For matrix A $(p \times p)$ with real eigenvalues, define F^A, the empirical distribution function of the eigenvalues of A, to be

$$F^A(x) \equiv (1/p) \cdot (\text{number of eigenvalues of } A \leq x).$$

For and p.d.f. G the Stieltjes transform of G is defined as

$$m_G(z) \equiv \int \frac{1}{\lambda - z} dG(\lambda), \quad z \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} : \Im z > 0\}.$$

Inversion formula

$$G\{[a,b]\} = (1/\pi) \lim_{\eta \to 0^+} \int_a^b \Im\, m_G(\xi + i\eta) d\xi$$

(a, b continuity points of G).

Notice

$$m_{F^A}(z) = (1/p) \mathsf{tr}\,(A - zI)^{-1}.$$

Theorem [S. (1995)]. Assume

a) For $n = 1, 2, \ldots$ $X_n = (X_{ij}^n)$, $n \times N$, $X_{ij}^n \in \mathbb{C}$, i.d. for all n, i, j, independent across i, j for each n, $\mathsf{E}|X_{1\,1}^1 - \mathsf{E}X_{1\,1}^1|^2 = 1$.

b) $N = N(n)$ with $n/N \to c > 0$ as $n \to \infty$.

c) T_n $n \times n$ random Hermitian nonnegative definite, with F^{T_n} converging almost surely in distribution to a p.d.f. H on $[0, \infty)$ as $n \to \infty$.

d) X_n and T_n are independent.

Let $T_n^{1/2}$ be the Hermitian nonnegative square root of T_n, and let $B_n = (1/N)T_n^{1/2}X_nX_n^*T_n^{1/2}$ (obviously $F^{B_n} = F^{(1/N)X_nX_n^*T_n}$). Then, almost surely, F^{B_n} converges in distribution, as $n \to \infty$, to a (nonrandom) p.d.f. F, whose Stieltjes transform $m(z)$ ($z \in \mathbb{C}^+$) satisfies

$$(*) \qquad m = \int \frac{1}{t(1 - c - czm) - z} dH(t),$$

in the sense that, for each $z \in \mathbb{C}^+$, $m = m(z)$ is the unique solution to $(*)$ in $\{m \in \mathbb{C} : -\frac{1-c}{z} + cm \in \mathbb{C}^+\}$.

We have

$$F^{(1/N)X^*TX} = (1 - \frac{n}{N})I_{[0,\infty)} + \frac{n}{N}F^{(1/N)XX^*T}$$
$$\xrightarrow{a.s.} (1-c)I_{[0,\infty)} + cF \equiv \underline{F}.$$

Notice m_F and $m_{\underline{F}}$ satisfy

$$\frac{1-c}{cz} + \frac{1}{c}m_{\underline{F}}(z) = m_F(z) = \int \frac{1}{-zm_{\underline{F}}t - z}dH(t).$$

Therefore, $\underline{m} = m_{\underline{F}}$ solves

$$z = -\frac{1}{\underline{m}} + c\int \frac{t}{1 + t\underline{m}}dH(t).$$

Facts on F:

1. The endpoints of the connected components (away from 0) of the support of F are given by the extrema of

$$f(\underline{m}) = -\frac{1}{\underline{m}} + c\int \frac{t}{1+t\underline{m}}dH(t) \quad \underline{m} \in \mathbb{R}$$

   [Marčenko and Pastur (1967), S. and Choi (1995)].

