Phase Transitions in the Quantum Conductance Problem

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In collaboration with
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- Chaotic cavities --> sample-to-sample fluctuations of observables

- Goal: full probability distribution

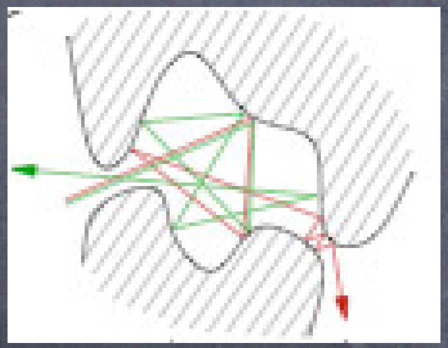
- Technique: N-fold integral for large N ---> canonical partition function of an auxiliary thermodynamical problem (Coulomb gas)

- Phase transitions in the gas --> weak non-analytic points in the distribution

- "A cavity of sub-micron dimensions, etched in a semiconductor is called a quantum dot" [C.W.J. Beenakker]

- "...is essentially a mesoscopic electron billiard, consisting of a ballistic cavity connected by two small holes to two electron reservoirs." [R.A. Jalabert et al.]

Left lead



Right lead

'N' electronic channels in each of the two leads

Main feature: sample-to-sample fluctuations of experimental observables ---> Statistical Theory

Main observables

- Conductance $$G = \lim_{V \to 0} \frac{\bar{I}}{V}$$
- Shot Noise $$P = 4 \int_0^\infty dt \, \overline{\delta I(t + t_0) \delta I(t_0)}$$

They fluctuate from sample-to-sample and will be regarded as random variables...

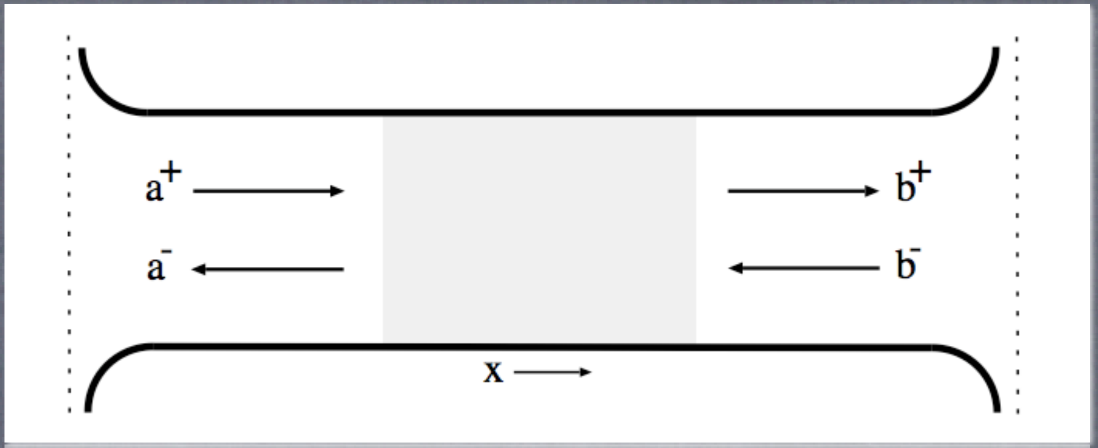
Scattering Matrix

$$\mathcal{S} = \begin{pmatrix} \mathbf{r} & \mathbf{t}' \\ \mathbf{t} & \mathbf{r}' \end{pmatrix}$$

2N

2N

$$c^{\text{out}} = \mathcal{S} c^{\text{in}}$$



$$c^{\text{in}} \equiv (a_1^+, a_2^+, \dots a_N^+, b_1^-, b_2^-, \dots b_N^-)$$

$$c^{\text{out}} \equiv (a_1^-, a_2^-, \dots a_N^-, b_1^+, b_2^+, \dots b_N^+)$$

Scattering Matrix: connects the incoming and outgoing wave function coefficients → Charge Conservation → Unitary or Orthogonal

$$\beta = 1, 2$$

Europhys. Lett., **27** (4), pp. 255-260 (1994)

Universal Quantum Signatures of Chaos in Ballistic Transport.

R. A. JALABERT (*), J.-L. PICHARD (**) and C. W. J. BEENAKKER (***)

VOLUME 73, NUMBER 1 PHYSICAL REVIEW LETTERS 4 JULY 1994

Mesoscopic Transport through Chaotic Cavities: A Random S-Matrix Theory Approach

Harold U. Baranger[1] and Pier A. Mello[2]

$$\mathcal{S} = \begin{pmatrix} \mathbf{r} & \mathbf{t}' \\ \mathbf{t} & \mathbf{r}' \end{pmatrix}$$

...is a random matrix drawn uniformly from the unitary (orthogonal) group

Landauer-Imry-Büttiker Theory

$$S = \begin{pmatrix} \mathbf{r} & \mathbf{t}' \\ \mathbf{t} & \mathbf{r}' \end{pmatrix}$$

