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Random crowds: diffusion of large matrices

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Outline

- Diffusion of large (huge) matrices (physics, telecommunication, life sciences..)
- Non-linear Smoluchowski-Fokker-Planck equations and shock waves
- Finite N as viscosity in the spectral flow – Burgers equations
- Order-disorder phase transition in large N YM theory, colored catastrophes and universality
- Summary

Motivation

Justification for novel approach

- Contemporary physical and other complex systems are characterized by huge matrices (Large N, MIMO systems, DNA data, dEEG data...)
- Systems evolve due to dynamic evolution as a function of exterior parameters (time, length of the wire, area of the surface, temperature ...)
- Can we achieve mathematical formulation for this setup?
- Can noise **improve** our understanding?

Two probability calculi

CLASSICAL

- pdf $< ... >= \int ... p(x) dx$
- Fourier transform $F(k)$ of pdf generates moments
- ln $F(k)$ of Fourier generates additive cumulants
- Gaussian – Non-vanishing second cumulant only

MATRICIAL (FRV for $N = \infty$)

- spectral measure $< ... >= \int ... P(H) dH$
- Resolvent $G(z) = \left\langle \mathrm{Tr} \frac{1}{z-H} \right\rangle$
- R-transform generates additive cumulants $G[R(z) + 1/z] = z$
- Wigner semicircle – Non-vanishing second cumulant only

Exact analogies for CLT, Lévy processes, Extreme values statistics...

Inviscid Burgers equation

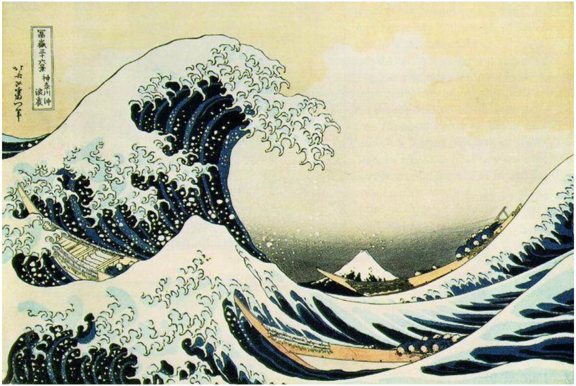
- $H_{ij} \to H_{ij} + \delta H_{ij}$ with $< \delta H_{ij} = 0 >$ and $< (\delta H_{ij})^2 >= (1 + \delta_{ij})\delta t$
- For eigenvalues x_i, random walk undergoes in the "electric field" (Dyson) $< \delta x_i >= \sum_{i \neq j} \left(\frac{1}{x_j - x_i} \right)$ and $< (\delta x_i)^2 >= \delta t$
- Resulting SFP equation for the resolvent in the limit $N = \infty$ and $\tau = Nt$ reads $\partial_\tau G(z, \tau) + G(z, \tau)\partial_z G(z, \tau) = 0$
- Non-linear, inviscid complex Burgers equation, very different comparing to Fick equation for the "classical" diffusion

Inviscid Burgers equation - details

- SFP eq:
  $\partial_t P(\{x_j\},t) = \frac{1}{2}\sum_i \partial^2_{ii} P(\{x_j\},t) - \sum_i \partial_i (E(x_i) P(\{x_j\},t))$
- Integrating, normalizing densities to 1 and rescaling the time $\tau = Nt$ we get
  $\partial_\tau \rho(x) + \partial_x \rho(x) P.V. \int dy \frac{\rho(y)}{x-y} =$
  $\frac{1}{2N}\partial^2_{xx}\rho(x) + P.V. \int dy \frac{\rho_c(x,y)}{x-y}$
- r.h.s. tends to zero in the large N limit
- $\frac{1}{x \pm i\epsilon} = P.V. \frac{1}{x} \mp i\pi\delta(x)$

Complex Burgers Equation and viscosity

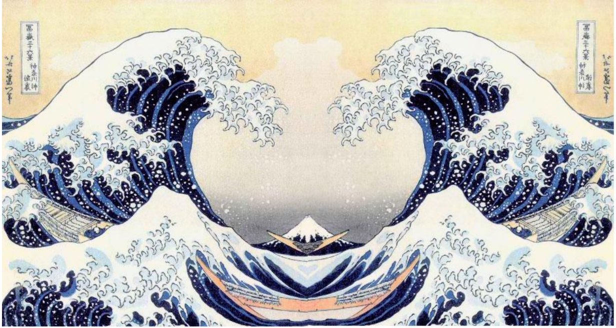
- Burgers equation for $G(z,\tau)$
  $\partial_\tau G + G\partial_z G = 0$
- Complex characteristics
  $G(z,\tau) = G_0(\xi[z,\tau)])$ $\quad$ $G_0(z) = G(\tau = 0, z) = \frac{1}{z}$
  $\xi = z - G_0(\xi)\tau \quad (\xi = x - vt)$, so solution reads
  $G(z,\tau) = G_0(z - \tau G(z,\tau))$
- Shock wave when $\frac{d\xi}{dz} = \infty$
- Universal preshock - expansion at the singularity for finite N
  We define $f(z,\tau) = \partial_z \ln < \det(z - H(\tau)) >$
  $\partial_\tau f + f\partial_z f = -\nu\partial_{zz} f \qquad \nu = \frac{1}{2N}$
- Exact (for any N) viscid Burgers equation with negative viscosity, similar equation for
  $g(z,\tau) = \partial_z \ln < 1/\det(z - H(\tau)) >$
- Universal oscillations anticipating the shock, contrary to smoothening of the shock in hydrodynamics – Airy universality



"Behind the Great Wave at Kanagawa" (by Hokusai) Color woodcut, Metropolitan Museum of Art, New York.