2. F has a continuous density away from the origin given by

$$\frac{1}{c\pi}\Im\underline{m}(x) \quad 0 < x \in \text{ support of } F$$

   where

$$\underline{m}(x) = \lim_{z\in\mathbb{C}^+\to x} m_{\underline{F}}(z)$$

   solves

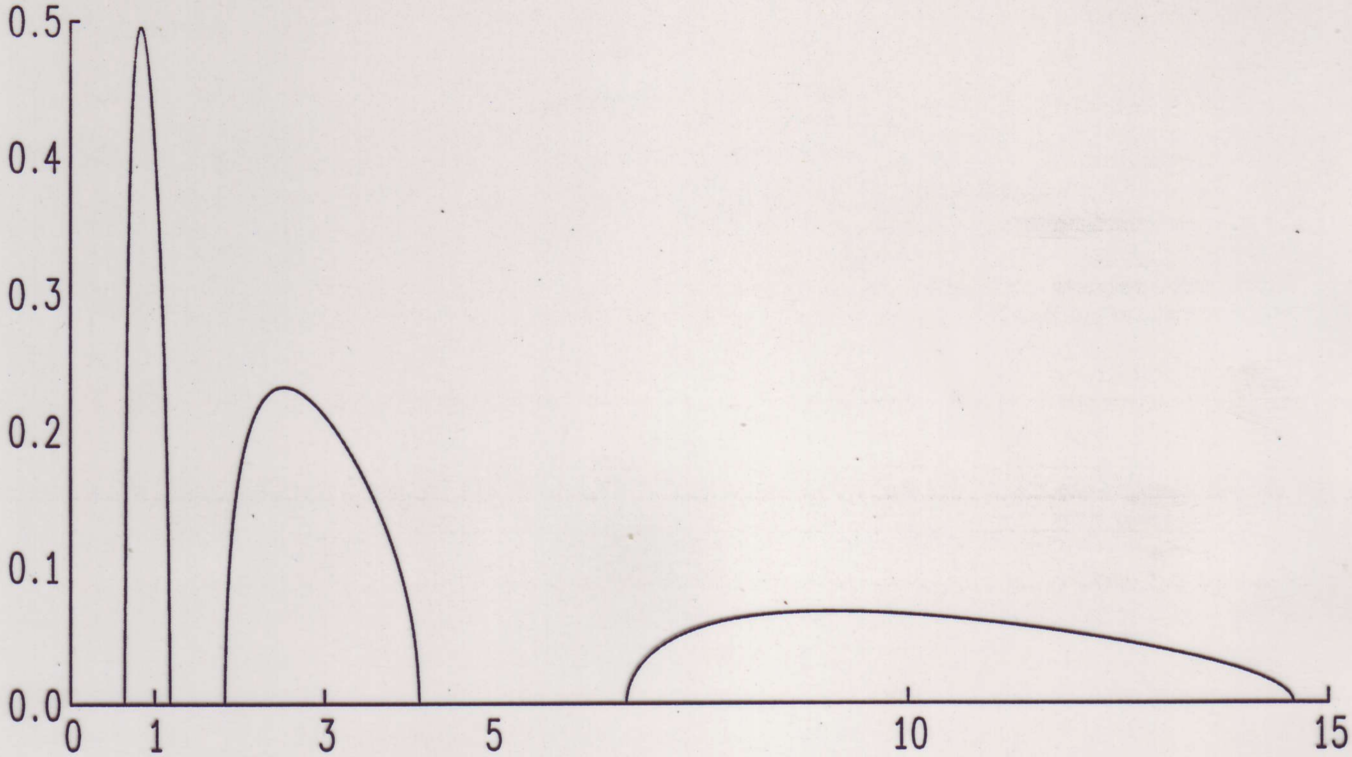
$$x = -\frac{1}{\underline{m}} + c\int \frac{t}{1+t\underline{m}}dH(t).$$

   (S. and Choi 1995).

