Eigenvalues and Singular Values of Products of Rectangular Gaussian Random Matrices

Z. Burda, A. Jarosz, G. L., M. A. Nowak and A. Swiech, *arXiv:1007.3594*

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23rd Marian Smoluchowski Symposium on Statistical Physics
Kraków, September 27, 2010

Agenda

- Products of random matrices
- Why rectangular Gaussian matrices?
- The eigenvalue density
  - ⋆ Planar diagrammatics for real and complex spectra
  - ⋆ Sketch of the derivation
  - ⋆ Numerical confirmations and finite size effects
- The singular value density
  - ⋆ Sketch of the derivation: the Free Random Variables multiplication law
  - ⋆ Results and numerical confirmation
- Conclusions and perspectives

Products of Random Matrices

- Spectral analysis of **products of Random Matrices** has emerged as a powerful tool in several disciplines

  A. Crisanti, G. Paladin and A. Vulpiani, *Products of Random Matrices in Statistical Physics*, Springer-Verlag (1993)

  H. Caswell, *Matrix Population Models*, Sinauer Assoc. Inc., Sunderland, MA (2001)

  A. M. Tulino and S. Verdu, *Random Matrix Theory and Wireless Communications*, NOW Publishers Inc. (2004)

- Starting point: products $\mathbf{P}_L = \mathbf{A}_1 \mathbf{A}_2 \dots \mathbf{A}_L$ of $N \times N$ **Girko-Ginibre matrices**

  Z. Burda, R. Janik and B. Wacław, *Phys. Rev. E* **81** (2010)

$$\mathrm{d}\mu(\mathbf{A}_l) \propto \mathrm{e}^{-\frac{N}{\sigma_l^2}\mathrm{Tr}\left(\mathbf{A}_l^\dagger \mathbf{A}_l\right)} \mathrm{D}\mathbf{A}_l, \qquad \mathrm{D}\mathbf{A}_l \doteq \prod_{a,b=1}^{N} \mathrm{d}(\mathrm{Re}[\mathbf{A}_l]_{ab})\mathrm{d}(\mathrm{Im}[\mathbf{A}_l]_{ab})$$

- Gaussian IID complex entries: $\langle [\mathbf{A}_l]_{ab} \rangle = 0, \quad \left\langle |[\mathbf{A}_l]_{ab}|^2 \right\rangle = \frac{\sigma_l^2}{N}, \quad \forall\, a, b$
- **Rotational symmetry** $(N \to \infty)$

$$\rho_{\mathbf{P}_L}(\lambda, \overline{\lambda}) = \begin{cases} \frac{1}{L\pi\sigma^2}\left|\frac{\lambda}{\sigma}\right|^{-2\left(1-\frac{1}{L}\right)} & \text{for} \quad |\lambda| \le \sigma \\ 0 & \text{for} \quad |\lambda| > \sigma \end{cases}$$

- **Universality**: GUE, GOE, "Gaussian elliptic ensembles",....

Products of rectangular Gaussian matrices

- Again: $\mathbf{P}_L = \mathbf{A}_1 \mathbf{A}_2 \dots \mathbf{A}_L$
- Each $\mathbf{A}_l$ is a rectangular $N_l \times N_{l+1}$ matrix ($N_1 = N_{L+1}$ to have eigenvalues)
- Probability measure:

$$\mathrm{d}\mu\left(\mathbf{A}_l\right) \propto \mathrm{e}^{-\frac{\sqrt{N_l N_{l+1}}}{\sigma_l^2}\mathrm{Tr}\left(\mathbf{A}_l^\dagger \mathbf{A}_l\right)} \mathrm{D}\mathbf{A}_l$$

- Variance scaling: $\left\langle |[\mathbf{A}_l]_{ab}|^2 \right\rangle = \frac{\sigma_l^2}{\sqrt{N_l N_{l+1}}}, \quad \forall\ a, b$
- **Planar diagrammatics** to derive the eigenvalue density under the **thermodynamical limit**:

$$N_l \to \infty \quad \text{with} \quad R_l \doteq \frac{N_l}{N_{L+1}} = \text{finite}$$