N

N

$$\mathbf{T} = \mathbf{t}\mathbf{t}^\dagger$$

Hermitian transmission matrix

$$0 \leq T_i \leq 1$$

Experimental observables are linear statistics on the random eigenvalues of the transmission matrix

$$\mathcal{A} = \sum_{i=1}^{N} f(T_i)$$

$$G = \sum_{i=1}^{N} T_i$$

Conductance

$$P = \sum_{i=1}^{N} T_i(1 - T_i)$$

Shot Noise

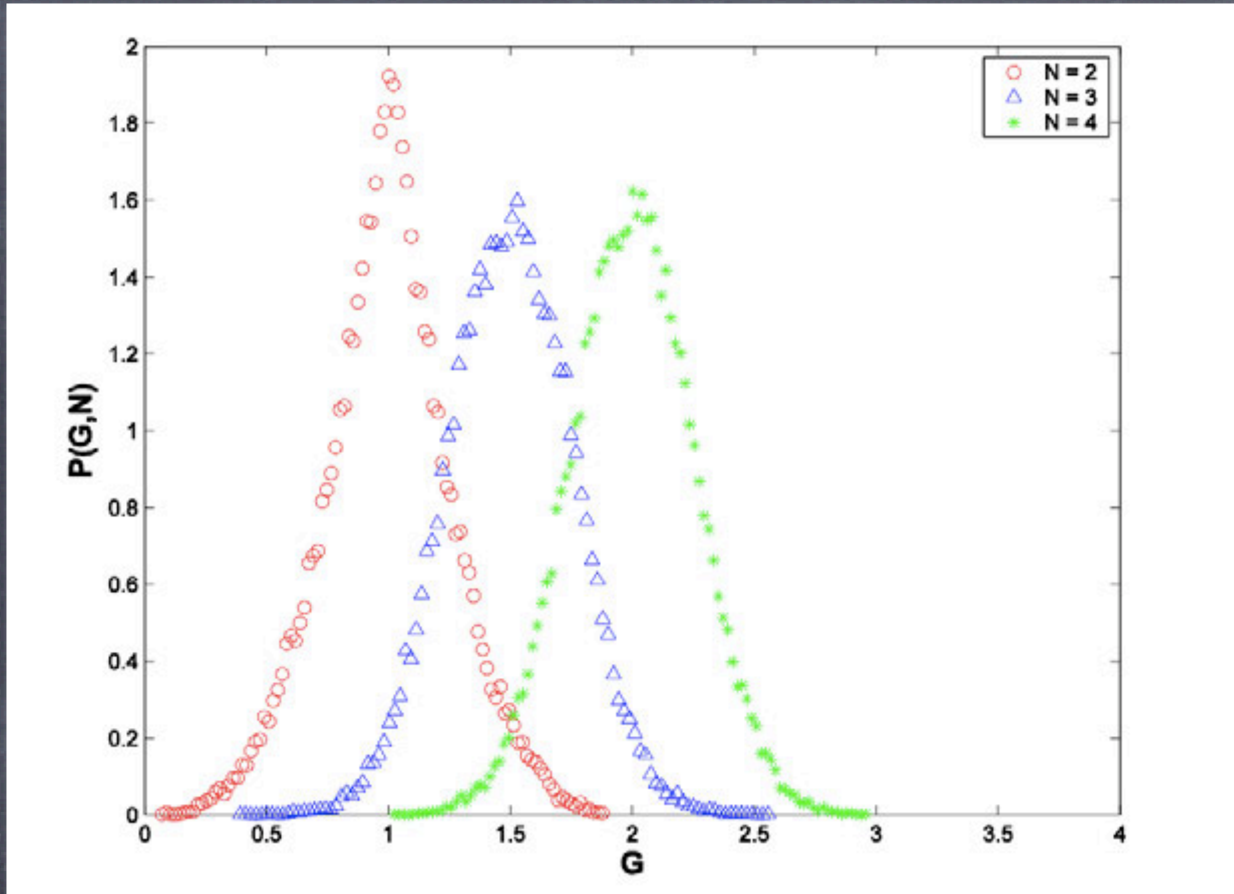
An Example for $\beta = 2$ and $N = 2$

$$
\begin{pmatrix}
-0.2277+0.0543i & 0.2360+0.4072i & -0.0093-0.1596i & -0.4345-0.7137i \\
-0.8358-0.0563i & -0.2372-0.1252i & -0.2902-0.3261i & 0.0002+0.1890i \\
-0.3391-0.1650i & 0.7410+0.0487i & -0.0378+0.4035i & 0.3733+0.0515i \\
0.3176+0.0206i & 0.2871-0.2694i & -0.5986-0.5111i & 0.2836-0.2090i
\end{pmatrix}
$$

$$
\mathbf{T} = tt^{\dagger} = \begin{pmatrix} 0.6937 & 0.0885+0.1682i \\ 0.0885-0.1682i & 0.2563 \end{pmatrix}
$$

