

# Distribution of Bipartite Entanglement of a Random Pure State

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see also [arXiv 1006.4091](#) ([long version](#))

# Plan

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- Bipartite entanglement of a random pure state

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- Reduced density matrix  $\rightarrow$  random matrix theory

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- Distribution of the Renyi entropy

Results

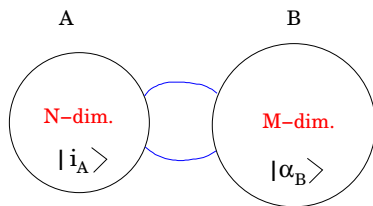
Coulomb gas technique for large systems

Phase transitions

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- Bipartite entanglement of a random pure state
- Reduced density matrix  $\rightarrow$  random matrix theory
- Distribution of the Renyi entropy
  - Results
  - Coulomb gas technique for large systems
  - Phase transitions
- Summary and Conclusions

# Coupled Bipartite System



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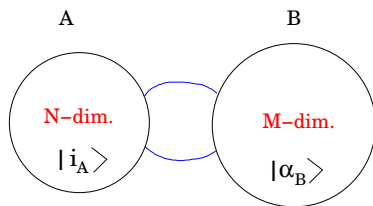
$$N \ll M$$

**Bipartite** quantum system  $A \times B$ : Hilbert space  $H_A \otimes H_B$

**subsystem A**: dimension  $N$  (the system to study)

**subsystem B**: dimension  $M \geq N$  ("environment")

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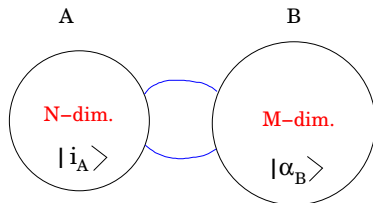
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**Large system**: limit  $N \gg 1$  and  $M \gg 1$  with  $M \approx N$ .

# Pure State and Reduced Density Matrix

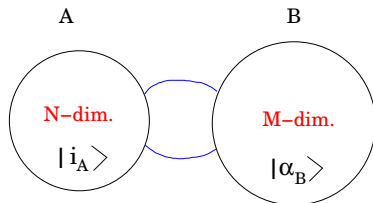


Coupled Bipartite System

$$N \ll M$$



# Pure State and Reduced Density Matrix



Coupled Bipartite System

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Any **pure** state of the full system:

$$|\psi\rangle = \sum_{i,\alpha} x_{i,\alpha} |i_A\rangle \otimes |\alpha_B\rangle$$

$X = [x_{i,\alpha}] \rightarrow (N \times M)$  rectangular **Coupling** matrix

# Pure State of Bipartite System



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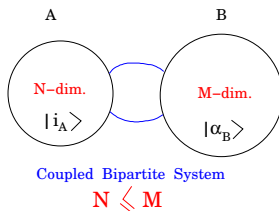
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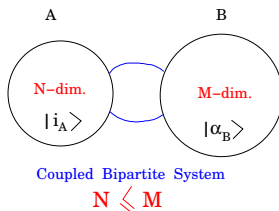
- Pure state:  $\hat{\rho} \neq \sum_k p_k |\psi_k\rangle\langle\psi_k|$  → not a Mixed state



# Reduced Density Matrix of subsystem A:

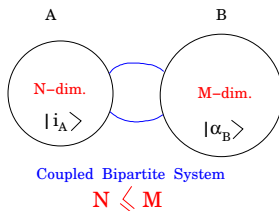


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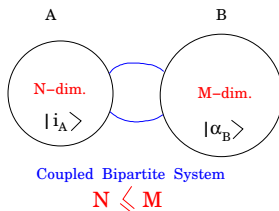
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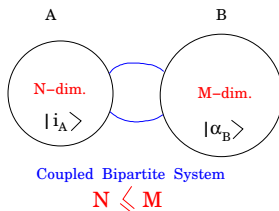


