{j-1}(1+x^2)}{(1-x)_+}\right] = C_F\left[\frac{3}{2} + \int_0^1 dx\, \frac{x^{j-1}+x^{j+1}-2}{1-x}\right] =$

$$= C_F\left[\frac{3}{2} - 2\gamma_E - \psi(j) - \psi(j+2)\right], \qquad \psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \text{ - digamma function}$$

$$\gamma_{qG} = \frac{1}{2}\left(\frac{1}{j} - \frac{2}{j+1} + \frac{2}{j+2}\right), \quad \gamma_{Gq} = C_F\left(\frac{2}{j-1} - \frac{2}{j} + \frac{1}{j+1}\right), \quad \gamma_{GG} = 2N_c\left[\frac{1}{j-1} - \frac{1}{j} + \frac{1}{j+1} - \frac{1}{j+2} - \gamma_E - \psi(j+1)\right] + \frac{11N_c - 2N_f}{6}$$

Solution non-singlet: $\tilde{f}(j,Q^2) = \tilde{f}(j,\mu_F^2)\exp\left\{\frac{\alpha_s}{2\pi}\gamma_{qq}(j)\ln\frac{Q^2}{\mu_F^2}\right\} = \tilde{f}(j,\mu_F^2)\left(\frac{Q^2}{\mu_F^2}\right)^{\frac{\alpha_s}{2\pi}\gamma_{qq}}$

Running coupling: $\alpha_s = \frac{b_0}{\ln\left(\frac{Q^2}{\Lambda^2}\right)}, \quad b_0 = \frac{12\pi}{11N_c - 2N_f}$

$$\alpha_s \int_{\mu_0^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2} \longrightarrow \int_{\mu_0^2}^{Q^2} \frac{dk_\perp^2}{k_\perp^2}\alpha_s(k_\perp^2) = b_0 \ln\left(\frac{\ln\frac{Q^2}{\Lambda^2}}{\ln\frac{\mu_0^2}{\Lambda^2}}\right)$$

$$\left(\ln\frac{Q^2}{\Lambda^2}\right)\frac{\partial f}{\partial \ln\frac{Q^2}{\Lambda^2}} = \frac{b_0}{2\pi}\int_x^1 \frac{dz}{z} P_{qq}(z)\, f\left(\frac{x}{z}\right) \qquad \text{after same steps}$$

Solution in Mellin space: $\tilde{f}(j,Q^2) = \tilde{f}(j,\mu_F^2)\left[\frac{\alpha_s(\mu_F^2)}{\alpha_s(Q^2)}\right]^{\frac{b_0}{2\pi}\gamma_{qq}}$

Small x limit: small x corresponds to the rightmost singularity in a complex j-plane : $\int_0^1 dx\, x^{j-1} \left(\frac{1}{x}\right)^{\alpha} = \frac{1}{j-\alpha}$

$$P_{Gq} \simeq \frac{2C_F}{j-1}, \quad \gamma_{GG} \simeq \frac{2N_c}{j-1} \qquad Q^2 \frac{\partial}{\partial Q^2} \begin{pmatrix} \Sigma \\ g \end{pmatrix} = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} \begin{pmatrix} 0 & 0 \\ \gamma_{Gq} & \gamma_{GG} \end{pmatrix} \begin{pmatrix} \Sigma \\ g \end{pmatrix}$$

$$Q^2 \frac{\partial}{\partial Q^2} g = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dz}{z} \left[\gamma_{Gq} \underbrace{\Sigma}_{\simeq 0} + \gamma_{GG}\, g \right]$$

Σ is small

with running coupling: $$\tilde{g}(j, Q^2) = \tilde{g}(j, \mu_F^2) \left[\frac{\alpha_s(\mu_F^2)}{\alpha_s(Q^2)} \right]^{\frac{b_0}{2\pi} \frac{2N_c}{j-1}}$$

Inverting Mellin transform: $$g(x, Q^2) = \int_{c-i\infty}^{c+i\infty} \frac{dj}{2\pi i}\, e^{j \ln\frac{1}{x} + \frac{a}{j-1}}\, \tilde{g}(j, \mu_F^2), \quad a = \frac{b_0 N_c}{\pi} \ln \frac{\alpha_s(\mu_F^2)}{\alpha_s(Q^2)}$$

saddle point: $$W'(j) = \left(j \ln\frac{1}{x} + \frac{a}{j-1} \right)' = \ln\frac{1}{x} - \frac{a}{(j-1)^2} = 0 \Rightarrow \quad j_s = 1 \pm \sqrt{\frac{a}{\ln 1/x}}$$