$$
\mathrm{eig}(\mathbf{T}) = [0.1852 \quad , \quad 0.7647] \in [0,1] \times [0,1]
$$

$$
\boxed{G = 0.1852 + 0.7647 = 0.9499}
$$

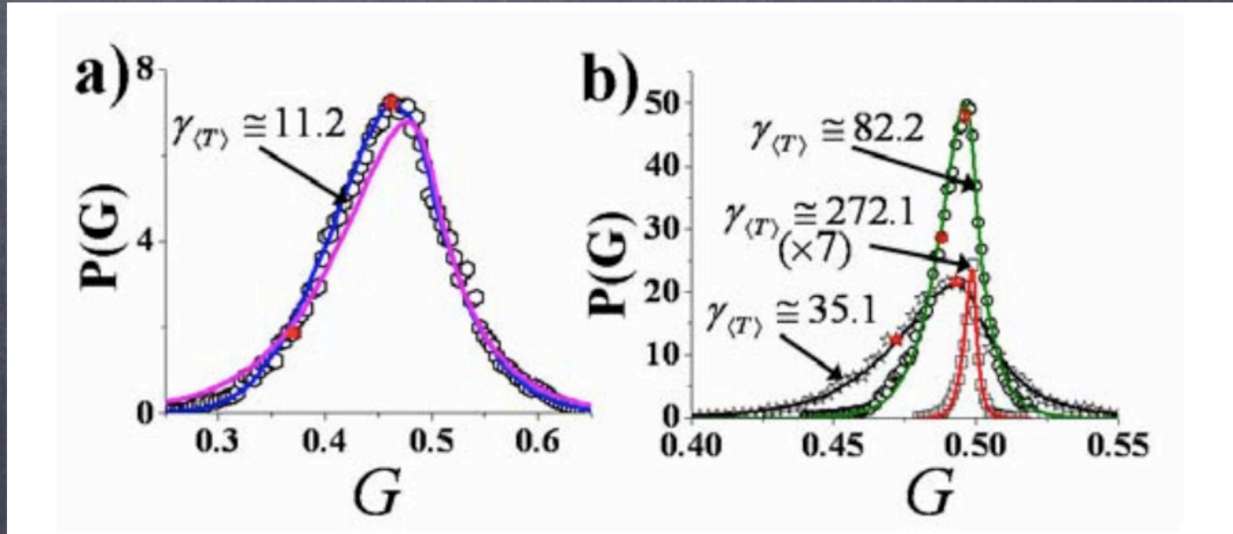


Numerics

Experiments

- Support $0 \leq G \leq N$
- Mean $\langle G \rangle = \frac{N}{2}$
- Variance $\mathrm{var}(G) = \frac{1}{8\beta}$

[Hemmady et al. 2006]



$$S = \begin{pmatrix} \mathbf{r} & \mathbf{t}' \\ \mathbf{t} & \mathbf{r}' \end{pmatrix}$$

N N

+

'S' is drawn uniformly from the unitary (orthogonal) group

$$\mathbf{T} = \mathbf{t}\mathbf{t}^{\dagger}$$

$$P(T_1, \ldots, T_N) = \frac{1}{Z} \prod_{j<k} |T_j - T_k|^{\beta} \prod_{i=1}^{N} T_i^{\beta/2-1}$$

Joint Probability Density of Transmission Eigenvalues

A physical realization of Jacobi ensemble of random matrices

*Muttalib, Pichard and Stone (1987)

*Mello, Pereira and Kumar (1988)

*Forrester (2006)

Probability Density of Conductance

$$G = \sum_{i=1}^{N} T_i$$

+

$$P(T_1, \ldots, T_N) = \frac{1}{Z} \prod_{j<k} |T_j - T_k|^\beta \prod_{i=1}^{N} T_i^{\beta/2-1}$$

$$\mathcal{P}(G, N) = \frac{1}{Z} \int_0^1 \cdots \int_0^1 dT_1 \cdots dT_N \prod_{j<k} |T_j - T_k|^\beta \prod_{i=1}^{N} T_i^{\beta/2-1} \delta\left(\sum_{i=1}^{N} T_i - G\right)$$