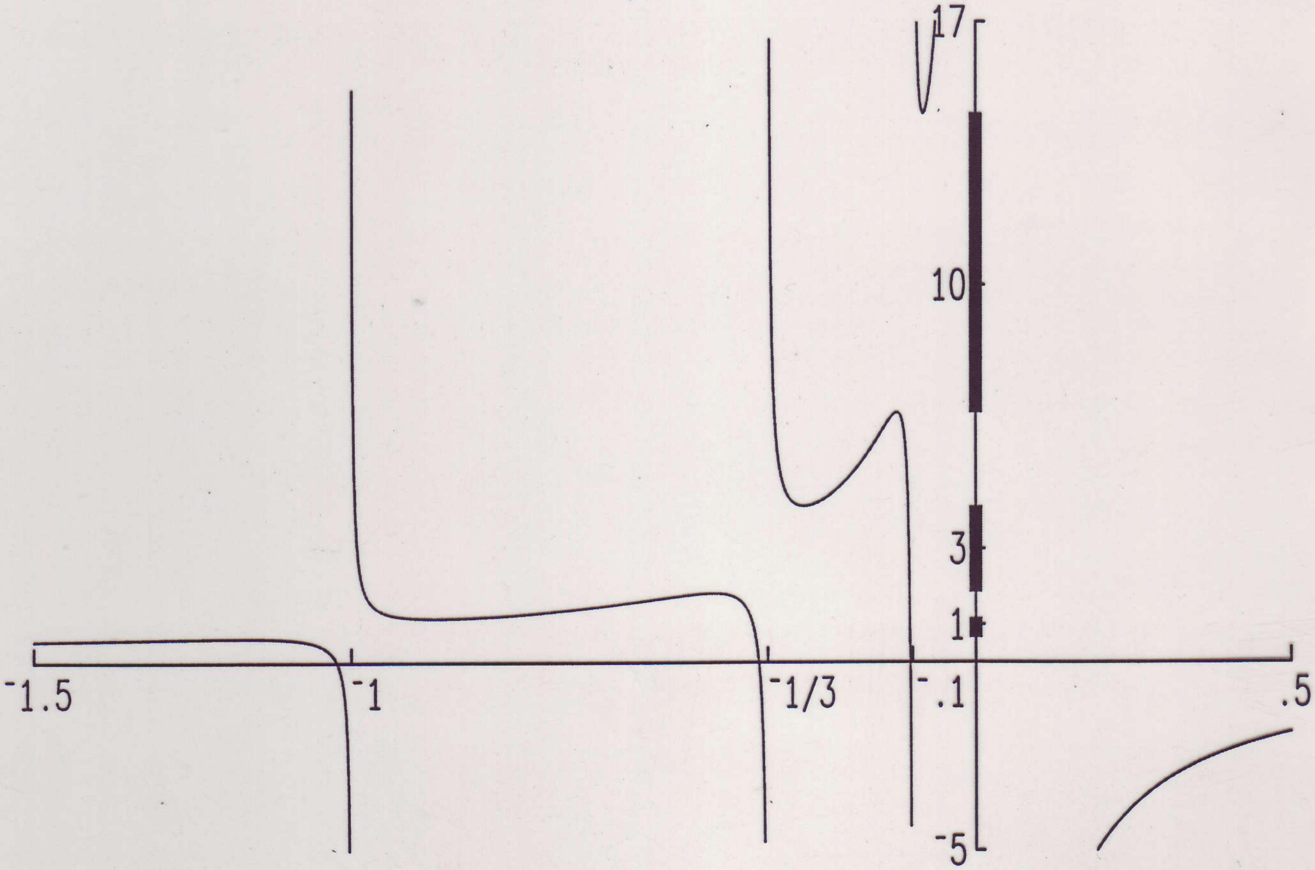
3. F' is analytic inside its support, and when H is discrete, has infinite slopes at boundaries of its support [S. and Choi (1995)].

4. c and F uniquely determine H.