+ sign dominates

small x, large Q^2

$$W''(j) = \frac{2 (\ln 1/x)^{3/2}}{\sqrt{a}} > 0$$

$$g(x, Q^2) \cong \tilde{g}(j_s, \mu_F^2)\, e^{W(j_s)} \int_{c-i\infty}^{c+i\infty} \frac{dj}{2\pi i}\, e^{\frac{1}{2} W''(j_s)(j-j_s)^2}$$

changing the integration contour $j - j_s = i\nu$

$$g(x,Q^2) = \tilde{g}(j_s,\mu_F^2)\exp\left\{\ln\frac{1}{x} + 2\sqrt{a\ln\frac{1}{x}}\right\}\frac{1}{\sqrt{2\pi W''(j_s)}} \Rightarrow$$

$$x g(x,Q^2) = \frac{\tilde{g}(j_s,\mu_F^2)}{2\sqrt{\pi}}\left\{\frac{b_0 N_c}{\pi}\ln\frac{\alpha_s(\mu_F^2)}{\alpha_s(Q^2)}\right\}^{1/4}\left[\ln\frac{1}{x}\right]^{-3/4}\exp\left\{2\sqrt{\frac{b_0 N_c}{\pi}\ln\left(\frac{\ln Q^2/\Lambda^2}{\ln\mu_F^2/\Lambda^2}\right)\ln\frac{1}{x}}\right\}$$

DLA double log approximation

Factorization: $$F_2(x,Q^2) = x\sum_i\int_x^1\frac{dz}{z}\left[C_i\left(\frac{x}{z},\alpha_s\right)f_i(z,Q^2) + C_g\left(\frac{x}{z},\alpha_s\right)g(z,Q^2)\right] + \mathcal{O}\left(\frac{\Lambda^2}{Q^2}\right)$$

DGLAP: $$\begin{cases} Q^2\frac{\partial}{\partial Q^2}f_i(x,Q^2) = \int_x^1\frac{dz}{z}\left[P_{qq}\left(\frac{x}{z},\alpha_s\right)f_i(z,Q^2) + P_{qG}\left(\frac{x}{z},\alpha_s\right)g(z,Q^2)\right] \\ Q^2\frac{\partial}{\partial Q^2}g(x,Q^2) = \int_x^1\frac{dz}{z}\left[P_{Gq}\left(\frac{x}{z},\alpha_s\right)f_i(z,Q^2) + P_{GG}\left(\frac{x}{z},\alpha_s\right)g(z,Q^2)\right]\end{cases}$$

Coefficient functions $C_{q,g}$ and splitting function P_{ij} can be computed in perturbation theory.

$$C_i(x,\alpha_s) = C_i^{(0)}(x) + \frac{\alpha_s}{2\pi}C_i^{(1)}(x) + \dots$$

$$P_{ij}(x,\alpha_s) = \frac{\alpha_s}{2\pi}P_{ij}^{(0)}(x) + \dots.$$

Non-perturbative input: distribution functions at the factorization scale $f_i(x,\mu_F^2)$, $g(x,\mu_F^2)$

P_{ij}, C_i are known at NNLO

NLO

$$P^{(1)}_{\rm ps}(x) = 4\, C_F n_f \left(\frac{20}{9}\frac{1}{x} - 2 + 6x - 4\mathrm{H}_0 + x^2\left[\frac{8}{3}\mathrm{H}_0 - \frac{56}{9}\right] + (1+x)\left[5\mathrm{H}_0 - 2\mathrm{H}_{0,0}\right]\right)$$