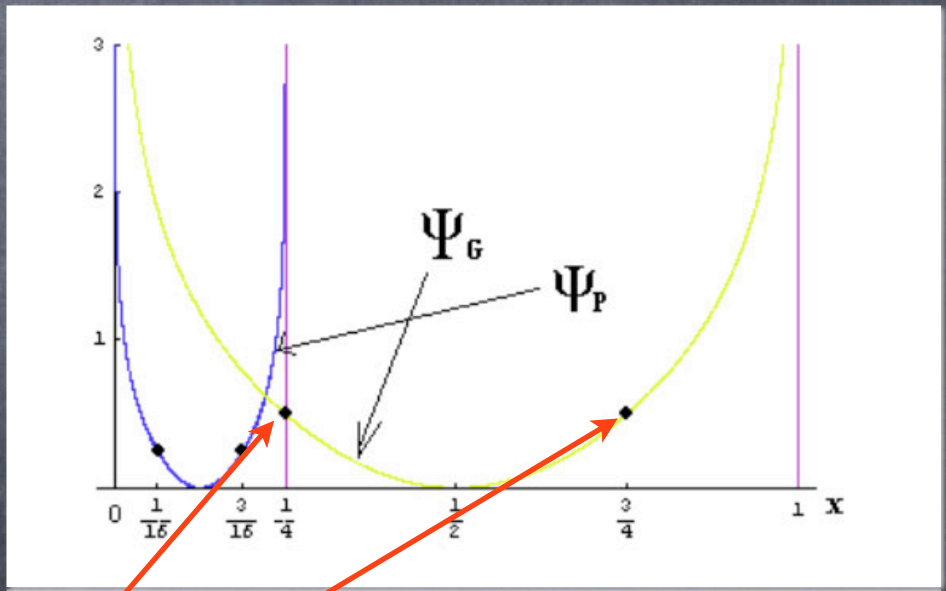
Goal: behaviour of this integral for large 'N'

Main result

$$\lim_{N\to\infty}\left[-\frac{2\log\mathcal{P}(G=Nx,N)}{\beta N^2}\right]=\Psi_G(x)$$

Rate Function

$$\Psi_G(x)=\begin{cases}\frac{1}{2}-\log(4x) & \text{for}\quad 0\le x\le\frac{1}{4}\\ 8\left(x-\frac{1}{2}\right)^2 & \text{for}\quad \frac{1}{4}\le x\le\frac{3}{4}\\ \frac{1}{2}-\log[4(1-x)] & \text{for}\quad \frac{3}{4}\le x\le 1\end{cases}$$



3rd derivative is discontinuous at critical points!

[Sommers et al. (2007); Khoruzhenko et al. (2009)]

$$P_g^{(\beta)}(g) = n! \sum_{m=1}^{\infty} \frac{2}{n} \sin\left(\frac{m\pi g}{n}\right) A^{(\beta)}(m) \tag{13}$$

where $A^{(\beta)}(m)$ is known at $\beta = 1, 2, 4$. For example,

$$A^{(2)}(m) = C_2 \operatorname{Im} \det B_{kl}^{(2)}(m) \tag{14}$$

is given by the imaginary part of the determinant of the following matrix:

$$B_{kl}^{(2)}(m) = \int_0^1 dT\, T^{\alpha+k+l-3} \mathrm{e}^{im\pi T/n}, \qquad \text{for} \quad k, l = 1, 2, \ldots, n\,. \tag{15}$$

$$\mathcal{F}_n(z) \propto z^{-n(n+\nu)} \int_{(0,z)^n} \prod_{j=1}^{n} d\lambda_j\, \lambda_j^{\nu}\, e^{-\lambda_j} \cdot \Delta_n^2(\boldsymbol{\lambda}), \tag{12}$$

one immediately derives [34, 35]:

$$\mathcal{F}_n(z) = \exp\left(\int_0^z dt \frac{\sigma_{\rm V}(t) - n(n+\nu)}{t}\right). \tag{13}$$

Here, $\sigma_{\rm V}(t)$ satisfies the Jimbo-Miwa-Okamoto form of the Painlevé V equation [36]

$$\begin{aligned} (t\sigma_{\rm V}'')^2 &+ [\sigma_{\rm V} - t\sigma_{\rm V}' + 2(\sigma_{\rm V}')^2 + (2n+\nu)\sigma_{\rm V}']^2 \\ &+ 4(\sigma_{\rm V}')^2(\sigma_{\rm V}'+n)(\sigma_{\rm V}'+n+\nu) = 0 \end{aligned} \tag{14}$$

$$\begin{aligned} [(N_{\rm L}+N_{\rm R})^2 - j^2](j+1)\kappa_{j+1}(g) &= 2\sum_{\ell=0}^{j-1}(3\ell+1)(j-\ell)^2 \binom{j}{\ell} \kappa_{\ell+1}(g)\kappa_{j-\ell}(g) \\ &- (N_{\rm L}+N_{\rm R})(2j-1)\, j\kappa_j(g) - j(j-1)(j-2)\,\kappa_{j-1}(g) \end{aligned}$$

This result shows that the Gaussian approximation for the conductance distribution is only valid for $|g - n/2| < n/4$. Away from this region, the conductance distribution exhibits long tails described by the exponential rather than the Gaussian law. Finally, it is straightforward to derive from the Toda lattice Eq. (10) that, in the vicinity $|g - g_*| \leq 1$ of the edges [33] $g_* = 0$ and $g_* = n$, the conductance distribution exhibits even slower, power-law decay [20, 23]

[Osipov and Kanzieper (2008)]

