Random Density Matrices and Fuss–Catalan distribution

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in collaboration with

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Dedicated to the memory of

Dr **Ryszard Zygadło,** **1964 - 2010**

Assistant Professor at Smoluchowski Institute of Physics,
Jagiellonian University,
Organiser of recent editions of **Smoluchowski Symposia**

Ensembles of random density operators

Mixed quantum state = density operator which is

a) **Hermitian**, $\rho = \rho^\dagger$,
b) **positive**, $\rho \geq 0$,
c) **normalized**, $\mathrm{Tr}\rho = 1$.

Let $\mathcal{M}_N$ denote the set of density operators of size N.

Ensembles of random states in $\mathcal{M}_N$

Let A be matrix from an arbitrary **ensemble** of **random matrices**.
Then

$$\rho = \frac{AA^\dagger}{\mathrm{Tr}AA^\dagger}$$

forms a **random quantum state**

The ensemble analyzed

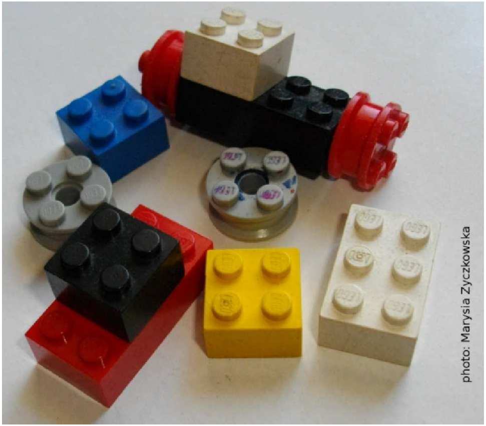
$$W_{k,s} \ := \ \Big(p_1 U_1 + p_2 U_2 + \cdots + p_k U_k \Big) G_1 \cdots G_s$$

where U_i are independent **Haar random unitary** matrices in $U(N)$, while G_i are independent (rectangular) random **Ginibre matrices** and $p = \{p_1, \ldots, p_k\}$ is a probability vector.
Define ensemble of normalized random density matrices of size N

$$\rho_{k,s} \ := W_{k,s} W_{k,s}^{\dagger} / \mathrm{Tr}(W_{k,s} W_{k,s}^{\dagger})$$

* 1) What **ensembles** can be generated in this way?
* 2) What are their **statistical properties** ?
* 3) How these random states may emerge in **quantum physics?**
* 4) How to generate **numerically** random matrices from certain ensembles, (e.g. **Bures** ensemble) ?

Having at your disposal **LEGO** pieces of **two kinds**:
a) **rectangular** pieces (random **Ginibre matrices**)
b) **round** pieces (Haar random **unitary matrices**)



What can you construct out of them ?

How random mixed states appear in quantum physics ?

Reduction of random pure states

1) Consider an ensemble of **random pure states** $|\psi\rangle$ of a **composite system** distributed according to a given measure μ.

2) Perform partial trace over a chosen subsystem B to get a **random mixed state**

$$\rho \ := \mathrm{Tr}_B |\psi\rangle\langle\psi|$$

Depending on the **structure** of the composite system, the initial **measure** μ in the space of the pure states and the choice of the **subsystem** B, over which the averaging is performed
one obtaines different **ensembles of random mixed states**.

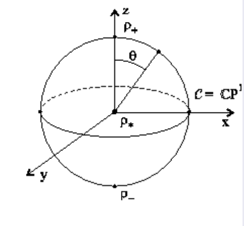
Pure states in a finite dimensional Hilbert space $\mathcal{H}_N$

Space of normalized complex pure states for an arbitrary N:

Since $\langle\psi|\psi\rangle = 1$ a **normalized** state belongs to the **sphere** S^{2N-1}.

Two states equal up to a phase are identified, $|\psi\rangle \sim e^{i\alpha}|\psi\rangle$, so the set of states is equivalent to the **complex projective space** $\mathbb{C}P^{N-1}$ of $2N-2$ real dimensions.

$N=2$: For **qubit** = quantum bit the word **geometry** can be treated literally!



$$|\psi\rangle = \cos\frac{\vartheta}{2}|1\rangle + e^{i\phi}\sin\frac{\vartheta}{2}|0\rangle$$

$\mathbb{C}P^1$ = **Bloch sphere of $N=2$ pure states**

Random Pure states in $\mathcal{H}_N$

'Quantum chaotic' dynamics **(pseudo-random evolution)**

described by a **random unitary** matrix U acting on a pure state produces (almost surely) a **'generic pure state'** $|\psi\rangle = U|\phi_0\rangle$.