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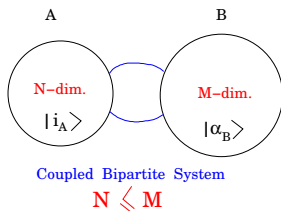


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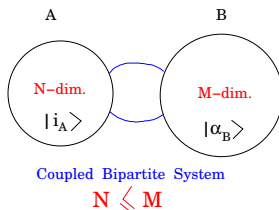


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$$= \sum_{i,j=1}^N W_{i,j} |i_A\rangle \langle j_A|$$

where the  $N \times N$  matrix:  $W = XX^\dagger$

# Eigenvalues of $W$ :

- In the diagonal representation

$$\hat{\rho}_A = \sum_{i,j=1}^N W_{i,j} |i_A\rangle\langle j_A| \rightarrow \sum_{i=1}^N \lambda_i |\lambda_i^A\rangle\langle \lambda_i^A|$$



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- Schmidt decomposition:

$$\boxed{|\psi\rangle = \sum_{i=1}^N \sqrt{\lambda_i} |\lambda_i^A\rangle \otimes |\lambda_i^B\rangle}$$

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(i) **Unentangled**

$$\lambda_{i_0} = 1, \lambda_j = 0 \quad \forall j \neq i_0$$

$$|\psi\rangle = |\lambda_{i_0}^A\rangle \otimes |\lambda_{i_0}^B\rangle$$

is **separable**

$$\hat{\rho}_A = |\lambda_{i_0}^A\rangle\langle\lambda_{i_0}^A| \text{ is pure}$$

(ii) **Maximally entangled**

$$\lambda_j = 1/N \text{ for all } j$$

(all eigenvalues equal)

$|\psi\rangle$  is a superposition of all product states

$$\hat{\rho}_A = \frac{1}{N} \sum_{i=1}^N |\lambda_i^A\rangle\langle\lambda_i^A| = \frac{1}{N} \mathbf{1}_A \text{ is completely mixed}$$

# Entanglement entropy

Subsystem  $A$  described by  $\hat{\rho}_A = \sum_{i=1}^N \lambda_i |\lambda_i^A\rangle\langle\lambda_i^A|$

- **Von Neuman entropy:**  $S_{\text{VN}} = -\text{Tr}[\hat{\rho}_A \ln \hat{\rho}_A] = -\sum_{i=1}^N \lambda_i \ln \lambda_i$

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- $S_{q \rightarrow 1} = S_{\text{VN}}$  and  $S_{q \rightarrow \infty} = -\ln(\lambda_{\max})$
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$S_{\text{VN}} = 0 = S_q$  is **minimal**

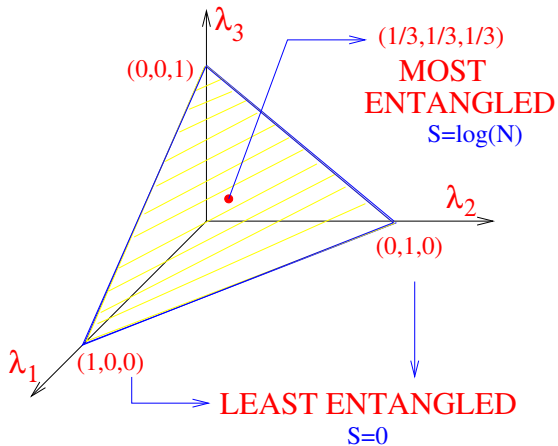
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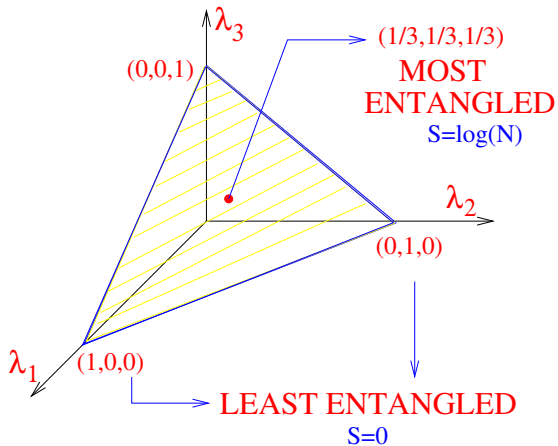
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# A Simple Diagram for $N=3$