Sketch of derivation (I)

$$\mathcal{P}(G,N) = \frac{1}{Z}\int_0^1 \cdots \int_0^1 dT_1 \cdots dT_N \prod_{j<k} |T_j - T_k|^\beta \prod_{i=1}^N T_i^{\beta/2-1} \delta\left(\sum_{i=1}^N T_i - G\right)$$

Laplace Transform

$$\langle e^{-\frac{\beta}{2}NpG} \rangle$$

Partition Function

$$e^{-\beta \mathcal{H}(\{T_i\};p,N)}$$

$$\int_0^\infty \mathcal{P}(G,N) e^{-\frac{\beta}{2}NpG} dG = \frac{1}{Z}\int_{[0,1]^N} \prod_{i=1}^N dT_i \exp\left(\frac{\beta}{2}\sum_{j\neq k} \log|T_j - T_k| + \left(\frac{\beta}{2}-1\right)\sum_{i=1}^N \log T_i - \frac{\beta}{2}pN\sum_{i=1}^N T_i\right)$$

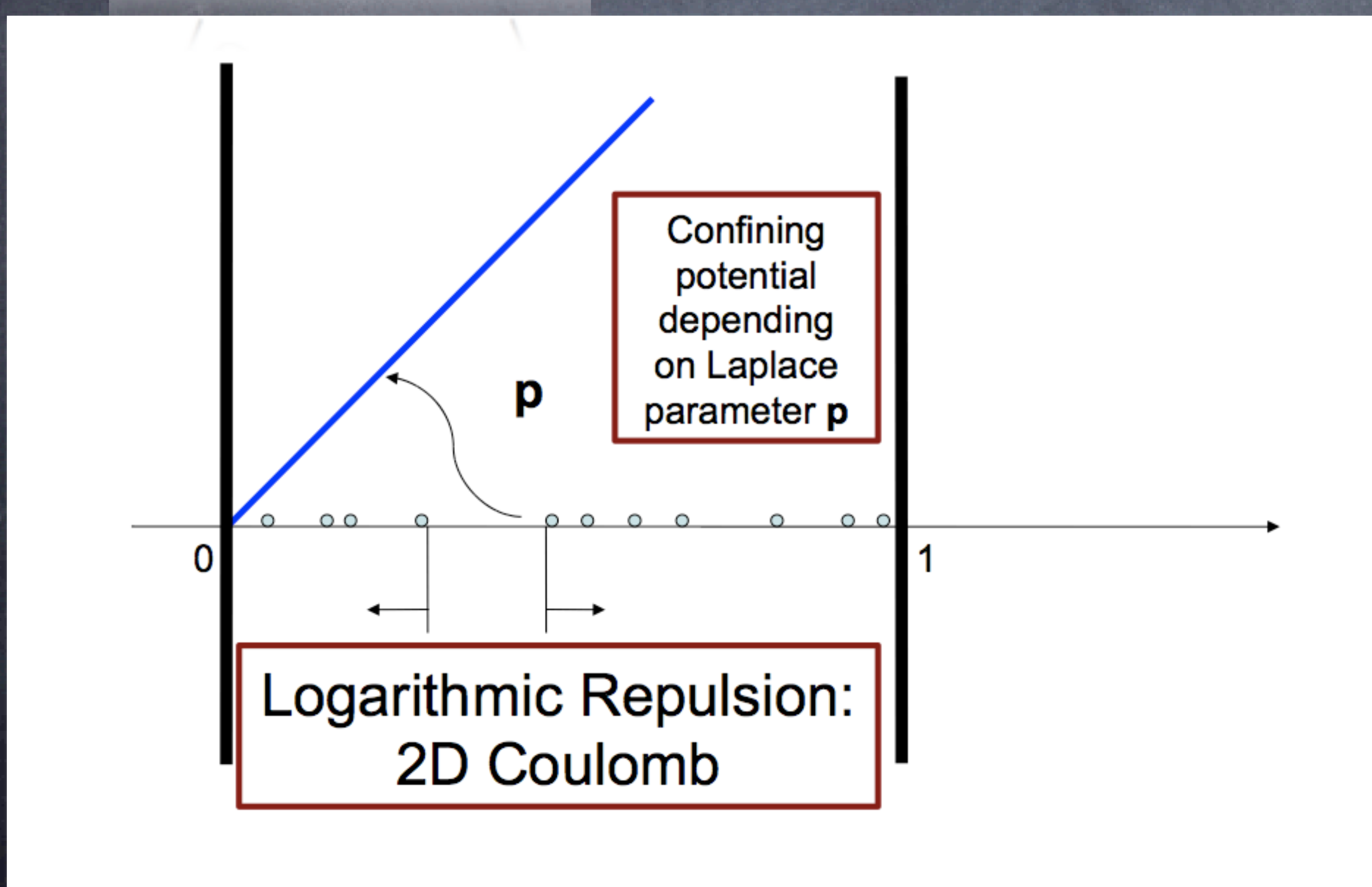
= 'N' $\Rightarrow$ $p \in \mathbb{R}$

Sketch of derivation (II)