Products of rectangular Gaussian matrices: motivation

- **Wireless telecommunication**: multiple-input multiple-output (MIMO) links
  R. R. Mueller, *IEEE Trans. Inf. Theor.* **48** (2002); E. Telatar, *Eur. Trans. Telecomm. ETT* **10**
  - ⋆ Input signal $\mathbf{x}$ travelling from $N_{\mathrm{tr.}}$ transmitters to $N_{\mathrm{rec.}}$ receivers

$$\mathbf{y} = \sqrt{\frac{\mathrm{SNR}}{N_{\mathrm{tr.}}}}\mathbf{A}\mathbf{x} + \boldsymbol{\eta}$$

  - ⋆ $\mathbf{A}$: rectangular Gaussian matrix
  - ⋆ Re-transmissions: effective propagation described by $\mathbf{P}_L = \mathbf{A}_L \dots \mathbf{A}_2\mathbf{A}_1$

- **Quantum entanglement**: random graphs states
  B. Collins, I. Nechita and K. Życzkowski, *J. Phys. A* **43** (2010)
  - ⋆ Edges: by-partite maximally entangled states
  - ⋆ Vertices: Couplings between systems
  - ⋆ Single link: one bi-partite maximally entangled state

Density matrix $\mathbf{Q} = \mathbf{A}^\dagger\mathbf{A}$; $\mathbf{A}\ (N_1 \times N_2)$; $|\psi\rangle = \sum_{a=1}^{N_1}\sum_{b=1}^{N_2} \mathbf{A}_{ab}|\alpha_a\rangle \otimes |\beta_b\rangle$

  - ⋆ **Loops**: the density matrices for subsystems sitting at the end vertices are $\mathbf{Q}_L = \mathbf{P}_L^\dagger\mathbf{P}_L$

- **Finance**: lagged correlation functions
  - ⋆ Risk management, portfolio selection

Diagrammatics for real spectra: Hermitian matrices

- Hermitian random matrix $\mathbf{H}$ ($N \times N$ with $N \to \infty$)

$$\textbf{GUE} \text{ ensemble}: \; \mathrm{d}\mu(\mathbf{H}) \propto \mathrm{e}^{-\frac{N}{2\sigma^2}\mathrm{Tr}\mathbf{H}^2} \mathrm{D}\mathbf{H}, \quad \mathrm{D}\mathbf{H} \doteq \prod_{i=1}^{N} \mathrm{d}\mathbf{H}_{ii} \prod_{i>j} \mathrm{d}(\mathrm{Re}\mathbf{H}_{ij})\mathrm{d}(\mathrm{Im}\mathbf{H}_{ij})$$

- **Eigenvalue density**: $\rho_{\mathbf{H}}(\lambda) \doteq \frac{1}{N}\sum_{a=1}^{N} \langle \delta(\lambda - \lambda_a) \rangle$
- **Matrix valued Green's function**: $\mathbf{G}_{\mathbf{H}}(z) \doteq \left\langle (\mathbf{Z} - \mathbf{H})^{-1} \right\rangle$, ($\mathbf{Z} = z\mathbf{1}_N$ and $z \in \mathbb{C}$)
- **Green's function**: $G_{\mathbf{H}}(z) \doteq \frac{1}{N}\mathrm{Tr}\mathbf{G}_{\mathbf{H}}(z) = \frac{1}{N}\sum_{a=1}^{N} \left\langle \frac{1}{z-\lambda_a} \right\rangle$

$$\rho_{\mathbf{H}}(\lambda) = -\frac{1}{\pi} \lim_{\epsilon \to 0^+} \mathrm{Im}\; G_{\mathbf{H}}(\lambda + \mathrm{i}\epsilon)$$