• Formally one defines an (unique) **Fubini–Study measure** μ on complex projective spaces which is **unitarily invariant**: for any (measurable) set A of states one requires $\mu(A) = \mu(U(A))$.

• This measure covers the entire space $\mathbb{C}P^{N-1}$ **uniformly**, and for $N = 2$ it is just equivalent to the uniform, **Lebesgue measure on the sphere** S^2.

How to obtain numerically a **random pure state** $|\psi\rangle$ **?**

a) Take a column (a row) of a **random unitary** U so that $|\psi\rangle = U|i\rangle$.
b) generate N **independent complex random numbers** z_i according to the **normal** distribution. Write $|\psi\rangle = \sum_{i=1}^{N} c_i|i\rangle$ where the expansion coefficients read $c_i = z_i/\sqrt{\sum_i |z_i|^2}$.

Ryszard with Ewa during an earlier Smoluchowski Symposium

Composed systems & entangled states

bi-partite systems: $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$

- **separable pure states:** $|\psi\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$
- **entangled pure states:** all states not of the above product form.

Two–qubit system: $d = 2 \times 2 = 4$

Maximally entangled **Bell state** $|\varphi^+\rangle := \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$

Entanglement measures

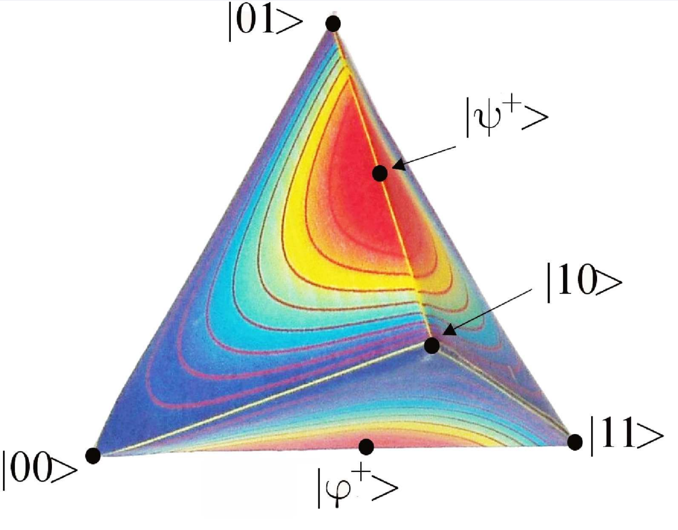
For any pure state $|\psi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ define its partial trace $\sigma = \mathrm{Tr}_B|\psi\rangle\langle\psi|$.
Definition: Entanglement entropy of $|\psi\rangle$ is equal to von Neumann entropy of the partial trace

$$E(|\psi\rangle) := -\mathrm{Tr}\,\sigma \ln \sigma$$

The more mixed partial trace, the more entangled initial pure state...

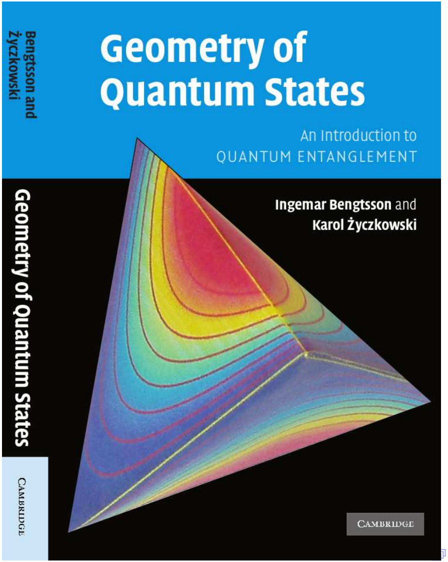
Entanglement of two real qubits

Entanglement entropy at the tetrahedron of $d = 4$ real pure states



More on this is can be found in

I. Bengtsson and K. Życzkowski, Geometry of Quantum States (Cambridge, 2006, 2008)



Generic pure states of a bi-partite system

'Two quNits' $=$ $N \times N$ quantum system

The space $\mathbb{C}P^{N^2-1}$ of all states in $\mathcal{H} = \mathcal{H}_N \otimes \mathcal{H}_N$ has $d_{\rm tot} = N^2 - 2$ dimensions.
The subspace of **separable** **(product) states** $\mathbb{C}P^{N-1} \times \mathbb{C}P^{N-1}$ has only $d_{\rm sep} = 2(N-2)$ dimensions. For large N we observe that $d_{\rm sep} \sim 2N << d_{\rm tot} \sim N^2$ so the **separable states** form a set of measure zero in the space of all states.

