Representations of Lie groups and random matrices

joint work with Benoît Collins

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Outline

“big representations of the unitary groups

behave like random matrices”

explanation: this happens because representation can be viewed

as a random matrix (with quantum entries)
Representations of $U(d)$

we say that $\Pi$ is a representation of the unitary group $U(d)$ if $\Pi: U(d) \rightarrow \text{End}(V)$ for some vector space $V$ is such that

$$\Pi(gh) = \Pi(g)\Pi(h),$$

we say that representation $\Pi$ is reducible if $V = V_1 \oplus V_2$ and $\Pi = \Pi_1 \oplus \Pi_2$,

irreducible representations of $U(d)$ are indexed by highest weights: tuples $\Lambda = (\lambda_1 \geq \cdots \geq \lambda_d)$, where $\lambda_1, \ldots, \lambda_d \in \mathbb{Z}$,

notation: $\epsilon\Lambda = (\epsilon\lambda_1, \ldots, \epsilon\lambda_d)$ for $\epsilon \in \mathbb{R}$
Representations of $U(d)$

Let reducible representation $\Pi$ of $U(d)$ be given.

$\Pi$ can be written as a sum of irreducible components.

We define random highest weight associated to $\Pi$ with distribution

$$P(\Lambda) = \frac{\text{multiplicity of } \Lambda \text{ in } \Pi \cdot \text{dimension of } \Lambda}{\text{dimension of } \Pi}$$
Part 1. representation theory of $U(d)$
$d$ is fixed
Problem: tensor product of representations

let \( \Pi^{(1)}, \Pi^{(2)} \) be irreducible representations of \( U(d) \)

Kronecker tensor product is a representation \( \Pi^{(1)} \otimes \Pi^{(2)} \) of \( U(d) \) defined by

\[
[\Pi^{(1)} \otimes \Pi^{(2)}](g) = [\Pi^{(1)}(g)] \otimes [\Pi^{(2)}(g)]
\]

\( \Pi^{(1)} \otimes \Pi^{(2)} = ? \)
Problem: tensor product of representations

let \((\epsilon_n)\) be a sequence of real numbers which converges to zero

let \((\Lambda_n^{(1)})\) and \((\Lambda_n^{(2)})\) be two sequences of highest weights such that

\[ \epsilon_n \Lambda_n^{(1)} \to \Lambda^{(1)}, \quad \epsilon_n \Lambda_n^{(2)} \to \Lambda^{(2)} \]

let \((\Pi_n^{(1)})\) and \((\Pi_n^{(2)})\) be irreducible representations of \(U(d)\) corresponding to the highest weights \((\Lambda_n^{(1)})\) and \((\Lambda_n^{(2)})\)

let \(\Lambda_n^{(3)}\) be the random highest weight associated to \(\Pi_n^{(1)} \otimes \Pi_n^{(2)}\)

\[ \epsilon_n \Lambda_n^{(3)} \to ? \]
Tensor product of representations: solution

Let $A^{(1)}$ and $A^{(2)}$ be independent, unitarily invariant hermitian $d \times d$ random matrices with deterministic eigenvalues $\Lambda^{(1)}$ and $\Lambda^{(2)}$.
Sketch of proof: quantum random variables

matrix algebra $M_k(\mathbb{C})$ can be viewed as algebra of quantum random variables

mean value $\mathbb{E}X = \frac{1}{k} \text{Tr} X$

if $X_1, X_2, \ldots$ are quantum random variables, their joint distribution is a collection of their mixed moments:

$\left(\mathbb{E}X_{i_1} \cdots X_{i_l}\right)_{i_1,\ldots,i_l}$
Sketch of proof

A representation of the Lie group $\Pi : U(d) \to \text{End}(V)$ gives a representation of the Lie algebra $\pi : u(d) \to \text{End}(V)$

\[
\pi = \begin{bmatrix}
\pi(e_{11}) & \cdots & \pi(e_{1d}) \\
\vdots & \ddots & \vdots \\
\pi(e_{d1}) & \cdots & \pi(e_{dd})
\end{bmatrix}
\]

can be viewed as a matrix with quantum entries

(spectral measure of $\pi$) $\approx$ (random highest weight $\Lambda$)
Sketch of proof

A representation of the Lie group $\Pi : U(d) \to \text{End}(V)$ gives a representation of the Lie algebra $\pi : u(d) \to \text{End}(V)$

$$
\begin{bmatrix}
\epsilon\pi(e_{11}) & \cdots & \epsilon\pi(e_{1d}) \\
\vdots & \ddots & \vdots \\
\epsilon\pi(e_{d1}) & \cdots & \epsilon\pi(e_{dd})
\end{bmatrix}
$$

can be viewed as a matrix with quantum entries

(spectral measure of $\epsilon\pi$) $\approx$ (random highest weight $\epsilon\Lambda$)
Sketch of proof: asymptotic commutativity

assume that $\epsilon \to 0$ and $\epsilon\pi$ is bounded

$$[\pi_{ij}, \pi_{kl}] = (\delta_{jk} \pi_{il} - \delta_{li} \pi_{kj})$$

so $\epsilon\pi$ converges (in distribution) to a matrix with commuting entries

this is the unitarily invariant random matrix with the distribution of eigenvalues given by the random highest weight $\epsilon\Lambda$

$$\Pi^{(3)} = \Pi^{(1)} \otimes \Pi^{(2)}$$

implies

$$\epsilon\pi^{(3)} \approx \epsilon\pi^{(1)} \otimes 1 + 1 \otimes \epsilon\pi^{(2)} \approx A^{(1)} + A^{(2)}$$
Sketch of proof: asymptotic commutativity

Assume that $\epsilon \to 0$ and $\epsilon \pi$ is bounded

$$[\epsilon \pi_{ij}, \epsilon \pi_{kl}] = \epsilon \left( \delta_{jk} \epsilon \pi_{il} - \delta_{li} \epsilon \pi_{kj} \right)$$

So $\epsilon \pi$ converges (in distribution) to a matrix with commuting entries.