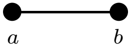
Power series expansion around infinite z

$$\mathbf{G}_{\mathbf{H}}(z) = \mathbf{Z}^{-1} + \left\langle \mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1} \right\rangle + \left\langle \mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1} \right\rangle + \dots$$

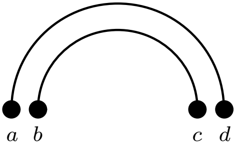
- ⋆ Only even moments survive
- ⋆ $\mathbf{Z}$ independent of $\mathbf{H} \to \left\langle \mathbf{H}_{a_1 a_2}\mathbf{H}_{a_3 a_4} \dots \mathbf{H}_{a_{2n-1} a_{2n}} \right\rangle$
- ⋆ **Wick's Theorem**: n-point correlation functions as sums of all possible contractions (2-point correlation functions) $\to \langle \mathbf{H}_{ab}\mathbf{H}_{cd} \rangle = \frac{\sigma^2}{N}\delta_{ad}\delta_{bc}$

Diagrammatics for real spectra: propagators

$$\mathbf{G_H}(z) = \mathbf{Z}^{-1} + \langle \mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1} \rangle + \langle \mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1} \rangle + \ldots$$



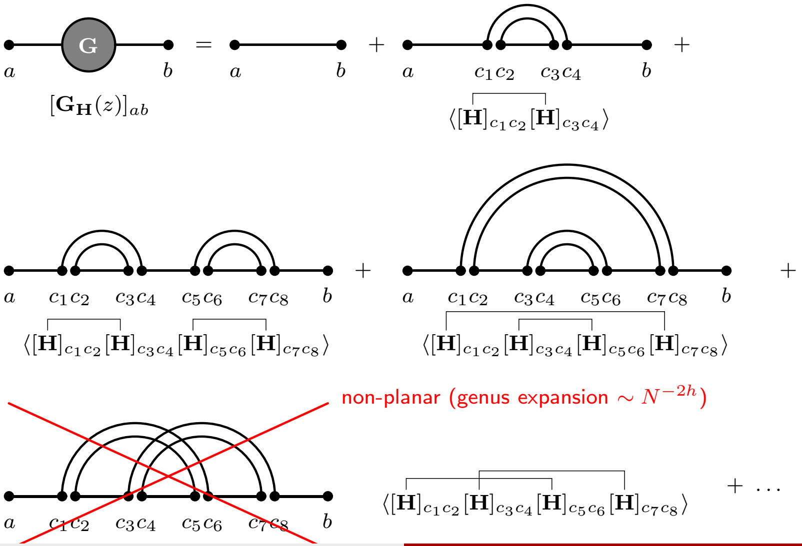
$$= \left[\mathbf{Z}^{-1}\right]_{ab} = \frac{1}{z}\delta_{ab}$$



$$= \langle \mathbf{H}_{ab}\mathbf{H}_{cd} \rangle = \frac{\sigma^2}{N}\delta_{ad}\delta_{bc}$$

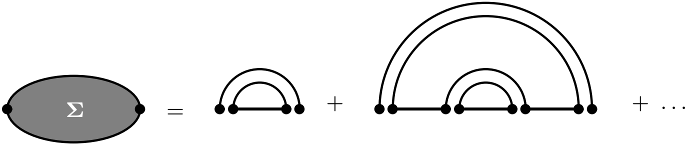
Diagrammatics for real spectra: the Green's function

$$\mathbf{G_H}(z) = \mathbf{Z}^{-1} + \langle \mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1} \rangle + \langle \mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1}\mathbf{H}\mathbf{Z}^{-1} \rangle + \ldots$$

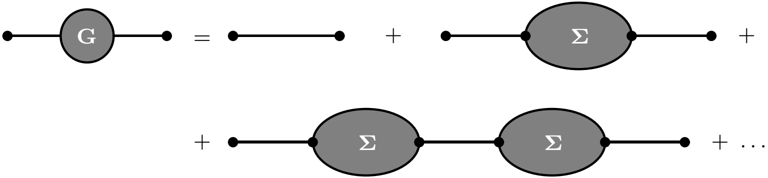


Dyson-Schwinger equations (1)