Thus a **'typical'** random state is **entangled**!

How much entangled?

Mean entropy of the reduced density matrix ρ

Let us call $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. Take any **pure state** $|\psi\rangle \in \mathcal{H}$ and define its partial trace $\rho := {\rm Tr}_B|\psi\rangle\langle\psi| = {\rm Tr}_A|\psi\rangle\langle\psi|$.
The **von Neumann entropy** S of the reduced **mixed state** ρ is a measure of **entanglement** of the initially **pure** bi-partite state $|\psi\rangle$.

Average entanglement entropy for a bipartite system

$N \times N$ system

$$\langle S(\psi)\rangle_\psi \;\approx\; \ln N - \frac{1}{2} + \mathcal{O}\left(\frac{\ln N}{N}\right)$$

$N \times K$ system: formula of Don Page (1993/1995)

valid for random states in $\mathcal{H}_N \otimes \mathcal{H}_K$ with $K \geq N$

$$\langle S(\psi)\rangle_\psi = \Psi(NK+1) - \Psi(K+1) - \frac{N-1}{2K} \approx \;\ln N - \frac{N}{2K}.$$

$N \times K$ system: probability measure

Let $\lambda = \{\lambda_1, \dots \lambda_N\}$ denote the spectrum of the reduced matrix $\rho := \mathrm{Tr}_B|\psi\rangle\langle\psi|$. If $|\psi\rangle$ is taken **uniformly** on $\mathcal{H}_N \otimes \mathcal{H}_K$ then

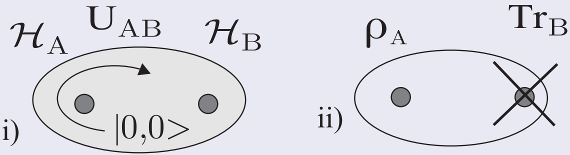
$$P_{N,K}(\lambda) = C_{N,K}\, \delta\big(1 - \textstyle\sum_i \lambda_i\big) \prod_i \lambda_i^{K-N} \prod_{i<j}(\lambda_i - \lambda_j)^2$$

Composed bi–partite systems on $\mathcal{H}_A \otimes \mathcal{H}_B$

Ensembles obtained by partial trace: **a) induced measure**

i) **natural measure** on the space of **pure states** obtained by acting on a fixed state $|0,0\rangle$ with a global random unitary U_{AB} of size NK

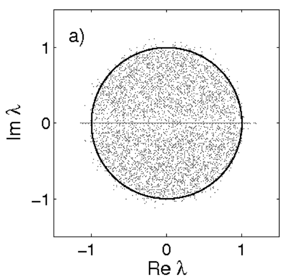
$$|\psi\rangle = \sum_{i=1}^{N} \sum_{j=1}^{K} G_{ij} |i\rangle \otimes |j\rangle$$



ii) partial trace over the K dimensional subsystem B gives $\rho_A = \mathrm{Tr}_B |\psi\rangle\langle\psi|$ and leads to the **induced measure** $P_{N,K}(\lambda)$ in the space of mixed states of size N. Integrating out all eigenvalues but λ_1 one arrives (for large N) at the **Marchenko–Pastur** distribution $P_c(x = N\lambda_1)$ with the parameter $c = K/N$.

Spectral properties of random matrices

Non-hermitian matrix G of size N of the Ginibre ensemble



Under normalization $\mathrm{Tr} GG^\dagger = N$
the spectrum of G fills **uniformly**
(for large N!) the **unit disk**

The so–called **circular law** of **Girko** !