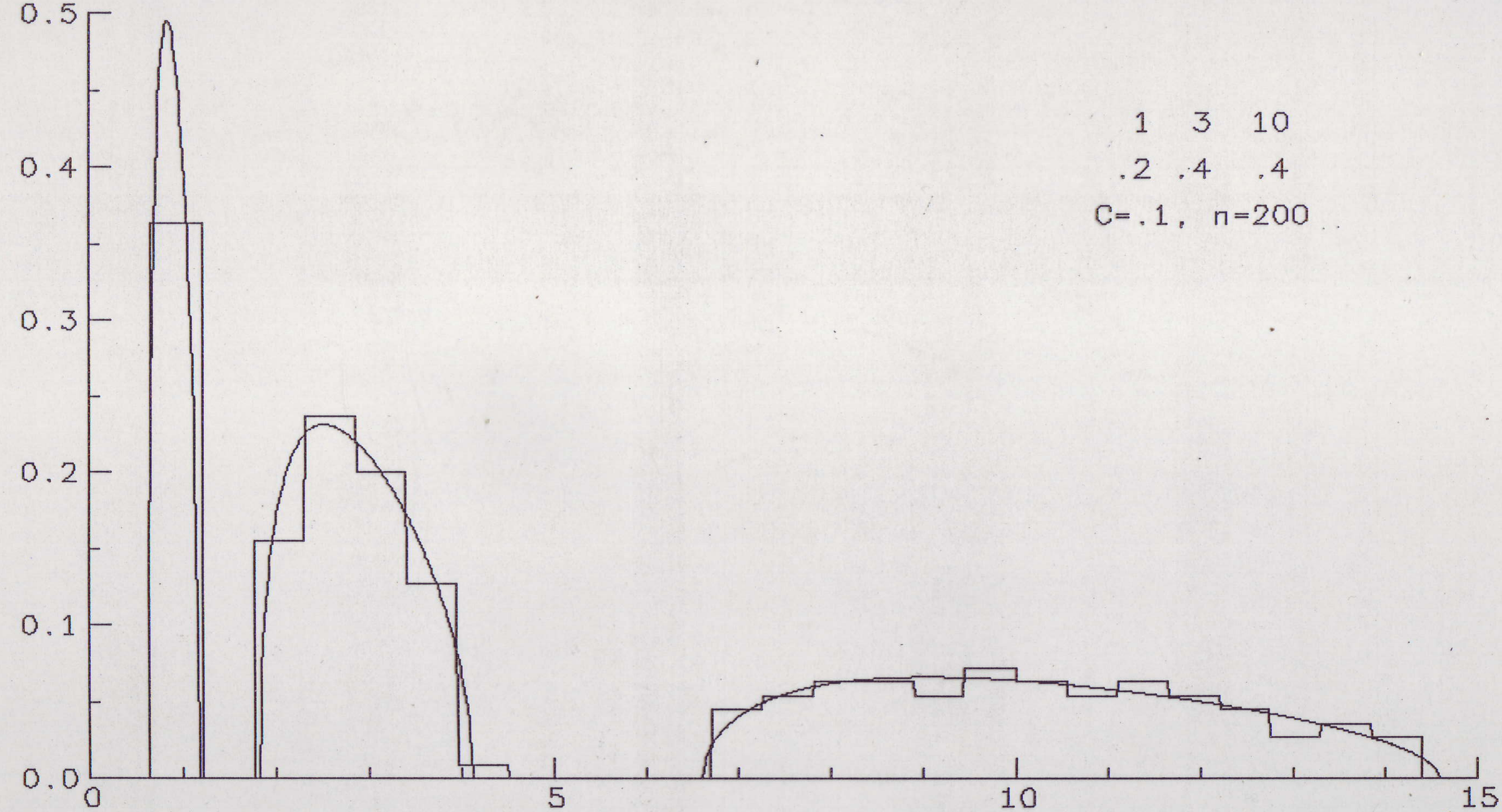
5. $F \xrightarrow{D} H$ as $c \to 0$ (complements $B_n \xrightarrow{a.s.} T_n$ as $N \to \infty$, n fixed).



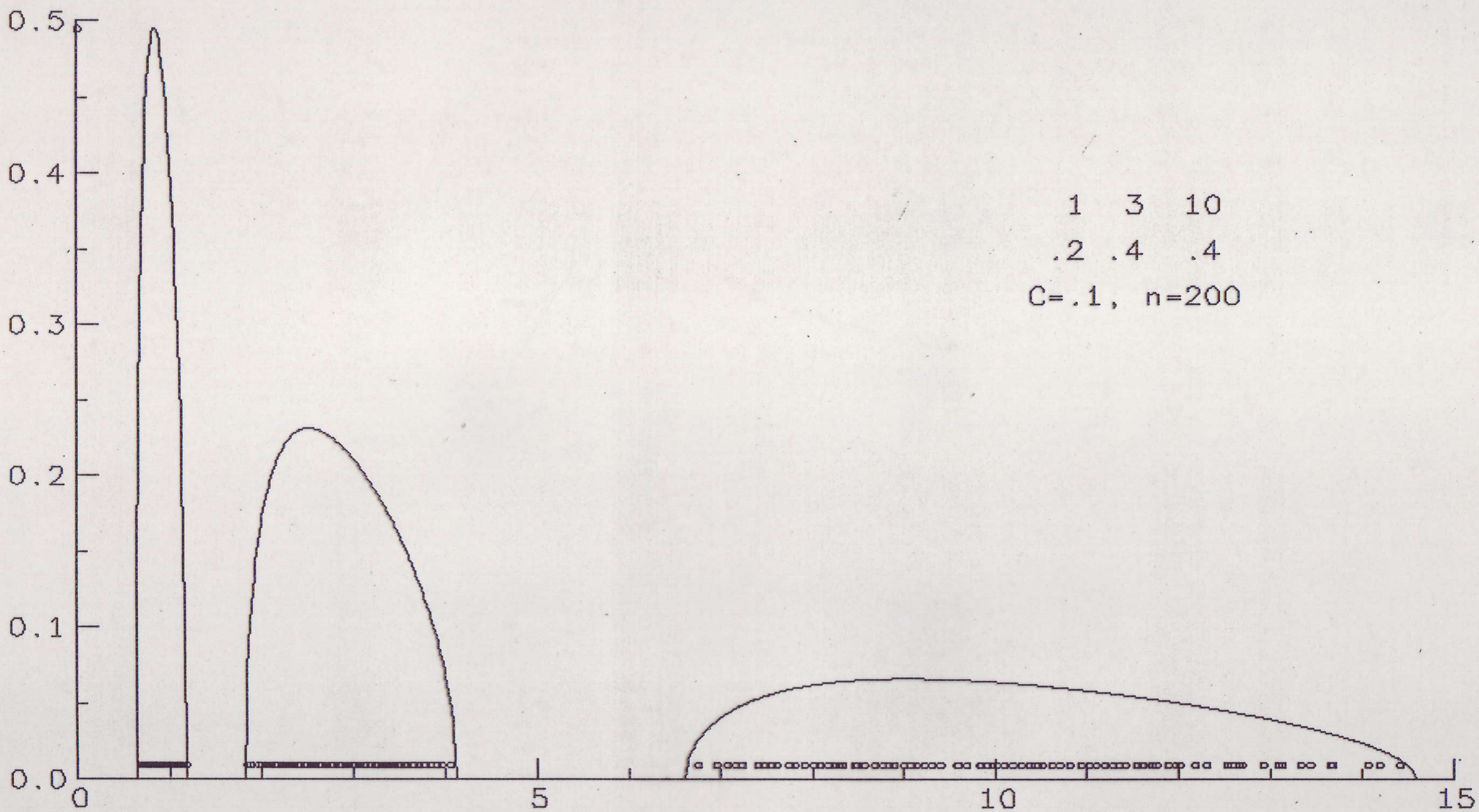
(a)