Self-energy: generating function for "one-line-irreducible-diagrams" (1LI)



$\mathbf{G_H}$ is made of 1LI diagrams and horizontal ($\mathbf{Z}^{-1}$) lines:

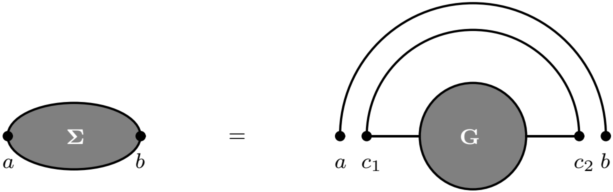


First Dyson-Schwinger equation

$$\mathbf{G_H}(z) = \mathbf{Z}^{-1} + \mathbf{Z}^{-1}\mathbf{\Sigma_H}(z)\mathbf{Z}^{-1} + \mathbf{Z}^{-1}\mathbf{\Sigma_H}(z)\mathbf{Z}^{-1}\mathbf{\Sigma_H}(z)\mathbf{Z}^{-1} + \ldots = (\mathbf{Z} - \mathbf{\Sigma_H}(z))^{-1}$$

Dyson-Schwinger equations (2)

Any 1LI diagram can be obtained by adding an external double arc to a planar diagram



Second Dyson-Schwinger equation

$$[\boldsymbol{\Sigma}_{\mathbf{H}}(z)]_{ab} = \sum_{c_1,c_2=1}^{N} [\mathbf{G}_{\mathbf{H}}(z)]_{c_1 c_2} \left\langle [\mathbf{H}]_{ac_1} [\mathbf{H}]_{c_2 b} \right\rangle = \sigma^2 \left(\frac{1}{N} \mathrm{Tr} \mathbf{G}_{\mathbf{H}}(z) \right) \delta_{ab}$$

The Dyson-Schwinger equations form a closed set from which an equation for the Green's function can be obtained

In this case:
$$G_{\mathbf{H}}(z) \left(z - \sigma^2 G_{\mathbf{H}}(z) \right) = 1$$

and the famous **Wigner's semi-circle** distribution can be derived from here.

Diagrammatics for complex spectra

E. Gudowska-Nowak, R. A. Janik, J. Jurkiewicz and M. A. Nowak, *Nucl. Phys. B* **670** (2003)

Z. Burda, R. A. Janik and B. Wacław, *Phys. Rev. E* **81** (2010)

- **Non-Hermitian matrices**: when $N \to \infty$ eigenvalues coalesce into 2-dimensional domains $\mathcal{D}$ in $\mathbb{C}$

$$\rho_{\mathbf{X}}(\lambda, \overline{\lambda}) = \frac{1}{N} \sum_{a=1}^{N} \left\langle \delta^{(2)}(\lambda - \lambda_a, \overline{\lambda} - \overline{\lambda}_a) \right\rangle$$

- 2-dimensional Dirac delta

$$\delta^{(2)}(z, \overline{z}) = \frac{1}{\pi} \lim_{\epsilon \to 0} \frac{\epsilon^2}{(|z|^2 + \epsilon^2)^2} = \frac{1}{\pi} \frac{\partial}{\partial \overline{z}} \lim_{\epsilon \to 0} \frac{\overline{z}}{(|z|^2 + \epsilon^2)}$$

- Mimicking this form: **non-holomorphic Green's function** (non linear in $\mathbf{X}$)

$$\begin{aligned} G_{\mathbf{X}}(z, \overline{z}) &\doteq \lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \sum_{a=1}^{N} \left\langle \frac{\overline{z} - \overline{\lambda_a}}{|z - \lambda_a|^2 + \epsilon^2} \right\rangle \\ &= \lim_{\epsilon \to 0} \lim_{N \to \infty} \frac{1}{N} \mathrm{Tr} \left\langle \frac{\overline{z}\mathbf{1}_N - \mathbf{X}^\dagger}{(z\mathbf{1}_N - \mathbf{X})\left(\overline{z}\mathbf{1}_N - \mathbf{X}^\dagger\right) + \epsilon^2 \mathbf{1}_N} \right\rangle \end{aligned}$$

