# Representations of Lie groups and random matrices

joint work with Benoît Collins

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# Outline

*"big representations of the unitary groups*
*behave like random matrices"*

explanation: *this happens because representation can be viewed as a random matrix (with quantum entries)*

# Representations of $U(d)$

we say that $\Pi$ is a representation of the unitary group $U(d)$
if $\Pi\colon U(d) \to \mathrm{End}(V)$ for some vector space $V$ is such that

$$\Pi(gh) = \Pi(g)\Pi(h),$$

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we say that representation $\Pi$ is reducible
if $V = V_1 \oplus V_2$ and $\Pi = \Pi_1 \oplus \Pi_2$,

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irreducible representations of $U(d)$ are indexed by highest weights:
tuples $\Lambda = (\lambda_1 \geq \cdots \geq \lambda_d)$, where $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}$,

notation: $\epsilon\Lambda = (\epsilon\lambda_1, \ldots, \epsilon\lambda_d)$ for $\epsilon \in \mathbb{R}$

# Representations of $U(d)$

let reducible representation $\Pi$ of $U(d)$ be given

$\Pi$ can be written as a sum of irreducible components

we defne random highest weight associated to $\Pi$ with distribution

$$P(\Lambda) = \frac{(\text{multiplicity of } \Lambda \text{ in } \Pi) \cdot (\text{dimension of } \Lambda)}{(\text{dimension of } \Pi)}$$

# Part 1.

# representation theory of $U(d)$
# $d$ is fixed

# Problem: tensor product of representations

let $\Pi^{(1)}, \Pi^{(2)}$ be irreducible representations of $U(d)$

Kronecker tensor product is a representation $\Pi^{(1)} \otimes \Pi^{(2)}$ of $U(d)$ defined by

$$\left[\Pi^{(1)} \otimes \Pi^{(2)}\right](g) = \left[\Pi^{(1)}(g)\right] \otimes \left[\Pi^{(2)}(g)\right]$$

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$$\Pi^{(1)} \otimes \Pi^{(2)} = ?$$

# Problem: tensor product of representations

let $(\epsilon_n)$ be a sequence of real numbers which converges to zero

let $(\Lambda_n^{(1)})$ and $(\Lambda_n^{(2)})$ be two sequences of highest weights such that

$$\epsilon_n \Lambda_n^{(1)} \to \Lambda^{(1)}, \quad \epsilon_n \Lambda_n^{(2)} \to \Lambda^{(2)}$$

let $(\Pi_n^{(1)})$ and $(\Pi_n^{(2)})$ be irreducible representations of $U(d)$ corresponding to the highest weights $(\Lambda_n^{(1)})$ and $(\Lambda_n^{(2)})$

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let $\Lambda_n^{(3)}$ be the random highest weight associated to $\Pi_n^{(1)} \otimes \Pi_n^{(2)}$

$$\epsilon_n \Lambda_n^{(3)} \to ?$$

# Tensor product of representations: solution

let $A^{(1)}$ and $A^{(2)}$ be independent, unitarily invariant hermitian $d \times d$ random matrices with deterministic eigenvalues $\Lambda^{(1)}$ and $\Lambda^{(2)}$

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**Theorem**

$$\epsilon_n \Lambda_n^{(3)} \xrightarrow{\textit{in distribution}} \textit{eigenvalues of } A^{(1)} + A^{(2)}$$

# Sketch of proof: quantum random variables

matrix algebra $M_k(\mathbb{C})$ can be viewed as
algebra of quantum random variables

mean value $\mathbb{E}X = \frac{1}{k} \operatorname{Tr} X$

if $X_1, X_2, \dots$ are quantum random variables, their joint distribution is a collection of their mixed moments:

$$\left(\mathbb{E} X_{i_1} \cdots X_{i_l}\right)_{i_1,\dots,i_l}$$

# Sketch of proof

a representation of the Lie group $\Pi : U(d) \to \mathrm{End}(V)$
gives a representation of the Lie algebra $\pi : \mathfrak{u}(d) \to \mathrm{End}(V)$

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$$\pi = \begin{bmatrix} \pi(e_{11}) & \cdots & \pi(e_{1d}) \\ \vdots & \ddots & \vdots \\ \pi(e_{d1}) & \cdots & \pi(e_{dd}) \end{bmatrix}$$

can be viewed as a matrix with quantum entries

(spectral measure of $\pi$) $\approx$ (random highest weight $\Lambda$)

# Sketch of proof

a representation of the Lie group $\Pi : U(d) \to \mathrm{End}(V)$
gives a representation of the Lie algebra $\pi : \mathfrak{u}(d) \to \mathrm{End}(V)$

---

$$\epsilon\pi = \begin{bmatrix} \epsilon\pi(e_{11}) & \cdots & \epsilon\pi(e_{1d}) \\ \vdots & \ddots & \vdots \\ \epsilon\pi(e_{d1}) & \cdots & \epsilon\pi(e_{dd}) \end{bmatrix}$$

can be viewed as a matrix with quantum entries

(spectral measure of $\epsilon\pi$) $\approx$ (random highest weight $\epsilon\Lambda$)