Diffusion of unitary matrices:

- Similar Burgers equation for $G(z, \tau)$ Durhuus, Olesen, Migdal, Makeenko, Kostov, Matytsin, Gross, Gopakumar, Douglas, Rossi, Kazakov, Voiculescu, Pandey, Shukla, Janik, Wieczorek, Neuberger
- Collision of two shock waves, since they propagate on the circle
- Universal preshock - expansion at the singularity for finite N
- Similar, exact (for any finite N) viscid Burgers equation with negative viscosity (for $< det >$ and $< 1/det >$)
- Universal, wild oscillations anticipating the shock, contrary to smoothening of the shock in standard hydrodynamics – here Pearcey universality

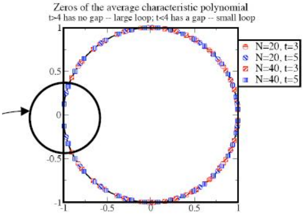


Colliding Great Waves at $\theta = \pi$ (by Hocus Pocusai, Microsoft Paint based on Hokusai woodcut)

Wilson loops in large N Yang-Mills theories (time $\equiv$ area)

Numerical studies on the lattice (Narayanan and Neuberger, 2006-2007)

- $W(c) = \left\langle P \exp(i \oint A_\mu dx^\mu) \right\rangle_{YM}$
- $Q_N(z, \mathcal{A}) \equiv \langle det(z - W(\mathcal{A})) \rangle$
- Double scaling limit...
- $z = -e^{-y}$
  $y = \frac{2}{12^{1/4} N^{3/4}} \xi$
  $\mathcal{A}^{-1} = \mathcal{A}^{*-1} + \frac{\alpha}{4\sqrt{3}} \frac{1}{N^{1/2}}$
- $Q_N(z, \mathcal{A}) \rightarrow$
  $\lim_{N \to \infty} \left(\frac{4N}{3}\right)^{1/4} Z_N(\Theta, \mathcal{A}) =$
  $= \int_{-\infty}^{+\infty} du e^{-u^4 - \alpha u^2 + \xi u}$

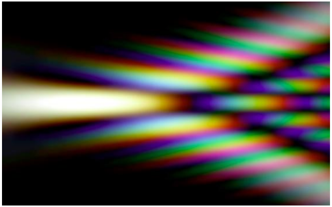


universality!

Closing of the gap is universal in $d = 2, 3, 4$

Universal scaling visualization - "classical" analogy

Caustics, illustration from Henrik Wann Jensen



Fold and cusp fringes, illustrations by Sir Michael Berry

Morphology of singularity (Thom, Berry, Howls)

GEOMETRIC OPTICS
(wavelength $\lambda = 0$)

- trajectories: rays of light
- intensity surface: caustic

WAVE OPTICS ($\lambda \to 0$)

$N \to \infty$ Yang-Mills
($\nu = \frac{1}{2N} = 0$)

- trajectories: characteristics
- singularities of spectral flow

FINITE N YM (viscosity $\nu \to 0$)

Universal Scaling, Arnold (μ) and Berry (σ) indices

"Wave packet" scaling
(interference regime)

- $\Psi = \frac{C}{\lambda^\mu}\Psi(\frac{x}{\lambda^{\sigma_x}}, \frac{y}{\lambda^{\sigma_y}})$
- fold $\mu = \frac{1}{6}$ $\sigma = \frac{2}{3}$ Airy
- cusp $\mu = \frac{1}{4}$ $\sigma_x = \frac{1}{2}$ $\sigma_y = \frac{3}{4}$ Pearcey

Yang-Lee zeroes scaling with N
(for $N \to \infty$)

- YL zeroes of Wilson loop
- $N^{2/3}$ scaling at the edge
- $N^{1/2}$ and $N^{3/4}$ scaling at the closure of the gap

Conclusions

- Powerful "spectral" formalism for matrix-valued diffusions (also for Ginibre-Girko matrices)
- Turbulence (in Kraichnan sense) as a mechanism for Haar measure in CUE
- Nonlinear effects, shock waves, universality
- New insight for several order-disorder transitions (e.g. Durhuus-Olesen transition)
- Multiple realizations of the universality, presumably also in several real complex systems
- New paradigm: for large matrices, noise is more helpful then distractive, improving predictability ("classical" limit)
- Hint for new mathematical structures?

More details: J.-P. Blaizot, MAN: 0911.3683, 0902.2223, PRL 101, 102001 and references therein.