(b)



0.5
0.4
0.3
0.2
0.1
0.0
0
5
10
15
1 3 10
.2 .4 .4
C=.1, n=200



0.5
0.4
0.3
0.2
0.1
0.0
0
5
10
15
1 3 10
.2 .4 .4
C=.1, n=200

$T_n = I_n \Longrightarrow F = F_c$, where, for $0 < c \le 1$, $F'_c(x) = f_c(x) =$

$$\frac{1}{2\pi c x}\sqrt{(x-b_1)(b_2-x)} \quad b_1 < x < b_2,$$

0 otherwise, where

$$b_1 = (1-\sqrt{c})^2, \quad b_2 = (1+\sqrt{c})^2,$$

and for $1 < c < \infty$,

$$F_c(x) = (1-(1/c))I_{[0,\infty)}(x) + \int_{b_1}^{x} f_c(t)dt.$$

Marčenko and Pastur (1967)

Grenander and S. (1977)

Multivariate F matrix: $T_n = ((1/N')\underline{X}_n\underline{X}_n^*)^{-1}$, $\underline{X}_n$ $n \times N'$ containing i.i.d. standardized entries, $n/N' \to c' \in (0,1) \Longrightarrow F = F_{c,c'}$,where, for $0 < c \le 1$, $F'_{c,c'}(x) = f_{c,c'}(x) =$

$$\frac{(1-c')\sqrt{(x-b_1)(b_2-x)}}{2\pi x(xc'+c)} \quad b_1 < x < b_2,$$

where

$$b_1 = \left[\frac{1-\sqrt{1-(1-c)(1-c')}}{1-c'}\right]^2, \quad b_2 = \left[\frac{1+\sqrt{1-(1-c)(1-c')}}{1-c'}\right]^2,$$

and for $1 < c < \infty$,

$$F_{c,c'}(x) = (1-(1/c))I_{[0,\infty)}(x) + \int_{b_1}^{x} f_{c,c'}(t)dt.$$

S. (1985)

Let, for any $d > 0$ and d.f. G, $F^{d,G}$ denote the limiting spectral d.f. of $(1/N)X^*TX$ corresponding to limiting ratio d and limiting F^{T_n} G.