This is the unitarily invariant random matrix with the distribution of eigenvalues given by the random highest weight $\epsilon \Lambda$

$$\Pi^{(3)} = \Pi^{(1)} \otimes \Pi^{(2)}$$

Implies

$$\epsilon \pi^{(3)} = \epsilon \pi^{(1)} \otimes 1 + 1 \otimes \epsilon \pi^{(2)} \approx A^{(1)} + A^{(2)}$$
Sketch of proof: asymptotic commutativity

assume that $\epsilon \to 0$ and $\epsilon \pi$ is bounded

$$[\epsilon \pi_{ij}, \epsilon \pi_{kl}] = \epsilon \left( \delta_{jk} \epsilon \pi_{il} - \delta_{li} \epsilon \pi_{kj} \right) \to 0$$

so $\epsilon \pi$ converges (in distribution) to a matrix with commuting entries

this is the unitarily invariant random matrix with the distribution of eigenvalues given by the random highest weight $\epsilon \Lambda$

$$\Pi^{(3)} = \Pi^{(1)} \otimes \Pi^{(2)}$$

implies

$$\epsilon \pi^{(3)} = \underbrace{\epsilon \pi^{(1)}}_{\approx A^{(1)}} \otimes 1 + 1 \otimes \underbrace{\epsilon \pi^{(2)}}_{\approx A^{(2)}}$$
It is trivial!

toy example:

decomposition of tensor product of two irreducible representations of $SO(3)$

$\longleftrightarrow$

addition of quantum angular momenta

classical limit:

$\hbar \rightarrow 0$

commutators vanish, we recover classical addition of angular momenta
Part 2.

representation theory of $U(d)$

$d \rightarrow \infty$
Spectral measure

spectral measure: for $\Lambda_n = (\lambda_1 \geq \cdots \geq \lambda_n)$ we set

$$\mu_{\Lambda_n} = \frac{\delta_{\lambda_1} + \cdots + \delta_{\lambda_n}}{n}$$

if $\Lambda_n$ is a random weight then its spectral measure is a random probability measure on $\mathbb{R}$

in a similar way, spectral measure for random matrices
Problem: tensor product of representations

let \((\Pi_n^{(1)})\) and \((\Pi_n^{(2)})\) be irreducible representations of \(U(n)\) corresponding to the highest weights \((\Lambda_n^{(1)})\) and \((\Lambda_n^{(2)})\)

assume that

\[\epsilon_n \Lambda_n^{(1)} \rightarrow \Lambda^{(1)}, \quad \epsilon_n \Lambda_n^{(2)} \rightarrow \Lambda^{(2)}\]

let \(\Lambda_n^{(3)}\) be the random highest weight associated to \(\Pi_n^{(1)} \otimes \Pi_n^{(2)}\)

\[\epsilon_n \Lambda_n^{(3)} \text{ in distribution} \rightarrow ?\]
Tensor product of representations: solution

let $A_n^{(1)}$ and $A_n^{(2)}$ be independent, unitarily invariant $n \times n$ hermitian random matrices with deterministic eigenvalues $\Lambda_n^{(1)}$ and $\Lambda_n^{(2)}$

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**Theorem**

*assume that $\epsilon_n n \to 0$ then

1. the spectral measure of $\epsilon_n \Lambda_n^{(3)}$,
2. the spectral measure of $A_n^{(1)} + A_n^{(2)}$

are asymptotically Gaussian with the same mean and the same global fluctuations*
Tensor product of representations: solution extended

I claim that if $\mu_n$ is

1. the spectral measure of $\epsilon_n \Lambda_n^{(3)}$,
2. the spectral measure of $A_n^{(1)} + A_n^{(2)}$,

and

$$M_{k,n} = \int x^k d\mu_n, \quad \left[ M_{k,n} \right]_0 = M_{k,n} - \mathbb{E}M_{k,n}$$

then

$$\lim_{n \to \infty} \mathbb{E}M_{k,n} \text{ exists for every } k \geq 1,$$

$$\left( n \left[ M_{k,n} \right]_0 \right)_{k \geq 1} \text{ converges to a Gaussian distribution}$$

and the limits are the same for both cases.
Sketch of proof

study unitarily invariant random matrices (with quantum entries)

find relationship between

- statistical properties of the spectral measure
- joint distribution of the entries of the matrix

if the non-commutativity of the entries is small, the matrix behaves like a non-quantum random matrix
Summary / open problems

- representation can be viewed as a random matrix with quantum entries
- (sometimes) the non-commutativity disappears
- asymptotically representation behaves like a usual random matrix

- can we use this idea to prove other connections between representations and random matrix theory?
Greg Kuperberg.  
Random words, quantum statistics, central limits, random matrices.  

Benoît Collins, Piotr Śniady.  
Representations of Lie groups and random matrices.  

Benoît Collins, Piotr Śniady.  
Asymptotic fluctuations of representations of the unitary groups.  
Preprint arXiv:0911.5546