- **Eigenvalue density**

$$\rho_{\mathbf{X}}(z, \overline{z}) = \frac{1}{\pi} \frac{\partial}{\partial \overline{z}} G_{\mathbf{X}}(z, \overline{z})$$

Diagrammatics for complex spectra: linearization

E. Gudowska-Nowak, R. A. Janik, J. Jurkiewicz and M. A. Nowak, *Nucl. Phys. B* **670** (2003)

$$G_{\mathbf{X}}(z,\overline{z}) = \lim_{\epsilon\to 0}\lim_{N\to\infty}\frac{1}{N}\mathrm{Tr}\left\langle \frac{\overline{z}\mathbf{1}_N - \mathbf{X}^\dagger}{(z\mathbf{1}_N - \mathbf{X})\left(\overline{z}\mathbf{1}_N - \mathbf{X}^\dagger\right) + \epsilon^2\mathbf{1}_N}\right\rangle$$

- Finite N, setting $\epsilon = 0$ (holomorphic sector): $G_{\mathbf{X}}(z,\overline{z}) = G_{\mathbf{X}}(z)$
- Only for $N\to\infty$ the limit $\epsilon\to 0$ yields a non-trivial result carrying information about the eigenvalue density within $\mathcal{D}$
- Non linearity in $G_{\mathbf{X}}$ $\to$ **doubling matrix sizes as a linearization trick**

$$\mathbf{G}^{\mathrm{D}}_{\mathbf{X}}(z,\overline{z}) \doteq \lim_{\epsilon\to 0}\lim_{N\to\infty}\left\langle\left(\mathbf{Z}^{\mathrm{D}}_{\epsilon} - \mathbf{X}^{\mathrm{D}}\right)^{-1}\right\rangle = \begin{pmatrix} \mathbf{G}^{zz}_{\mathbf{X}} & \mathbf{G}^{z\overline{z}}_{\mathbf{X}} \\ \mathbf{G}^{\overline{z}z}_{\mathbf{X}} & \mathbf{G}^{\overline{z}\overline{z}}_{\mathbf{X}} \end{pmatrix}$$

$$\mathbf{Z}^{\mathrm{D}}_{\epsilon} \equiv \begin{pmatrix} z\mathbf{1}_N & \mathrm{i}\epsilon\mathbf{1}_N \\ \mathrm{i}\epsilon\mathbf{1}_N & \overline{z}\mathbf{1}_N \end{pmatrix} \qquad \mathbf{X}^{\mathrm{D}} \equiv \begin{pmatrix} \mathbf{X} & \mathbf{0}_N \\ \mathbf{0}_N & \mathbf{X}^\dagger \end{pmatrix}$$

- The Green's function is now encoded as part of a "larger" object with **linear structure in $\mathbf{X}$**. It is recovered as:

$$G_{\mathbf{X}}(z,\overline{z}) = \frac{1}{N}\mathrm{Tr}\mathbf{G}^{zz}_{\mathbf{X}}$$

- Planar diagrammatics is now to be applied to $\mathbf{G}^{\mathrm{D}}_{\mathbf{X}}$
- The non-diagonal terms ($\mathbf{G}^{z\overline{z}}_{\mathbf{X}}$ and $\mathbf{G}^{\overline{z}z}_{\mathbf{X}}$) contain information on $\partial\mathcal{D}$

Linearization for products of matrices (1)