Hermitian, positive matrix $\rho = GG^\dagger$ of the Wishart ensemble

Let $x = N\lambda_i$, where $\{\lambda_i\}$ denotes the spectrum of ρ. As $\mathrm{Tr}\rho = 1$ so $\langle x \rangle = 1$. Distribution of the spectrum $P(x)$ is asymptotically given by the **Marchenko–Pastur law**

$$\pi^{(1)}(x) = P_{\mathrm{MP}}(x) = \frac{1}{2\pi}\sqrt{\frac{4}{x} - 1} \quad \text{for} \quad x \in [0,4]$$

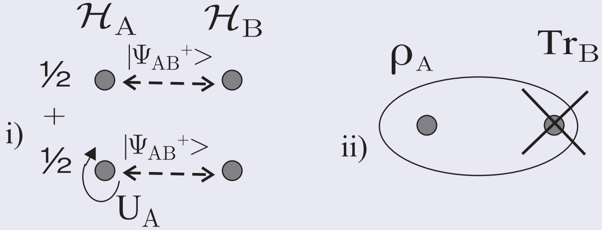
Composed bi–partite systems II

b) Arcsine ensemble

i) Consider a superposition of **two maximally entangled** states on $\mathcal{H}_N \otimes \mathcal{H}_N$

$|\phi\rangle = |\psi^+_{AB}\rangle + (U_A \otimes \mathbb{1}_N)|\psi^+_{AB}\rangle$, where $|\psi^+_{AB}\rangle = (1/\sqrt{N})\sum_{i=1}^{N} |i,i\rangle$,

while $U_A \in U(N)$ is a **Haar random unitary matrix** with phases α_i.



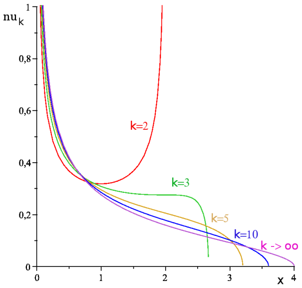
ii) The reduced state $\rho_A = \frac{\mathrm{Tr}_B|\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle} = \frac{2\mathbb{1}+U_A+U_A^\dagger}{2N+\mathrm{Tr}(U_A+U_A^\dagger)}$.

has the spectrum $\lambda_i = (1+\cos\alpha_i)/N$ for $i=1,\ldots,N$. Thus for large N the spectral density has the form of the **arcsine distribution**,

$P_{\mathrm{arc}}(x) = \frac{1}{\pi\sqrt{x(2-x)}}$ with support $x \in [0,2]$, where $x = N\lambda$.

c) Generalization for k states

i) Superposition of k **maximally entangled** states on $\mathcal{H}_N \otimes \mathcal{H}_N$
$|\phi\rangle = \sum_{i=1}^{k} (U_i \otimes \mathbb{1}_N)|\psi^+_{AB}\rangle$,
where $U_i \in U(N)$ are independent **Haar random unitary matrices**.
ii) The reduced state $\rho_A = \frac{\mathrm{Tr}_B |\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle} = \frac{(U_1+\cdots+U_k)(U_1^\dagger+\cdots+U_k^\dagger)}{\mathrm{Tr}(U_1+\cdots+U_k)(U_1^\dagger+\cdots+U_k^\dagger)}$. is asymptotically characterized by the leads to a **Kesten** distribution
$P_\mathrm{k}(x) = \frac{1}{2\pi} \frac{\sqrt{4k(k-1)x - k^2x^2}}{kx - x^2}$
which belongs to **free Meixner laws** (**Bożejko, Bryc** 2006)



with support $x \in [0, 4(k-1)/k]$, where $x = N\lambda$.

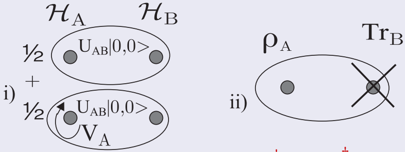
Observe that for $k \to \infty$ the distribution P_k tends to **Marchenko-Pastur** $\pi^{(1)}$, as the renormalized sum of many **independent** random unitaries bahaves as a **Ginibre** matrix .

d) Bures ensemble

i) Consider a superposition of two pure states: a random state $|\psi_1\rangle$ and the same state transformed by a **local unitary V_A**,

$$|\phi\rangle := (\mathbb{1} \otimes \mathbb{1} + V_A \otimes \mathbb{1})|\psi_1\rangle, \quad \text{where } |\psi_1\rangle = U_{AB}|0,0\rangle$$

while $V_A \in U(N)$ and $U_{AB} \in U(N^2)$ are **Haar random unitary matrices**.



ii) The reduced state $\rho_B = \frac{(\mathbb{1}+V_A)GG^\dagger(\mathbb{1}+V_A^\dagger)}{\mathrm{Tr}[(\mathbb{1}+V_A)GG^\dagger(\mathbb{1}+V_A^\dagger)]}$ is distributed according to the **Bures measure**, $P_B(\lambda_1, ...\lambda_N) = C_N^B \prod_i \lambda_i^{-1/2} \prod_{i<j}^{1...N} \frac{(\lambda_i-\lambda_j)^2}{\lambda_i+\lambda_j}$