$$\int_0^\infty \mathcal{P}(G,N)\, e^{-\frac{\beta}{2}NpG}\, dG = \frac{1}{Z}\int_{[0,1]^N}\prod_{i=1}^N dT_i \exp\left(\frac{\beta}{2}\sum_{j\neq k}\log|T_j-T_k| + \cancel{\left(\frac{\beta}{2}-1\right)\sum_{i=1}^N \log T_i} - \frac{\beta}{2}pN\sum_{i=1}^N T_i\right)$$

$$\langle e^{-\frac{\beta}{2}NpG}\rangle$$

$\mathcal{O}(N^2)$ $\cancel{\mathcal{O}(N)}$ $\mathcal{O}(N^2)$



Mapping: Laplace transform of sought probability --> canonical partition function of an auxiliary problem

Sketch of derivation (III)

Dyson (1962); Dean & Majumdar (2006,2008)

$$\int_{[0,1]^N} \prod_{i=1}^N dT_i \to \int \mathcal{D}[\varrho]$$

$$\sum_{i=1}^N \phi(T_i) \to \int dT\, \varrho(T)\phi(T)$$

$$\int_0^\infty \mathcal{P}(G,N) e^{-\frac{\beta}{2}NpG} dG \simeq \frac{Z_p(N)}{Z_0(N)}$$

$$Z_p(N) \propto \int \mathcal{D}[\varrho_p] e^{-\frac{\beta}{2}N^2 S[\varrho_p]}$$

Saddle Point!

$$S[\varrho_p] = p\int_0^1 \varrho_p(T)\, T dT + B\left[\int_0^1 \varrho_p(T) dT - 1\right] - \int_0^1\int_0^1 dT dT' \varrho_p(T)\varrho_p(T') \log|T-T'|.$$

$$\frac{\delta S[\varrho_p]}{\delta \varrho_p} = 0 \Rightarrow \varrho_p^\star(x)$$

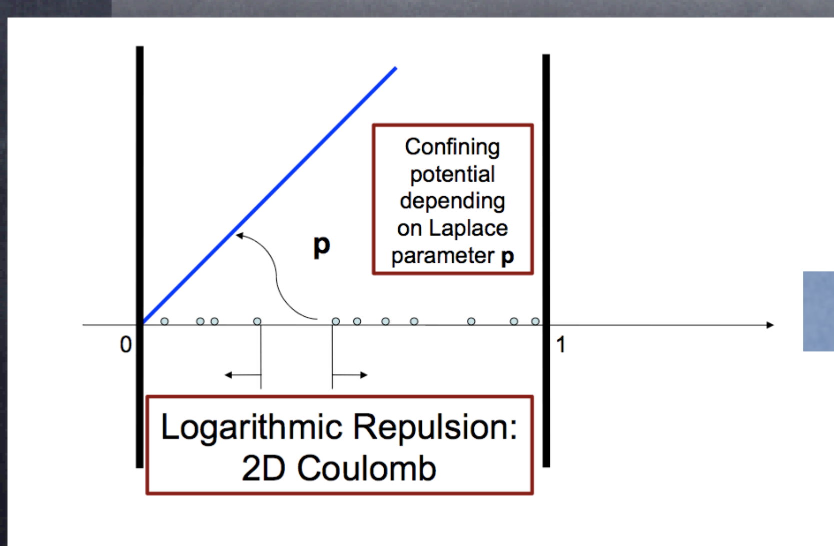
Sketch of derivation (IV)

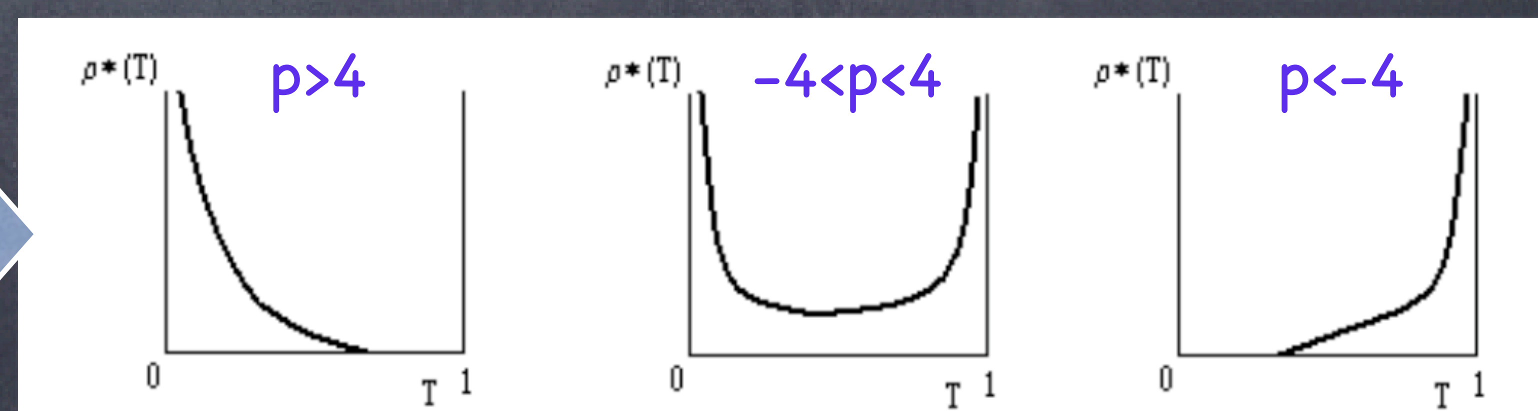
$$\frac{p}{2} = \mathrm{Pr}\int_0^1 \frac{\varrho_p^\star(T')}{T-T'}dT'$$