$$P^{(1)}_{\rm qg}(x) = 4\, C_A n_f \left(\frac{20}{9}\frac{1}{x} - 2 + 25x - 2p_{\rm qg}(-x)\mathrm{H}_{-1,0} - 2p_{\rm qg}(x)\mathrm{H}_{1,1} + x^2\left[\frac{44}{3}\mathrm{H}_0 - \frac{218}{9}\right] \right.$$
$$\left. +4(1-x)\left[\mathrm{H}_{0,0} - 2\mathrm{H}_0 + x\mathrm{H}_1\right] - 4\zeta_2 x - 6\mathrm{H}_{0,0} + 9\mathrm{H}_0\right) + 4\, C_F n_f \left(2p_{\rm qg}(x)\left[\mathrm{H}_{1,0} + \mathrm{H}_{1,1} + \mathrm{H}_2 \right.\right.$$
$$\left.\left. -\zeta_2\right] + 4x^2\left[\mathrm{H}_0 + \mathrm{H}_{0,0} + \frac{5}{2}\right] + 2(1-x)\left[\mathrm{H}_0 + \mathrm{H}_{0,0} - 2x\mathrm{H}_1 + \frac{29}{4}\right] - \frac{15}{2} - \mathrm{H}_{0,0} - \frac{1}{2}\mathrm{H}_0\right)$$

$$P^{(1)}_{\rm gq}(x) = 4\, C_A C_F \left(\frac{1}{x} + 2p_{\rm gq}(x)\left[\mathrm{H}_{1,0} + \mathrm{H}_{1,1} + \mathrm{H}_2 - \frac{11}{6}\mathrm{H}_1\right] - x^2\left[\frac{8}{3}\mathrm{H}_0 - \frac{44}{9}\right] + 4\zeta_2 - 2 \right.$$
$$\left. -7\mathrm{H}_0 + 2\mathrm{H}_{0,0} - 2\mathrm{H}_1 x + (1+x)\left[2\mathrm{H}_{0,0} - 5\mathrm{H}_0 + \frac{37}{9}\right] - 2p_{\rm gq}(-x)\mathrm{H}_{-1,0}\right) - 4\, C_F n_f \left(\frac{2}{3}x \right.$$
$$\left. -p_{\rm gq}(x)\left[\frac{2}{3}\mathrm{H}_1 - \frac{10}{9}\right]\right) + 4\, C_F{}^2 \left(p_{\rm gq}(x)\left[3\mathrm{H}_1 - 2\mathrm{H}_{1,1}\right] + (1+x)\left[\mathrm{H}_{0,0} - \frac{7}{2} + \frac{7}{2}\mathrm{H}_0\right] - 3\mathrm{H}_{0,0} \right.$$
$$\left. +1 - \frac{3}{2}\mathrm{H}_0 + 2\mathrm{H}_1 x\right)$$

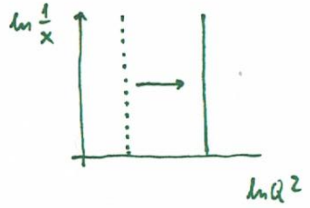
$$P^{(1)}_{\rm gg}(x) = 4\, C_A n_f \left(1 - x - \frac{10}{9}p_{\rm gg}(x) - \frac{13}{9}\left(\frac{1}{x} - x^2\right) - \frac{2}{3}(1+x)\mathrm{H}_0 - \frac{2}{3}\delta(1-x)\right) + 4\, C_A{}^2 \left(27 \right.$$
$$+(1+x)\left[\frac{11}{3}\mathrm{H}_0 + 8\mathrm{H}_{0,0} - \frac{27}{2}\right] + 2p_{\rm gg}(-x)\left[\mathrm{H}_{0,0} - 2\mathrm{H}_{-1,0} - \zeta_2\right] - \frac{67}{9}\left(\frac{1}{x} - x^2\right) - 12\mathrm{H}_0$$
$$\left. -\frac{44}{3}x^2\mathrm{H}_0 + 2p_{\rm gg}(x)\left[\frac{67}{18} - \zeta_2 + \mathrm{H}_{0,0} + 2\mathrm{H}_{1,0} + 2\mathrm{H}_2\right] + \delta(1-x)\left[\frac{8}{3} + 3\zeta_3\right]\right) + 4\, C_F n_f \left(2\mathrm{H}_0 \right.$$
$$\left. +\frac{2}{3}\frac{1}{x} + \frac{10}{3}x^2 - 12 + (1+x)\left[4 - 5\mathrm{H}_0 - 2\mathrm{H}_{0,0}\right] - \frac{1}{2}\delta(1-x)\right) .$$

NNLO

QCD fits:

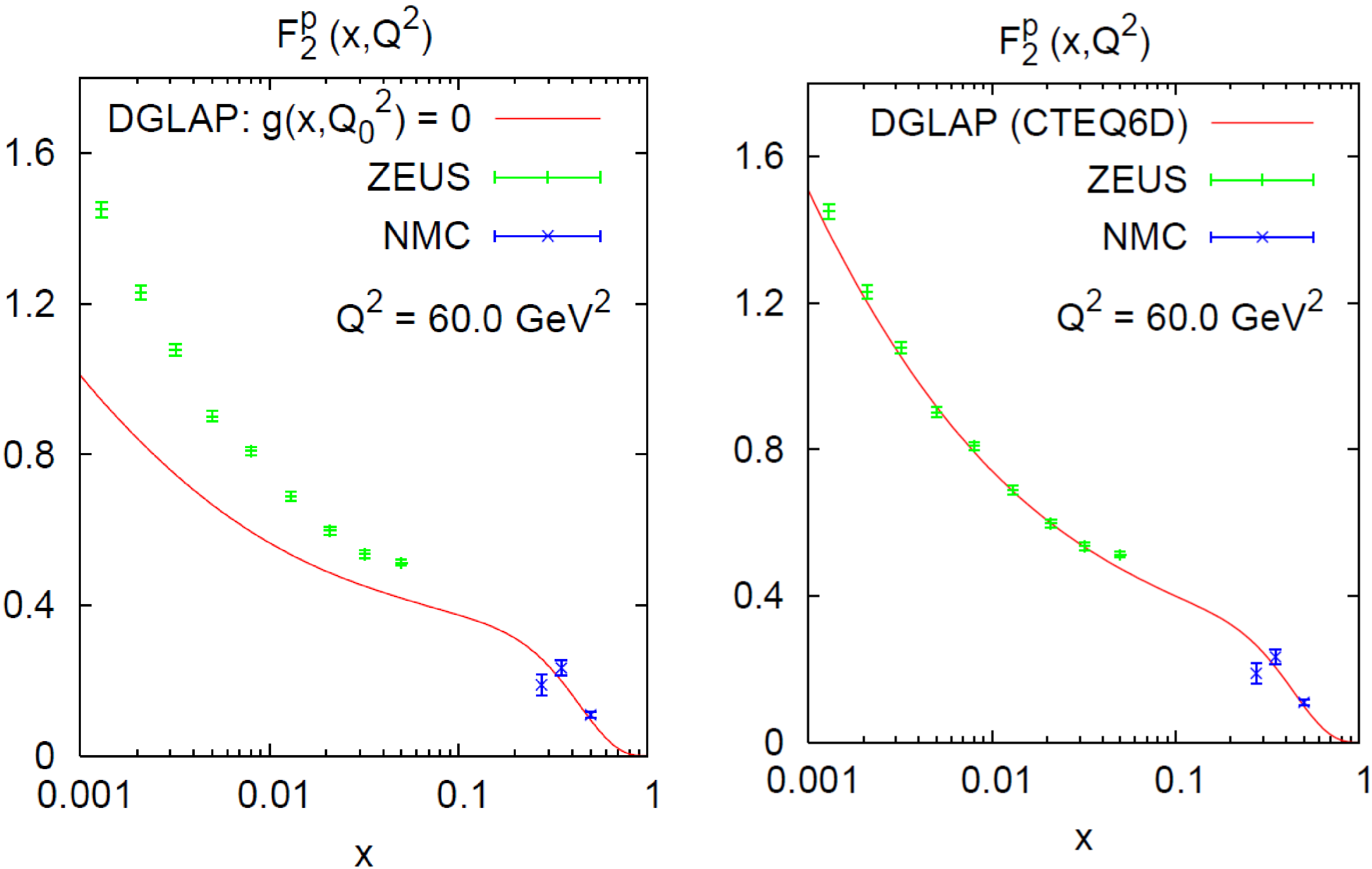
(1) Parametrization of PDFs at the factorization scale: $f(x, \mu_F^2; a_1, \dots, a_n)$, $g(x, \mu_F^2; b_1, \dots, b_n)$