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# Random Pure State: Haar Measure

$$|\psi\rangle = \sum_{i,\alpha} x_{i,\alpha} |i_A\rangle \otimes |\alpha_B\rangle = \sum_{i=1}^N \sqrt{\lambda_i} |\lambda_i^A\rangle \otimes |\lambda_i^B\rangle$$



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- **random pure state**  $\rightarrow$  where  $X = [x_{i,\alpha}]$  are **uniformly distributed** among the sets of  $\{x_{i,\alpha}\}$  satisfying  $\sum_{i,\alpha} |x_{i,\alpha}|^2 = \text{Tr}\rho = 1$

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- Equivalently, for the  $N \times M$  matrix

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- In the basis  $|i^A\rangle$  of  $H_A$ :  $\hat{\rho}_A = W = XX^\dagger$

$\rightarrow$  Distribution of the eigenvalues  $\lambda_i$  of  $\hat{\rho}_A$ ?

$\rightarrow$  **Distribution of the entanglement entropies**  $S_{\text{VN}}, S_q$ ?

# Joint PDF of Eigenvalues

- the joint pdf  $P(\lambda_1, \lambda_2, \dots, \lambda_N)$   
where  $\{\lambda_1, \lambda_2, \dots, \lambda_N\} \rightarrow$  eigenvalues of the Wishart matrix

$$W = \mathbf{X}\mathbf{X}^\dagger \quad (\mathbf{X} \rightarrow N \times M \text{ Gaussian random matrix})$$

with an additional constraint

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- Joint distribution of Wishart eigenvalues (James '64):

$$P(\{\lambda_i\}) \propto \exp \left[ -\frac{\beta}{2} \sum_{i=1}^N \lambda_i \right] \prod_i \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j < k} |\lambda_j - \lambda_k|^\beta$$

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where  $\{\lambda_1, \lambda_2, \dots, \lambda_N\} \rightarrow$  eigenvalues of the Wishart matrix

$$W = XX^\dagger \quad (X \rightarrow N \times M \text{ Gaussian random matrix})$$

with an additional constraint

$$\sum_{i=1}^N \lambda_i = 1$$

- Joint distribution of Wishart eigenvalues (James '64):

$$P(\{\lambda_i\}) \propto \exp \left[ -\frac{\beta}{2} \sum_{i=1}^N \lambda_i \right] \prod_i \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j < k} |\lambda_j - \lambda_k|^\beta \times \delta \left( \sum_{i=1}^N \lambda_i - 1 \right)$$

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- Given this pdf, what is the distribution of the Renyi entropy:

$$S_q = \frac{1}{1-q} \ln \left[ \sum_{i=1}^N \lambda_i^q \right]$$

# Random Pure State: Well Studied

- Average von Neumann entropy:  $\langle S_{VN} \rangle \rightarrow \ln N - \frac{N}{2M}$  for  $M \sim N \gg 1$   
→ **Near Maximal**

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[Lubkin ('78), Zyczkowski & Sommers (2001, 2004), Cappellini, Sommers and Zyczkowski (2006), Giraud (2007), Znidaric (2007), S.M., Bohigas and Lakshminarayan (2008), Kubotini, Adachi and Toda (2008), Chen, Liu and Zhou (2010), Vivo (2010), ...]