(Osipov, Sommers, Życzkowski, 2010) characterized by the **Bures distribution**,

$$P_B(x) = \frac{1}{4\pi\sqrt{3}} \left[\left(\frac{a}{x} + \sqrt{\left(\frac{a}{x}\right)^2 - 1} \right)^{2/3} - \left(\frac{a}{x} - \sqrt{\left(\frac{a}{x}\right)^2 - 1} \right)^{2/3} \right]$$

where $a = 3\sqrt{3}$. Square matrix G of size N from the **Ginibre ensemble** is obtained from the first column of U_{AB} od size N^2 which acts on $|0,0\rangle$.

Composed mutipartite systems & projections

a) Four-partite system & $\pi^{(2)}$ distribution

Take a four-partite product state,

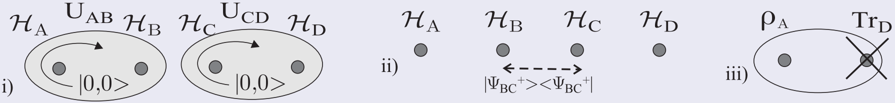
$$|\psi_0\rangle = |0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C \otimes |0\rangle_D =: |0,0,0,0\rangle \ \in \ \mathcal{H}_{\mathcal{N}}^{\otimes 4}.$$

i) Apply two random unitary matrices U_{AB} and U_{CD} of size N^2,

$$|\psi\rangle = U_{AB} \otimes U_{CD} |\psi_0\rangle \ = \ \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} G_{ij} E_{kl} \ |i\rangle_A \otimes |j\rangle_B \otimes |k\rangle_C \otimes |l\rangle_D$$

ii) Consider projector $P := \mathbb{1}_A \otimes |\Psi^+_{BC}\rangle\langle\Psi^+_{BC}| \otimes \mathbb{1}_D$

on the maximally entangled state, $|\Psi^+_{BC}\rangle = \frac{1}{\sqrt{N}} \ \sum_{\mu=1}^{N} |\mu\rangle_B \otimes |\mu\rangle_C$



The spectrum of the iii) reduced state $\rho_A \ = \ \frac{\mathrm{Tr}_D|\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle} = \frac{GEE^\dagger G^\dagger}{\mathrm{Tr}\ GEE^\dagger G^\dagger}$
consists of squared singular values of the product GE
of **two independent Ginibre matrices**, so the spectral density
is described by the **Fuss-Catalan distribution** $\pi^{(2)}(x)$.

b) 2*s*-partite system & $\pi^{(s)}$ Fuss-Catalan distribution

Take a 2*s*-partite product state,

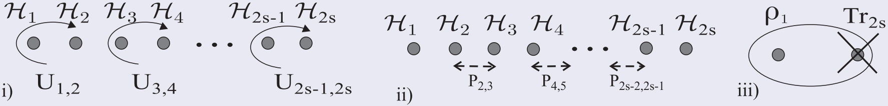
$$|\psi_0\rangle = |0\rangle_1 \otimes \cdots \otimes |0\rangle_{2s} \in \mathcal{H}_{\mathcal{N}}{}^{\otimes 2s}.$$

i) Apply *s* random unitary matrices $U_{1,2}$, $U_{3,4}$,.. $U_{2s-1,2s}$ of size N^2 each,

$$|\psi\rangle U_{1,2} \otimes \cdots U_{2s-1,2s}|0,\ldots,0\rangle = \sum_{i_1,\ldots i_{2s}} (G_1)_{i_1,i_2} \cdots (G_s)_{i_{2s-1},i_{2s}} |i_1,\ldots,i_{2s}\rangle$$

ii) Project onto the product of $(s-1)$ maximally entangled states,

$$P_s := \mathbb{1}_1 \otimes |\Psi^+_{2,3}\rangle\langle\Psi^+_{2,3}| \otimes \cdots \otimes |\Psi^+_{2s-2,2s-1}\rangle\langle\Psi^+_{2s-2,2s-1}| \otimes \mathbb{1}_{2s}$$



The spectrum of the iii) reduced state

$$\rho_A = \frac{\mathrm{Tr}_{2s}|\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle} = \frac{G_1 G_2 \cdots G_s (G_1 G_2 \cdots G_s)^\dagger}{\mathrm{Tr}\,[G_1 G_2 \cdots G_s (G_1 G_2 \cdots G_s)^\dagger]}$$

consists of squared singular values of the product $G_1 \cdots G_s$ of *s* **independent Ginibre matrices**, so the spectral density is described by the **Fuss-Catalan distribution** $\pi^{(s)}(x)$.