- Back to the product $\mathbf{P}_L = \mathbf{A}_1\mathbf{A}_2\ldots\mathbf{A}_L$ (each $\mathbf{A}_l$ being a rectangular $N_l \times N_{l+1}$ matrix filled with IID Gaussian complex entries)
- **Linearization**: $N_{\text{tot.}} \times N_{\text{tot.}}$ block matrix ($N_{\text{tot.}} \doteq N_1 + N_2 + \ldots N_L$):

$$\widetilde{\mathbf{P}} \doteq \begin{pmatrix} \mathbf{0} & \mathbf{A}_1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{A}_{L-1} \\ \mathbf{A}_L & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \end{pmatrix}$$

$$\widetilde{\mathbf{P}}^L = \begin{pmatrix} \mathbf{A}_1\mathbf{A}_2\ldots\mathbf{A}_{L-1}\mathbf{A}_L & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2\mathbf{A}_3\ldots\mathbf{A}_L\mathbf{A}_1 & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{A}_L\mathbf{A}_1\ldots\mathbf{A}_{L-2}\mathbf{A}_{L-1} \end{pmatrix}$$

- $\mathbf{P}_L$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{N_1}$ (assuming w.l.o.g. $N_1 \leq N_l \ \forall\ l \neq 1$)
- $\widetilde{\mathbf{P}}^L$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{N_1}$ ($L-$ fold degenerate) plus $N_{\text{tot.}} - LN_1$ zero modes
- $\widetilde{\mathbf{P}}$ has eigenvalues $\sqrt[L]{\lambda_1}$, $\sqrt[L]{\lambda_2}, \ldots,$ $\sqrt[L]{\lambda_{N_1}}$ ($L-$ fold degenerate) plus $N_{\text{tot.}} - LN_1$ zero modes

Linearization for products of matrices (2)

$$\rho_{\widetilde{\mathbf{P}}}(\lambda, \overline{\lambda}) = \frac{LN_1}{N_{\mathrm{tot.}}} \underbrace{\left(\frac{1}{N_1} \sum_{a=1}^{N_1} \left\langle \delta^{(2)} \left(\lambda - \sqrt[L]{\lambda_a}, \overline{\lambda} - \overline{\sqrt[L]{\lambda_a}} \right) \right\rangle \right)}_{\text{Density of the } L-\text{th roots of the eigenvalues of } \mathbf{P}_L} + \left(1 - \frac{LN_1}{N_{\mathrm{tot.}}} \right) \delta^{(2)}(\lambda, \overline{\lambda})$$

$$\Downarrow$$

$$M_{\widetilde{\mathbf{P}}}(z, \overline{z}) = \frac{LN_1}{N_{\mathrm{tot.}}} M_{\mathbf{P}_L}(z^L, \overline{z}^L)$$

- Here $M_{\widetilde{\mathbf{P}}}$ and $M_{\mathbf{P}_L}$ are **non-holomorphic** $M-$**transforms**:

$$M_{\widetilde{\mathbf{P}}}(z, \overline{z}) = zG_{\widetilde{\mathbf{P}}}(z, \overline{z}) - 1, \quad M_{\mathbf{P}_L}(z, \overline{z}) = zG_{\mathbf{P}_L}(z, \overline{z}) - 1$$

- The same relation can be proved also in the **holomorphic sector** by expanding the $M-$transforms into the series of moments

$$M_{\widetilde{\mathbf{P}}}(z) = \frac{1}{N_{\mathrm{tot.}}} \sum_{n \geq 1} \frac{1}{z^n} \left\langle \mathrm{Tr} \widetilde{\mathbf{P}}^n \right\rangle, \quad M_{\mathbf{P}_L}(z) = \frac{1}{N_1} \sum_{n \geq 1} \frac{1}{z^n} \left\langle \mathrm{Tr} \mathbf{P}_L^n \right\rangle$$

- Thus, solving the spectral problem for $\widetilde{\mathbf{P}}$ (which is linear w.r.t. the constituent matrices and allows for the Dyson-Schwinger approach) is **equivalent** to solving the spectral problem for $\mathbf{P}_L$