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- Laplace transform of the purity ( $q = 2$ ) distribution for large  $N$  [Facchi et. al. (2008, 2010)]

# Results: Entropy Distribution

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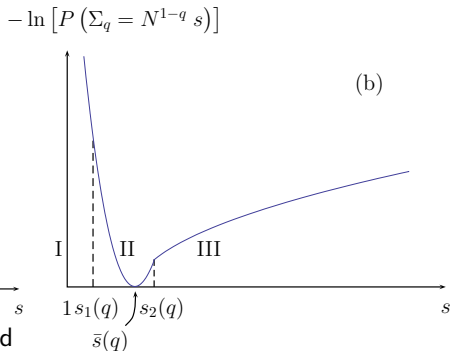
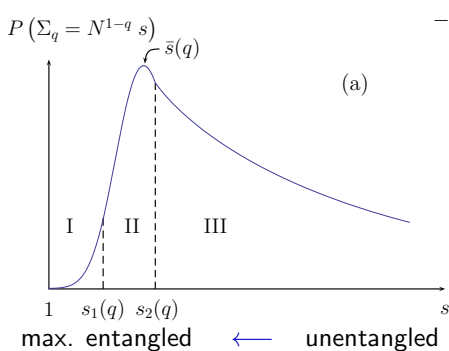
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For  $q = 2$ ,  $\langle \Sigma_2 \rangle \approx \frac{2}{N}$  (purity); For  $q \rightarrow 1$ ,  $\langle S_{VN} \rangle \approx \ln N - \frac{1}{2}$



# Results: pdf of $\Sigma_q = \sum_i \lambda_i^q$

$$P(\Sigma_q = N^{1-q} s) \approx \begin{cases} \exp\{-\beta N^2 \Phi_I(s)\} & \text{for } 1 \leq s < s_1(q) \\ \exp\{-\beta N^2 \Phi_{II}(s)\} & \text{for } s_1(q) < s < s_2(q) \\ \exp\{-\beta N^{1+\frac{1}{q}} \Psi_{III}(s)\} & \text{for } s > s_2(q) \end{cases}$$



# Computation of the pdf of $\Sigma_q$ : Coulomb gas method

- PDF of  $\Sigma_q = \sum_i \lambda_i^q$

$$P(\Sigma_q, N) = \int P(\lambda_1, \dots, \lambda_N) \delta\left(\sum_i \lambda_i^q - \Sigma_q\right) \left(\prod_i d\lambda_i\right).$$

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- The joint pdf of the eigenvalues can be interpreted as a **Boltzmann weight** at inverse temperature  $\beta$ :

$$\begin{aligned} P(\lambda_1, \dots, \lambda_N) &\propto \delta\left(\sum_i \lambda_i - 1\right) \prod_{i=1}^N \lambda_i^{\frac{\beta}{2}(M-N+1)-1} \prod_{i<j} |\lambda_i - \lambda_j|^\beta \\ &\propto \exp\{-\beta E[\{\lambda_i\}]\} \end{aligned}$$

where

$$E[\{\lambda_i\}] = -\gamma \sum_{i=1}^N \ln \lambda_i - \sum_{i<j} \ln |\lambda_i - \lambda_j| \quad (\text{with } \sum_i \lambda_i = 1)$$

→ effective energy of a **2D** Coulomb gas of charges

# Charge Density and Effective Energy

$$E\{\lambda_i\} = -\gamma \sum_{i=1}^N \ln \lambda_i - \sum_{i < j} \ln |\lambda_i - \lambda_j| \quad \text{with} \quad \sum_i \lambda_i = 1$$

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$$\{\lambda_i\} \longrightarrow \rho(x) = \frac{1}{N} \sum_i \delta(x - \lambda_i N)$$

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with **effective energy**

$$\begin{aligned} E_s[\rho] = & -\frac{1}{2} \int_0^\infty \int_0^\infty dx dx' \rho(x) \rho(x') \ln |x - x'| \\ & + \mu_0 \left( \int_0^\infty dx \rho(x) - 1 \right) \\ & + \mu_1 \left( \int_0^\infty dx x \rho(x) - 1 \right) + \mu_2 \left( \int_0^\infty dx x^q \rho(x) - s \right) \end{aligned}$$

where  $\mu_0$ ,  $\mu_1$  and  $\mu_2$  are Lagrange multipliers

# Saddle Point Solution

$$E_s[\rho] = -\frac{1}{2} \int_0^\infty \int_0^\infty dx dx' \rho(x) \rho(x') \ln|x-x'| + \mu_0 \left( \int_0^\infty dx \rho(x) - 1 \right) \\ + \mu_1 \left( \int_0^\infty dx x \rho(x) - 1 \right) + \mu_2 \left( \int_0^\infty dx x^q \rho(x) - s \right)$$