Inverse Electrostatic Problem

Flowchart:

$$\varrho_p^\star(x) \Rightarrow S[\varrho_p^\star] \Rightarrow \langle e^{-\frac{\beta}{2}NpG}\rangle \approx e^{-\frac{\beta}{2}N^2 \overbrace{[S[\varrho_p^\star]-S[\varrho_0^\star]]}^{J_G(p)}}$$



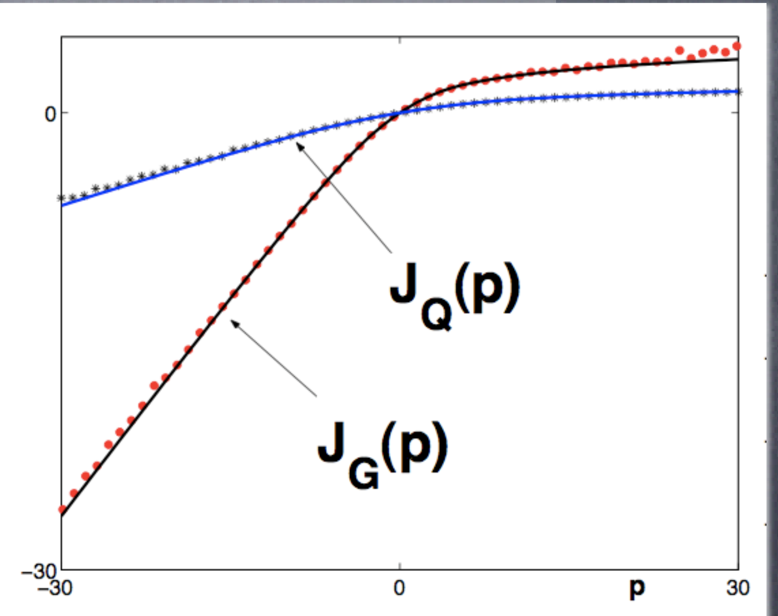


F. G. Tricomi, Integral Equations (1957)

Sketch of derivation (V)

$$\langle e^{-\frac{\beta}{2}NpG}\rangle \approx e^{-\frac{\beta}{2}N^2 \overbrace{\left[S[\varrho_p^\star]-S[\varrho_0^\star]\right]}^{J_G(p)}}$$

$$J_G(p) = \begin{cases} -\frac{p^2}{32}+\frac{p}{2} & -4\le p\le 4 \\ 3/2+\log(p/4) & p\ge 4 \\ 3/2+p+\log(-p/4) & p\le -4 \end{cases}$$