Theorem [Bai and S. (1998)]. Assume:

a) X_{ij}, $i,j = 1,2,...$ are i.i.d. random variables in $\mathbb{C}$ with $\mathsf{E}X_{11} = 0$, $\mathsf{E}|X_{11}|^2 = 1$, and $\mathsf{E}|X_{11}|^4 < \infty$.

b) $N = N(n)$ with $c_n = n/N \to c > 0$ as $n \to \infty$.

c) For each n T_n is an $n \times n$ Hermitian nonnegative definite satisfying $H_n \equiv F^{T_n} \xrightarrow{D} H$, a p.d.f.

d) $\|T_n\|$, the spectral norm of T_n is bounded in n.

e) $B_n = (1/N)T_n^{1/2}X_nX_n^*T_n^{1/2}$, $T_n^{1/2}$ any Hermitian square root of T_n, $\underline{B}_n = (1/N)X_n^*T_nX_n$, where $X_n = (X_{ij})$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,N$.

f) The interval $[a,b]$ with $a > 0$ lies in an open interval outside the support of F^{c_n,H_n} for all large n.

Then

$$\mathsf{P}(\text{ no eigenvalue of } B_n \text{ appears in } [a,b] \text{ for all large } n\ ) = 1.$$

Theorem [Bai and S. (1999)]. Assume (a)–(f) of the previous theorem.

1) If $c[1 - H(0)] > 1$, then x_0, the smallest value in the support of $F^{c,H}$, is positive, and with probability one $\lambda_N^{B_n} \to x_0$ as $n \to \infty$. The number x_0 is the maximum value of the function

$$z(m) = -\frac{1}{m} + c \int \frac{t}{1+tm} dH(t)$$

for $m \in \mathbb{R}^+$.

2) If $c[1 - H(0)] \leq 1$, or $c[1 - H(0)] > 1$ but $[a, b]$ is not contained in $[0, x_0]$ then $m_{F^{c,H}}(b) < 0$. Let for large n integer $i_n \geq 0$ be such that

$$\lambda_{i_n}^{T_n} > -1/m_{F^{c,H}}(b) \qquad \text{and} \qquad \lambda_{i_n+1}^{T_n} < -1/m_{F^{c,H}}(a)$$

(eigenvalues arranged in non-increasing order). Then

$$\mathsf{P}(\lambda_{i_n}^{B_n} > b \quad \textit{and} \quad \lambda_{i_n+1}^{B_n} < a \quad \text{for all large } n\ ) = 1.$$

From the work of X. Mestre (2008):

For fixed n, N, and $H_n = F^{T_n}$, let $\underline{m} = \underline{m}(z) = m_{F^{c_n, H_n}}(z)$. Then

$$\begin{aligned} z = z(\underline{m}) &= -\frac{1}{\underline{m}} + c_n \int \frac{t}{1 + t\underline{m}} dH_n(t) \\ &= \frac{1}{\underline{m}}(c_n - 1) - \frac{c_n}{\underline{m}^2} \int \frac{1}{t + \frac{1}{\underline{m}}} dH_n(t) \\ &= \frac{1}{\underline{m}}(c_n - 1) - \frac{c_n}{\underline{m}^2} m_{H_n}(-\tfrac{1}{\underline{m}}). \end{aligned}$$

Suppose T_n has positive eigenvalue t_1 with multiplicity n_1. Then on any contour in $\mathbb{C}$ positively oriented, encircling only eigenvalue t_1 of T_n we have

$$-\frac{n}{n_1} \frac{1}{2\pi i} \oint y m_{H_n}(y) dy = -\frac{n}{n_1} \frac{1}{2\pi i} \oint y \int \frac{1}{\lambda - y} dH_n(\lambda) dy$$

$$= \frac{n}{n_1} \frac{1}{2\pi i} \int \oint \frac{y}{y - \lambda} dy dH_n(\lambda) = \frac{n}{n_1} \int_{\{t_1\}} \lambda dH_n(\lambda) = t_1.$$

Substitute $\underline{m} = -\frac{1}{y}$. Then

$$\begin{aligned} t_1 &= \frac{n}{n_1}\frac{1}{2\pi i}\oint \frac{1}{\underline{m}} m_{H_n}(-\tfrac{1}{\underline{m}})\frac{1}{\underline{m}^2} d\underline{m} \\ &= \frac{n}{n_1}\frac{1}{c_n}\frac{1}{2\pi i}\oint \frac{1}{\underline{m}}\left(\frac{1}{\underline{m}}(c_n - 1) - z(\underline{m})\right) d\underline{m} \\ &= -\frac{N}{n_1}\frac{1}{2\pi i}\oint \frac{z(\underline{m})}{\underline{m}} d\underline{m}, \end{aligned}$$