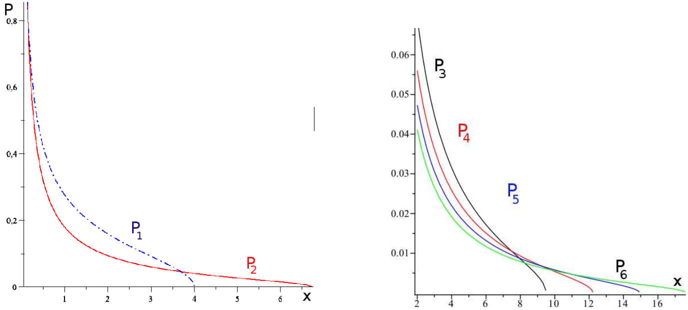
Fuss-Catalan distribution $\pi^{(s)}$

defined for an integer number s is characterized by its **moments**

$$\int x^p \pi^{(s)}(x)dx = \frac{1}{sp+1}\binom{sp+p}{p} \quad =: FC_p^{(s)}$$

equal to the **generalized Fuss-Catalan numbers** .

The density $\pi^{(s)}$ is analitic on the support $[0,(s+1)^{s+1}/s^s]$, while for $x \to 0$ it behaves as $1/(\pi x^{s/(s+1)})$.



The same moments decribe (asymptotically) distribution of singular values for s–th power of Ginibre G^s, (**Alexeev, Götze, Tikhomirov** 2010)

Fuss-Catalan distributions $\pi^{(s)}$

The moments of $\pi^{(s)}$ are equal to Fuss-Catalan numbers.
Using inverse **Mellin transform** one can represent $\pi^{(s)}$ by the **Meijer** G**–function**, which in this case reduces to s **hypergeometric functions**

Exact explicit expressions for FC $\pi^{(s)}$

$s=1$, $\pi^{(1)}(x) = \frac{1}{\pi\sqrt{x}}\ {}_1F_0\left(-\frac{1}{2};\ ;\ \frac{1}{4}x\right) = \frac{\sqrt{1-x/4}}{\pi\sqrt{x}}$, **Marchenko–Pastur**

$s=2$, $\pi^{(2)}(x) = \frac{\sqrt{3}}{2\pi x^{2/3}}\ {}_2F_1\left(-\frac{1}{6},\frac{1}{3};\ \frac{2}{3};\ \frac{4x}{27}\right) - \frac{\sqrt{3}}{6\pi x^{1/3}}\ {}_2F_1\left(\frac{1}{6},\frac{2}{3};\ \frac{4}{3};\ \frac{4x}{27}\right) =$

$$= \frac{\sqrt[3]{2}\sqrt{3}}{12\pi}\ \frac{\sqrt[3]{2}\left(27+3\sqrt{81-12x}\right)^{\frac{2}{3}} - 6\sqrt[3]{x}}{x^{\frac{2}{3}}\left(27+3\sqrt{81-12x}\right)^{\frac{1}{3}}}$$ **Fuss–Catalan**

Arbitrary s, $\Rightarrow$ $\pi^{(s)}(x)$ is a superposition of
s **hypergeometric functions,**

$$\pi^{(s)}(x) = \sum_{j=1}^{s} \beta_j\ {}_sF_{s-1}\left(a_1^{(j)},\dots,a_s^{(j)};\ b_1^{(j)},\dots,b_{s-1}^{(j)};\ \alpha_j x\right) \quad .$$

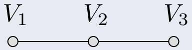
(Penson, Życzkowski, 2010)

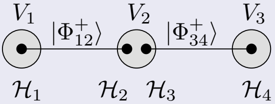
Multi–partite systems: graphs

Graph random states

Consider a graph Γ consisting of m edges $B_1, \dots B_m$ and k vertices $V_1, \dots V_k$. It represents a composite **quantum system** consisting of $2m$ sub–systems described in the Hilbert space with $2m$–fold tensor product $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{2m}$ of dimension N^{2m}.
Each **edge** represents the **maximally entangled state** $|\Phi^+\rangle$ in both subspaces, while each **vertex** represents a **random unitary matrix U** (**Haar measure ='generic' Hamiltonian**), coupling connected systems.