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$$\int_0^\infty dx' \rho_c(x') \ln|x - x'| = \mu_0 + \mu_1 x + \mu_2 x^q \equiv V(x)$$

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- Taking derivative with respect to  $x$  gives

$$\mathcal{P} \int_0^\infty dx' \frac{\rho_c(x')}{x-x'} = \mu_1 + q \mu_2 x^{q-1} = V'(x)$$

# Steps for Computing the PDF of $\Sigma_q$

(i) find the **solution**  $\rho_c(x)$  of the integral equation

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**Tricomi's** solution: density  $\rho_c(x)$  with finite support  $[L_1, L_2]$ :

$$\rho_c(x) = \frac{1}{\pi \sqrt{x-L_1} \sqrt{L_2-x}} \left[ C - \mathcal{P} \int_{L_1}^{L_2} \frac{dy}{\pi} \frac{\sqrt{y-L_1} \sqrt{L_2-y}}{x-y} V'(y) \right],$$

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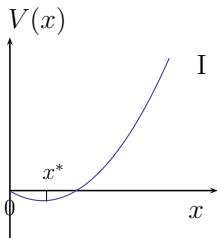
(iii) evaluate the **saddle point energy**  $E_s[\rho_c]$

# Phase Transitions

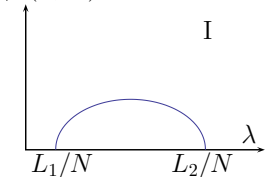
**Regime I:**

$$1 < s < s_1$$

$$(s_1(q=2) = 5/4)$$



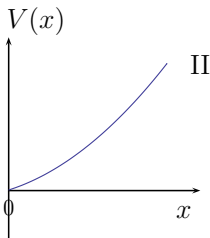
(a)  
 $\rho_c(\lambda, N)$



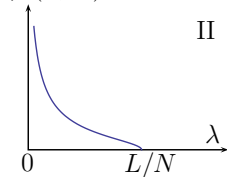
**Regime II:**

$$s_1 < s < s_2$$

$$(s_2(q=2) = 2 + \frac{2^{4/3}}{N^{1/3}})$$



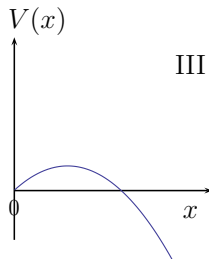
(b)  
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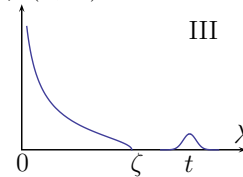
**Regime III:**

$$s > s_2$$

(weakly entangled)



(c)  
 $\rho_c(\lambda, N)$



## $q = 2$ : Purity

- **Regime I:**  $1 < s < 5/4$

$$\rho_c(x) = \frac{\sqrt{L_2 - x} \sqrt{x - L_1}}{2\pi(s - 1)},$$

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- **Regime II:**  $5/4 < s < 2 + \frac{2^{4/3}}{N^{1/3}}$

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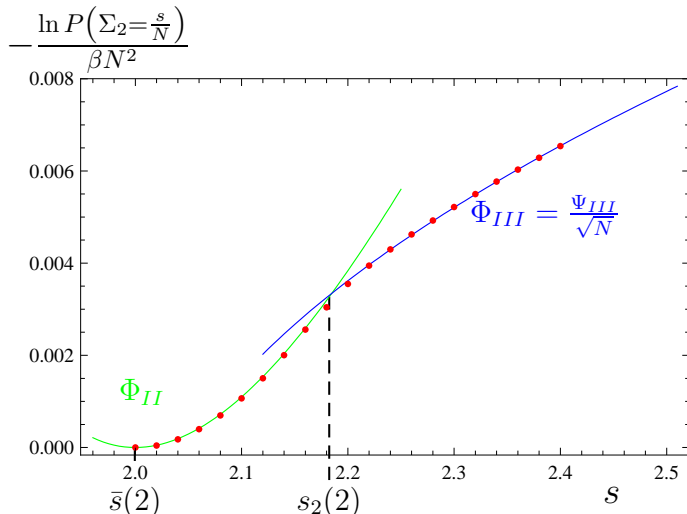
- **Regime III:**  $s > 2 + \frac{2^{4/3}}{N^{1/3}}$  Continuous density with support  $[0, \zeta]$  with