# Sketch of proof: asymptotic commutativity

assume that $\epsilon \to 0$ and $\epsilon\pi$ is bounded

$$[\ \pi_{ij},\ \ \pi_{kl}] = \ \left(\delta_{jk}\ \ \pi_{il} - \delta_{li}\ \ \pi_{kj}\right)$$

so $\epsilon\pi$ converges (in distribution) to a matrix with commuting entries

this is the unitarily invariant random matrix with the distribution of eigenvalues given by the random highest weight $\epsilon\Lambda$

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$$\Pi^{(3)} = \Pi^{(1)} \otimes \Pi^{(2)}$$

implies

$$\epsilon\pi^{(3)} = \underbrace{\epsilon\pi^{(1)}}_{\approx A^{(1)}} \otimes 1 + 1 \otimes \underbrace{\epsilon\pi^{(2)}}_{\approx A^{(2)}}$$

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# Sketch of proof: asymptotic commutativity

assume that $\epsilon \to 0$ and $\epsilon\pi$ is bounded

$$[\epsilon\pi_{ij}, \epsilon\pi_{kl}] = \epsilon\left(\delta_{jk}\ \epsilon\pi_{il} - \delta_{li}\ \epsilon\pi_{kj}\right) \to 0$$

so $\epsilon\pi$ converges (in distribution) to a matrix with commuting entries

this is the unitarily invariant random matrix with the distribution of eigenvalues given by the random highest weight $\epsilon\Lambda$

---

$$\Pi^{(3)} = \Pi^{(1)} \otimes \Pi^{(2)}$$

implies

$$\epsilon\pi^{(3)} = \underbrace{\epsilon\pi^{(1)}}_{\approx A^{(1)}} \otimes 1 + 1 \otimes \underbrace{\epsilon\pi^{(2)}}_{\approx A^{(2)}}$$

# It is trivial!

toy example:

decomposition of tensor product of two irreducible representations of $SO(3)$ $\longleftrightarrow$ addition of quantum angular momenta

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classical limit:

$$\hbar \to 0$$

commutators vanish, we recover classical addition of angular momenta

# Part 2.

# representation theory of $U(d)$ $d \to \infty$

# Spectral measure

spectral measure: for $\Lambda_n = (\lambda_1 \geq \cdots \geq \lambda_n)$ we set

$$\mu_{\Lambda_n} = \frac{\delta_{\lambda_1} + \cdots + \delta_{\lambda_n}}{n}$$

if $\Lambda_n$ is a random weight then its spectral measure is a random probability measure on $\mathbb{R}$

in a similar way, spectral measure for random matrices

# Problem: tensor product of representations

let $(\Pi_n^{(1)})$ and $(\Pi_n^{(2)})$ be irreducible representations of $U(n)$ corresponding to the highest weights $(\Lambda_n^{(1)})$ and $(\Lambda_n^{(2)})$

assume that

$$\epsilon_n \Lambda_n^{(1)} \to \Lambda^{(1)}, \quad \epsilon_n \Lambda_n^{(2)} \to \Lambda^{(2)}$$

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let $\Lambda_n^{(3)}$ be the random highest weight associated to $\Pi_n^{(1)} \otimes \Pi_n^{(2)}$

$$\epsilon_n \Lambda_n^{(3)} \xrightarrow{\text{in distribution}} ?$$

# Tensor product of representations: solution

let $A_n^{(1)}$ and $A_n^{(2)}$ be independent, unitarily invariant $n \times n$ hermitian random matrices with deterministic eigenvalues $\Lambda_n^{(1)}$ and $\Lambda_n^{(2)}$

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## Theorem

*assume that $\epsilon_n n \to 0$*

*then*

1. *the spectral measure of $\epsilon_n \Lambda_n^{(3)}$,*
2. *the spectral measure of $A_n^{(1)} + A_n^{(2)}$*

*are asymptotically Gaussian with the same mean and the same global fluctuations*

# Tensor product of representations: solution extended

I claim that if $\mu_n$ is

1. the spectral measure of $\epsilon_n \Lambda_n^{(3)}$,
2. the spectral measure of $A_n^{(1)} + A_n^{(2)}$,

and

$$M_{k,n} = \int x^k d\mu_n, \qquad \left[M_{k,n}\right]_0 = M_{k,n} - \mathbb{E}M_{k,n}$$

then

$$\lim_{n\to\infty} \mathbb{E}M_{k,n} \text{ exists for every } k \geq 1,$$

$$\left(n\left[M_{k,n}\right]_0\right)_{k\geq 1} \text{ converges to a Gaussian distribution}$$

and the limits are the same for both cases

# Sketch of proof

study unitarily invariant random matrices (with quantum entries)

find relationship between

- statistical properties of the spectral measure
- joint distribution of the entries of the matrix

if the non-commutativity of the entries is small, the matrix behaves like a non-quantum random matrix

# Summary / open problems

- representation can be viewed as a random matrix with quantum entries
- (sometimes) the non-commutativity disappears
- asymptotically representation behaves like a usual random matrix

- can we use this idea to prove other connections between representations and random matrix theory?

# Advertisement

Greg Kuperberg.
Random words, quantum statistics, central limits, random matrices.
Methods Appl. Anal., 9(1):99–118, 2002

Benoît Collins, Piotr Śniady.
Representations of Lie groups and random matrices.
Trans. Amer. Math. Soc. 361 (2009), 3269-3287.

Benoît Collins, Piotr Śniady.
Asymptotic fluctuations of representations of the unitary groups.
Preprint arXiv:0911.5546