A simple example: three vertices & two edges





We define a **random state** $|\psi\rangle = (\mathbf{U_1} \otimes \mathbf{U_{23}} \otimes \mathbf{U_4})\, |\mathbf{\Phi^+_{12}}\rangle \otimes |\mathbf{\Phi^+_{34}}\rangle$
where $|\Phi^+_{kj}\rangle$ denotes the maximally entangled state in subspaces k, j.

Multi–partite graph systems: mixed states

Partial trace over certain subspaces

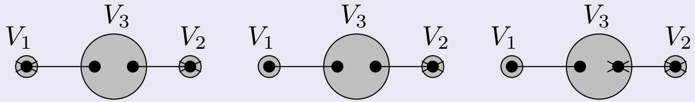
Consider an **ensemble of random pure states** $|\psi\rangle$ corresponding to a given graph Γ. Select a fixed **subset** T of subspaces and define a (random) **mixed state** $\rho(T) = \mathrm{Tr}_T|\psi\rangle\langle\psi|$.

Tasks

• Determine the **spectral properties** of the ensemble of mixed states $\rho(T)$ associated with the graph Γ.

• Find the mean **entropy** $\langle S(\rho)\rangle_\psi$ of the reduced state ρ averaged over the ensemble of graph random pure states $|\psi\rangle_{\Gamma,T}$.

Examples of partial trace for the graph Γ

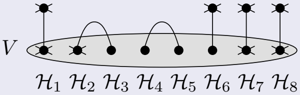


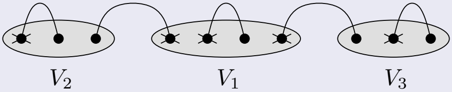
The partial trace is taken over all the subspaces T represented by open symbols.

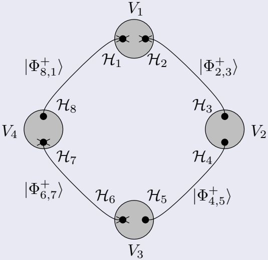
Graphs and random multi–partite systems

Partial trace over certain subspaces

For ensembles of **random states** associated with certain **graphs** Γ and selected subspaces T – cross $(\times)$ – over which the partial trace takes place







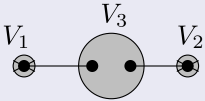
one can compute **moments of the traces** $\mu_q := \langle \mathrm{Tr}\rho^q \rangle_\psi$
and then obtain bounds for the **average entropy** $\langle S \rangle = \langle -\mathrm{Tr}\rho \ln \rho \rangle_\psi$.

Collins, Nechita, Życzkowski, *J. Phys. A*, (2010)

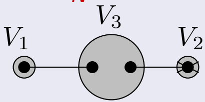
Spectral properties of random mixed states I

Example 1: 2 bonds, 4 subsystems and one bi-partite interaction U_0

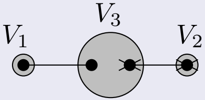
a) $\pi^{(0)}$ – maximaly mixed state $\rho = \frac{1}{N}\mathbb{1}$ with **entropy** $S(\rho) = \ln N$



or



b) $\pi^{(1)}$ random mixed state generated according to the induced measure



with **entropy** $S(\rho) \approx \ln N - 1/2$

Let $|\psi\rangle = \sum_i \sum_j G_{ij}|i\rangle \otimes |j\rangle$ be a **random pure state.**

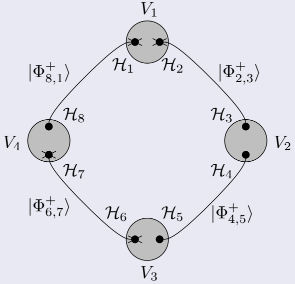
Then G is a random matrix of **Ginibre ensemble** consisting of independent complex Gaussian entries normalized as $|G|^2 = \mathrm{Tr}\, GG^\dagger = 1$.

The distribution of eigenvalues of a **non–hermitian matrix** G is given by the **Girko circular law**, while positive **Wishart** matrices $\rho = \mathrm{Tr}_B|\psi\rangle\langle\psi| = GG^\dagger$ are described by **Marchenko-Pastur** law $\pi^{(1)}$.

Spectral properties of random mixed states II

Example 2: 4 bonds, 8 subsystems and four bi-partite interactions V_i

c) $\pi^{(2)}$ random mixed state generated by the 4–**cycle graph**



After partial trace over **crossed** subsystems the random mixed state has the structure

$$\rho = \alpha G_2 G_1 G_1^\dagger G_2^\dagger,$$

where G_1 and G_2 are independent **Ginibre** matrices and $\alpha = 1/\mathrm{Tr}\, G_2 G_1 G_1^\dagger G_2^\dagger$.