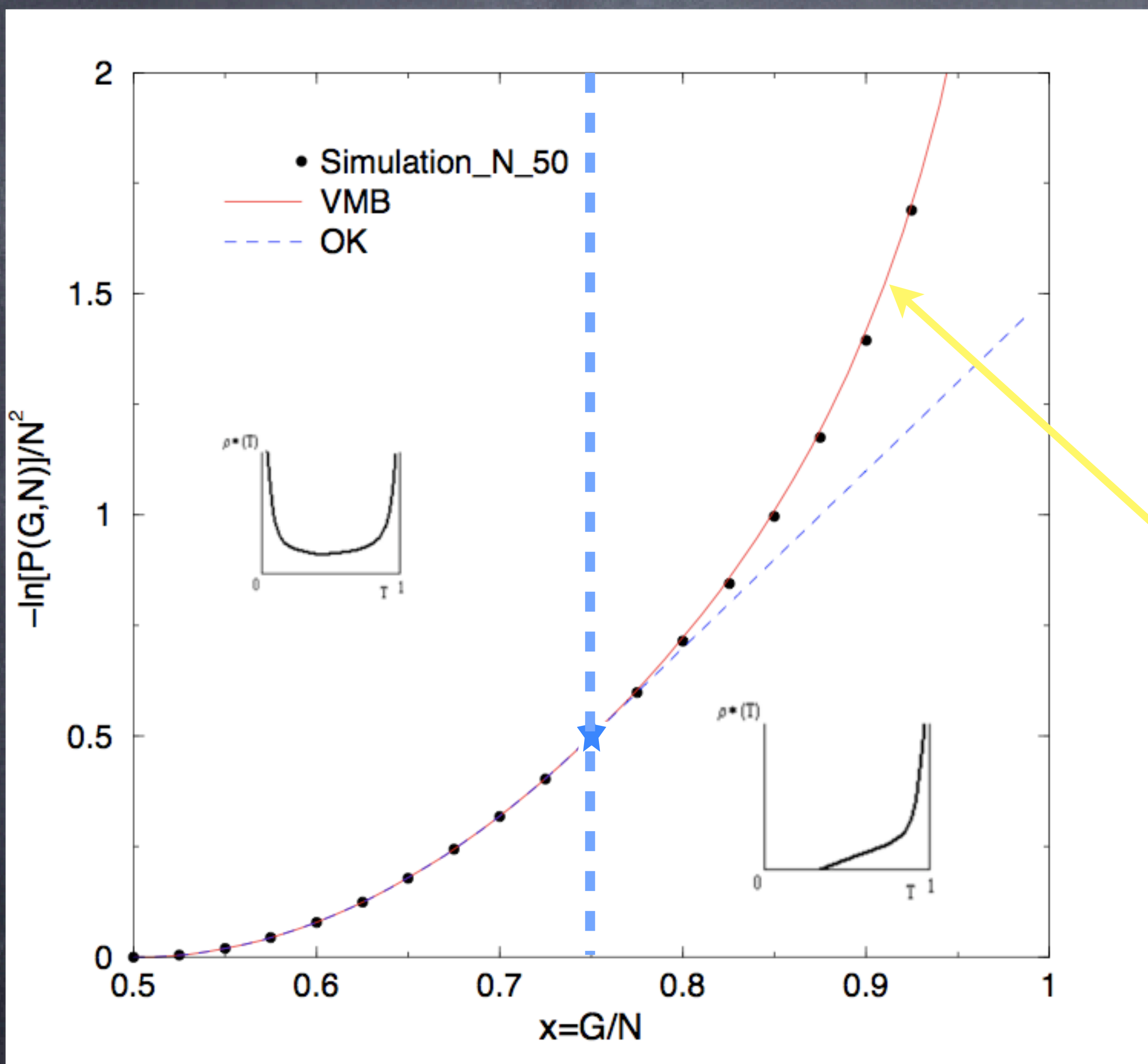


From Laplace space to real space... ---> Gärtner-Ellis Theorem

$$\mathcal{P}(G,N) \approx e^{-\frac{\beta}{2}N^2\Psi_G\left(\frac{G}{N}\right)} \quad \Psi_G(x) = \max_p\left[J_G(p) - p\,x\right]$$

Numerical Simulations

[C. Nadal]



$$\Psi_G(x) = \begin{cases} \frac{1}{2} - \log(4x) & \text{for} \quad 0 \le x \le \frac{1}{4} \\ 8\left(x - \frac{1}{2}\right)^2 & \text{for} \quad \frac{1}{4} \le x \le \frac{3}{4} \\ \frac{1}{2} - \log[4(1-x)] & \text{for} \quad \frac{3}{4} \le x \le 1 \end{cases}$$

Digression: largest Schmidt eigenvalue of entangled random pure states

$$\mathcal{P}(G,N) = \frac{1}{Z}\int_0^1 \cdots \int_0^1 dT_1 \cdots dT_N \prod_{j<k} |T_j - T_k|^\beta \prod_{i=1}^{N} T_i^{\beta/2-1} \delta\left(\sum_{i=1}^{N} T_i - G\right)$$

$$\lambda_i = xT_i \qquad G = 1/x$$

Probability Density

$$\mathcal{P}\left(\frac{1}{x}, N\right) \propto x^\nu Q_N(x)$$

Cumulative Distribution

Schmidt Eigenvalues of Entangled Random Pure States

$$Q_N(x) = \mathrm{Prob}[\lambda_{\max} \leq x]$$

$$\mathcal{P}(\lambda_1, \ldots, \lambda_N) \propto \delta\left(\sum_{i=1}^{N} \lambda_i - 1\right) \prod_{i=1}^{N} \lambda_i^\alpha \prod_{j<k} |\lambda_j - \lambda_k|^\beta$$

Summary

- Random Scattering Matrix approach to quantum transport in chaotic cavities
- Full probability distribution of experimental observables (conductance, shot noise, moments) when number of electronic channels is large
- Solution: canonical partition function of an auxiliary thermodynamical system (Coulomb gas)
- Phase Transitions in the equilibrium gas density

  → Weak non-analytic points in the distribution

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Distributions of Conductance and Shot Noise and Associated Phase Transitions

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Probability distributions of linear statistics in chaotic cavities and associated phase transitions

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FAST TRACK COMMUNICATION

Transmission eigenvalue densities and moments in chaotic cavities from random matrix theory

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Largest Schmidt eigenvalue of entangled random pure states and conductance distribution in chaotic cavities.*

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