the contour contained in the negative real portion of $\mathbb{C}$, encircling $-\frac{1}{t_1}$ and no other $-\frac{1}{t_j}$, t_j an eigenvalue of T_n.
Suppose exact separation occurs for the eigenvalues of B_n for all n large, associated with t_1. Then the contour can be chosen so that it intersects the real line at two places $\underline{m}_a < \underline{m}_b$ for which $x_a = z(\underline{m}_a)$ and $x_b = z(\underline{m}_b)$ are outside the support of F^{c_n,H_n}, and $[x_a, x_b]$ contains only the support of F^{c_n,H_n} associated with t_1. Then, with substitution $\underline{m} = \underline{m}(z)$ we have

$$t_1 = -\frac{N}{n_1}\frac{1}{2\pi i}\oint \frac{z\underline{m}'(z)}{\underline{m}(z)} dz,$$

the contour, $\mathcal{C}$, only containing the support of F^{c_n,H_n} associated with t_1.

Let $\underline{m}_n = m_{F^{(1/N)X_n^* T_n X_n}}$. We have, with probability 1,

$$\sup_{z\in\mathcal{C}} \max |\underline{m}(z) - \underline{m}_n(z)|, |\underline{m}'(z) - \underline{m}_n'(z)| \to 0,$$

as $n \to \infty$. Thus

$$-\frac{N}{n_1}\frac{1}{2\pi i}\oint \frac{z\underline{m}_n'(z)}{\underline{m}_n(z)}dz$$

can be taken as an estimate of t_1. This quantity equals

$$\frac{N}{n_1}\left(\sum_{\lambda_j\in[x_a,x_b]}\lambda_j - \sum_{\mu_j\in[x_a,x_b]}\mu_j\right),$$

where λ_j's are the eigenvalues of B_n, μ_j's are the zeros of $\underline{m}_n(z)$. We have

$$\underline{m}_n(z) = \frac{1}{N}\sum_{j=1}^{n}\frac{1}{\lambda_j - z} + \frac{N-n}{N}\frac{1}{-z} = 0$$

$$\Longleftrightarrow \frac{1}{N}\sum_{j=1}^{n}\frac{\lambda_j}{\lambda_j - z} = 1.$$

The solutions are the eigenvalues of the matrix

$$\mathrm{Diag}(\lambda_1,\ldots,\lambda_n) - N^{-1}ss^*,$$

where $s = (\sqrt{\lambda_1},\ldots,\sqrt{\lambda_n})^*$.

Population eigenvalues	1	3	10
Estimates	.9946	2.9877	10.0365

Theorem [Bai, S. (2009)]. Replace assumption a) in S. (1995) with: For $n = 1, 2, \ldots$ $X_n = (X_{ij}^n)$, $n \times N$, $X_{ij}^n \in \mathbb{C}$ are independent with common mean, unit variance, such that for any $\eta > 0$

$$\frac{1}{\eta^2 nN} \sum_{ij} \mathsf{E}(|X_{ij}^n|^2 I(|X_{ij}^n| \geq \eta\sqrt{n})) \to 0$$

as $n \to \infty$. Then the conclusion of S. (1995) remains true.

Theorem [Couillet, S., Bai, Debbah (to appear in *IEEE Transactions on Information Theory*)]. Replace assumption a) in Bai and S. (1998) with:

1) X_{ij}, $i, j = 1, 2, \ldots$ are independent random variables in $\mathbb{C}$ with $\mathsf{E}X_{1\,1} = 0$ and $\mathsf{E}|E_{1\,1}|^2 = 1$.
2) There exists a $K > 0$ and a random variable X with finite fourth moment such that, for any $x > 0$

$$\frac{1}{n_1 n_2} \sum_{i \leq n_1, j \leq n_2} \mathsf{P}(|X_{ij}| > x) \leq K\mathsf{P}(|X| > x)$$

for any positive integers n_1, n_2.
3) There is a positive function $\psi(x) \uparrow \infty$ as $x \to \infty$, and $M > 0$, such that

$$\max_{ij} \mathsf{E}[|X_{ij}|^2 \psi(|X_{ij}|)] \leq M.$$

Then the conclusions of Bai and S. (1998,1999) remain true.