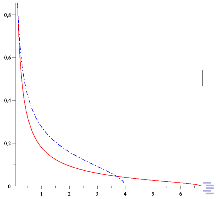
Mixed states with spectrum given by the

Fuss-Catalan distribution $\pi^{(2)}(x)$

characterized by mean **entropy**

$S(\rho) \approx \ln N - 5/6$

$P_{\mathbf{MP}}(x) = \pi^{(1)}(x)$ and $\pi^{(2)}(x)$.



Spectral properties of the ensembles analyzed

Spectral density $P(x)$ of the rescaled eigenvalue $x = N\lambda$

matrix W	$P(x)$	$x \to 0$	support	mean entropy
$\mathbb{1}$	$\pi^{(0)}$	–	$\{1\}$	0
$\mathbb{1} + U$	arcsine	$x^{-1/2}$	$[0, 2]$	$\ln 2 - 1 \approx -0.307$
G	**M.-P.** $\pi^{(1)}$	$x^{-1/2}$	$[0, 4]$	$-1/2 = -0.5$
$(\mathbb{1} + U)G$	**Bures**	$x^{-2/3}$	$[0, 3\sqrt{3}]$	$-\ln 2 \approx -0.693$
$G_1 G_2$	**F–C** $\pi^{(2)}$	$x^{-2/3}$	$[0, 6\frac{3}{4}]$	$-5/6 \approx -0.833$
...	...	...	...	...
$G_1 \cdots G_s$	F–C $\pi^{(s)}$	$x^{-s/(s+1)}$	$[0, b_s]$	$-\sum_{j=2}^{s+1} \frac{1}{j}$

Table: Ensembles of random mixed states obtained as normalized Wishart matrices, $\rho = WW^\dagger / \mathrm{Tr} WW^\dagger$. Here $b_s = (s+1)^{s+1}/s^s$ and the mean entropy $\langle S \rangle = -\int x \ln x P(x) dx$.

Generalized ensemble of random states

Let

$$W_{k,s} \ := \ \Big(U_1 + U_2 + \cdots + U_k\Big) G_1 \cdots G_s$$

where U_i are independent **Haar random unitary** matrices,
while G_i are independent random **Ginibre matrices**.
Define generalized ensemble of normalized random density matrices

$$\rho_{k,s} \ := W_{k,s}W_{k,s}^{\dagger}/\mathrm{Tr}(W_{k,s}W_{k,s}^{\dagger})$$

Special cases:

$s = 0, \ k = 1 \quad \Rightarrow$ **maximally mixed state**
$s = 0, \ k = 2 \quad \Rightarrow$ **arcsine ensemble**
$s = 0, \ k = k \quad \Rightarrow$ k–**Kesten ensemble**
$s = 1, \ k = 1 \quad \Rightarrow$ **Hilbert-Schmidt ensemble**
$s = 1, \ k = 2 \quad \Rightarrow$ **Bures ensemble**
$s = s, \ k = 1 \quad \Rightarrow$ s – **Fuss Catalan ensemble**

Concluding remarks

- **Random pure state** can be obtained from any initial state $|0\rangle$ by a generic unitary evolution operator U, (corresponding e.g. to a **quantized chaotic evolution**), $|\psi\rangle = U|0\rangle$.
- **Random mixed state** of size N from the **induced ensemble** (which leads to **Marchenko-Pastur** spectral density) is obtained by the partial trace of a composite system in an initially random pure state.
- **'Biased'** ensembles of random pure states + **partial trace** lead to other ensembles of **random states**, including **(Arcsine, k–Kesten, Bures, s–Fuss-Catalan)**.
- With any **graph** one can associate an **ensemble of random pure states**. Selecting a set A of subsystems we define an ensemble of mixed states ρ by performing the **partial trace** over them. Graphs leading (asymptotically) to **Fuss–Catalan** distributions $\pi^{(s)}(x)$ are identified for any $s = 0, 1, 2, ...$
- Explicit exact expresstions for the distribution **Fuss–Catalan** distributions $\pi^{(s)}(x)$ are derived.

Working with pieces of the two kinds:
a) **rectangular** pieces (random **Ginibre matrices**)
b) **round** pieces (Haar random **unitary matrices**)
one can construct...

many various ensembles of mixed quantum states !