$\zeta \approx \frac{4}{N}$  and separated  $\lambda_{\max} \gg \zeta$ :  $\lambda_{\max} = t \approx \frac{\sqrt{s-2}}{\sqrt{N}}$

$$P\left(\Sigma_2 = \frac{s}{N}, N\right) \approx e^{-\beta N^{\frac{3}{2}} \Psi_{III}(s)}, \quad \Psi_{III}(s) = \frac{\sqrt{s-2}}{2}$$

# Numerical simulations

**Monte Carlo Simulations** (non-standard Metropolis algorithm):  
pdf of  $\Sigma_2$  (purity) for  $N = 1000$  (second phase transition)

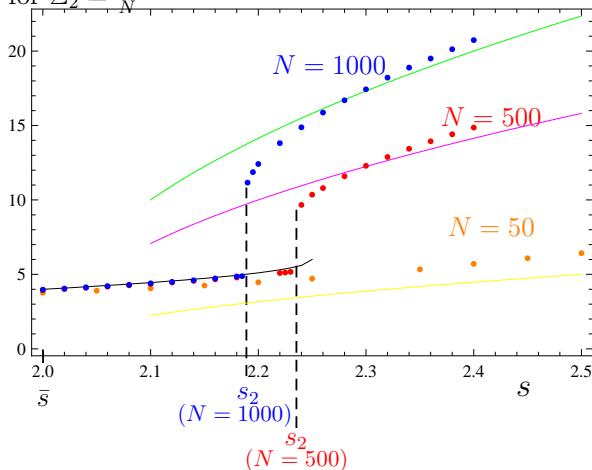


# Numerical Simulations (2)

Jump of the maximal eigenvalue (rightmost charge) at  $s = s_2$  for  $q = 2$  and different values of  $N$

$$N t = N \lambda_{\max}$$

for  $\Sigma_2 = \frac{s}{N}$



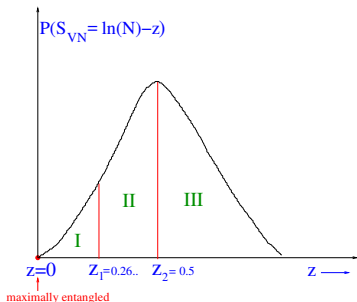
# The limit $q \rightarrow 1$ — von Neumann Entropy

$$S_{VN} = S_{q \rightarrow 1} = - \sum_{i=1}^N \lambda_i \ln(\lambda_i) = \ln(N) - z$$

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$$P(S_{VN} = \ln(N) - z) \approx \begin{cases} \exp \{ -\beta N^2 \phi_I(z) \} & \text{for } 0 \leq z < z_1 = \frac{2}{3} - \ln\left(\frac{3}{2}\right) = 0.26 \\ \exp \{ -\beta N^2 \phi_{II}(z) \} & \text{for } 0.26.. < z < z_2 = 1/2 \\ \exp \left\{ -\beta \frac{N^2}{\ln(N)} \phi_{III}(z) \right\} & \text{for } z > z_2 = 1/2 \end{cases}$$



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- For finite  $N$  and  $M$ , see P. Vivo, arXiv: 1009.1517

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## References:

C. Nadal, S.M. and M. Vergassola, **Phys. Rev. Lett.** 104, 110501 (2010)  
see also **arXiv 1006.4091** (**long version**)