Extension to power estimation of multiple signal sources in multi-antenna fading channels (Couillet, S., Bai, Debbah):

Consider K entities transmitting data. Transmitter $k \in \{1, \ldots, K\}$ has (unknown) transmission power P_k with n_k antennas. They transmit data to N sensing devices (receiver). The multiple antenna channel matrix between transmitter k and the receiver is denoted by $H_k \in \mathbb{C}^{N \times n_k}$, where the entries of $\sqrt{N} H_k$ are i.i.d. standardized.

At time instant $m \in \{1, \ldots, M\}$, transmitter k emits signal $x_k^{(m)} \in \mathbb{C}^{n_k}$, entries independent and standardized, independent for different m's. At the same time the receive signal is impaired by additive noise $\sigma w^{(m)} \in \mathbb{C}^N$ $(\sigma > 0)$, the entries of $w^{(m)}$ are i.i.d. standardized (independent across m). Therefore at time m the receiver senses the signal

$$y^{(m)} = \sum_{k=1}^{K} \sqrt{P_k} H_k x_k^{(m)} + \sigma w^{(m)}.$$

Therefore, with $Y = [y^{(1)}, \dots, y^{(M)}] \in \mathbb{C}^{N\times M}$, $X_k = [x_k^{(1)}, \dots, x_k^{(M)}]$ $\in \mathbb{C}^{n_k \times M}$, and $W = [w^{(1)}, \dots, w^{(M)}] \in \mathbb{C}^{N\times M}$ we have

$$Y = \sum_{k=1}^{K} \sqrt{P_k} H_k X_k + \sigma W \quad = \quad H P^{1/2} X + \sigma W,$$

where, with $n = n_1 + \cdots + n_K$, $H = [H_1, \dots, H_K]$,

$$X = \begin{bmatrix} X_1 \\ \vdots \\ X_K \end{bmatrix} \in \mathbb{C}^{n\times M},$$

and $P^{1/2}$ is the positive square root of the $n \times n$ diagonal matrix P having first n_1 diagonal entries equal to P_1, next n_2 diagonal matrices equal to P_2, etc.

Goal is to estimate the P_k's. Notice Y is the first N rows of

$$\begin{pmatrix} HP^{1/2} & I_N \\ 0_1 & 0_2 \end{pmatrix} \begin{pmatrix} X \\ W \end{pmatrix},$$

(I_N $N \times N$ identity matrix, 0_1, $n \times n$, 0_2 $n \times N$ zero matrices) so previous results apply.

Theorem. Assume σ and K are fixed, $M/N \to c > 0$, and each $N/n_k \to c_k > 0$, as $N \to \infty$. Let $B_N = (1/M)YY^*$. Then, almost surely, F^{B_N} converges in distribution, as $N \to \infty$, to a (nonrandom) p.d.f., whose Stieltjes transform, $m_F(z)$ ($z \in \mathbb{C}^+$) satisfies

$$m_F(z) = cm_{\underline{F}}(z) + (c-1)\frac{1}{z},$$

where $m_{\underline{F}}$ is the unique solution with positive imaginary part to the equation

$$\frac{1}{m_{\underline{F}}} = -\sigma^2 + \frac{1}{f} - \sum_{k=1}^{K} \frac{1}{c_k} \frac{P_k}{1 + P_k f}$$

with

$$f = (1-c)m_{\underline{F}} - czm_{\underline{F}}^2.$$

Theorem. Assuming $M > N$, $n < N$, $P_1 < P_2 < \cdots < P_K$, and certain assumptions on the size of c, and the c_k's, exact separation occurs. Let λ_i denote the i-th smallest eigenvalue of B_N and $s = (\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_N})^T$. Then with probability 1 $\hat{P}_k \to P_k$ as $N \to \infty$ where

$$\hat{P}_k = \frac{NM}{n_k(M-N)} \sum_{i \in \mathcal{N}_k} (\eta_i - \mu_i),$$

where $\mathcal{N}_k = \{N - \sum_{i=k}^{K} n_i + 1, \ldots, N - \sum_{i=k+1}^{K} n_i\}$, the η_i's are the ordered eigenvalues of $\operatorname{diag}(\lambda_1, \ldots, \lambda_N) - (1/N)ss^*$, and the μ_i's are the ordered eigenvalues of $\operatorname{diag}(\lambda_1, \ldots, \lambda_N) - (1/M)ss^*$.