(2) DGLAP evolution

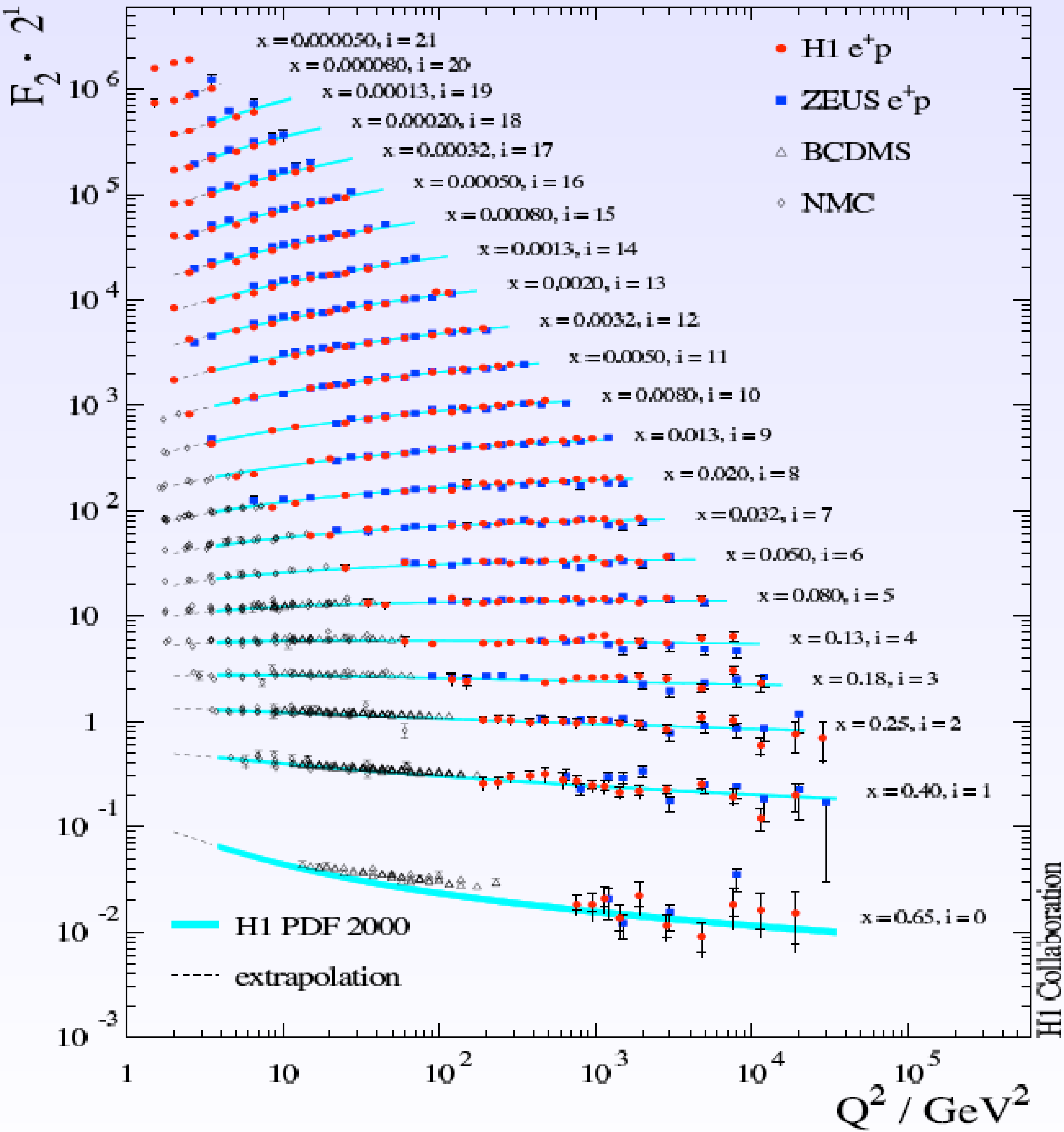


(3) Fit parameters to reproduce the data

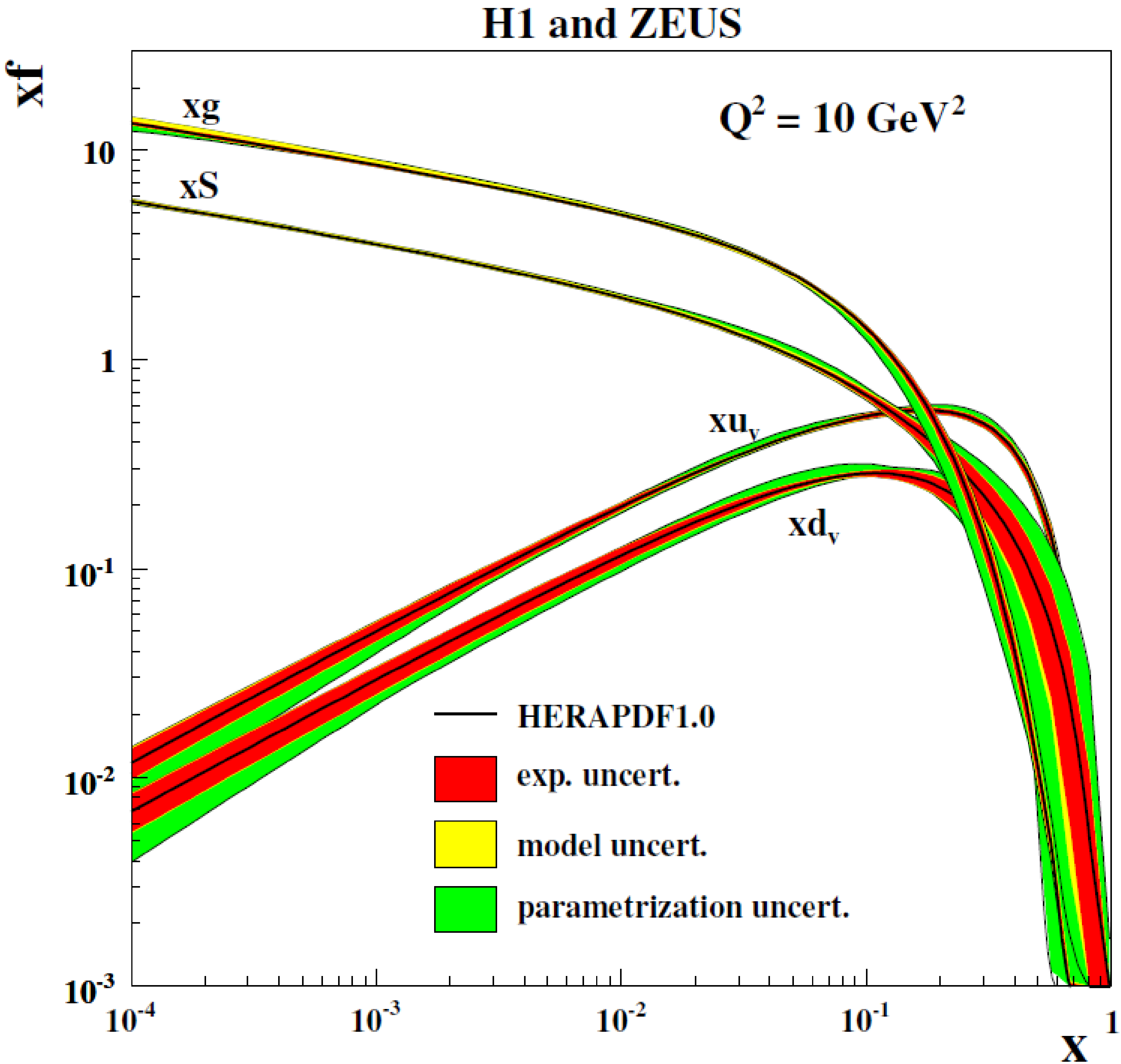
(4) Pure DGLAP is not sufficient: massive quarks requires special treatment



G. Salam, Lectures on QCD, School of High Energy Physics, Bautzen 2009.



$F_2 \cdot 2^i$
x = 0.000050, i = 21
x = 0.000080, i = 20
x = 0.00013, i = 19
x = 0.00020, i = 18
x = 0.00032, i = 17
x = 0.00050, i = 16
x = 0.00080, i = 15
x = 0.0013, i = 14
x = 0.0020, i = 13
x = 0.0032, i = 12
x = 0.0050, i = 11
x = 0.0080, i = 10
x = 0.013, i = 9
x = 0.020, i = 8
x = 0.032, i = 7
x = 0.050, i = 6
x = 0.080, i = 5
x = 0.13, i = 4
x = 0.18, i = 3
x = 0.25, i = 2
x = 0.40, i = 1
x = 0.65, i = 0
H1 e^+p
ZEUS e^+p
BCDMS
NMC
H1 PDF 2000
extrapolation
H1 Collaboration
Q^2 / GeV^2


H1 and ZEUS
xf
$Q^2 = 10\ GeV^2$
xg
xS
xu_v
xd_v
HERAPDF1.0
exp. uncert.
model uncert.
parametrization uncert.
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
1
x