The eigenvalue density: sketch of the derivation (1)

- $\mathbf{P}_L = \mathbf{A}_1\mathbf{A}_2\ldots\mathbf{A}_L$, each $\mathbf{A}_l$ drawn from $\mathrm{d}\mu(\mathbf{A}_l) \propto \mathrm{e}^{-\frac{\sqrt{N_l N_{l+1}}}{\sigma_l^2}\mathrm{Tr}\left(\mathbf{A}_l^\dagger \mathbf{A}_l\right)}\mathrm{D}\mathbf{A}_l$
- $2-$point correlation functions:

$$\left\langle [\mathbf{A}_l]_{ab}[\mathbf{A}_m^\dagger]_{cd}\right\rangle = \frac{\sigma_l^2}{\sqrt{N_l N_{l+1}}}\delta_{lm}\delta_{ad}\delta_{bc}$$

- Doube matrix size to use Dyson-Schwinger equations

$$\widetilde{\mathbf{P}} = \begin{pmatrix} \mathbf{0} & \mathbf{A}_1 & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_2 & \ldots & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{A}_{L-1} \\ \mathbf{A}_L & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \end{pmatrix} \longrightarrow \widetilde{\mathbf{P}}^{\mathrm{D}} = \begin{pmatrix} \widetilde{\mathbf{P}} & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbf{P}}^\dagger \end{pmatrix}$$

- Propagator structure:

$$\left\langle \left[\widetilde{\mathbf{P}}^{\mathrm{D}}\right]_{l,l+1} \left[\widetilde{\mathbf{P}}^{\mathrm{D}}\right]_{\overline{l+1},\overline{l}} \right\rangle = \frac{\sigma_l^2}{\sqrt{N_l N_{l+1}}}\mathbf{1}_{N_l} \otimes \mathbf{1}_{N_{l+1}}$$

The eigenvalue density: sketch of the derivation (2)

- First Dyson-Schwinger equation:

$$\mathbf{G}^{\mathrm{D}} = \left(\mathbf{Z}^{\mathrm{D}} - \mathbf{\Sigma}^{\mathrm{D}}\right)^{-1}$$

- Second Dyson-Schwinger equation (depending on propagator structure):

$$\left[\mathbf{\Sigma}^{\mathrm{D}}\right]_{l\bar{l}} = \underbrace{\frac{\sigma_l^2}{\sqrt{N_l N_{l+1}}} \mathrm{Tr}\left[\mathbf{G}^{\mathrm{D}}\right]_{l+1,\overline{l+1}}}_{\doteq \alpha_l} \mathbf{1}_{N_l} \; ; \quad \left[\mathbf{\Sigma}^{\mathrm{D}}\right]_{\bar{l}l} = \underbrace{\frac{\sigma_{l-1}^2}{\sqrt{N_{l-1} N_l}} \mathrm{Tr}\left[\mathbf{G}^{\mathrm{D}}\right]_{\overline{l-1},l-1}}_{\doteq \beta_l} \mathbf{1}_{N_l}$$

$$\mathbf{G}^{\mathrm{D}} = \left(\begin{array}{c|c} \mathbf{G}^{ww} & \mathbf{G}^{w\overline{w}} \\ \hline \mathbf{G}^{\overline{w}w} & \mathbf{G}^{\overline{ww}} \end{array}\right) = \left(\begin{array}{ccc|ccc} \left[\mathbf{G}^{\mathrm{D}}\right]_{11} & \cdots & \left[\mathbf{G}^{\mathrm{D}}\right]_{1L} & \left[\mathbf{G}^{\mathrm{D}}\right]_{1\bar{1}} & \cdots & \left[\mathbf{G}^{\mathrm{D}}\right]_{1\bar{L}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \left[\mathbf{G}^{\mathrm{D}}\right]_{L1} & \cdots & \left[\mathbf{G}^{\mathrm{D}}\right]_{LL} & \left[\mathbf{G}^{\mathrm{D}}\right]_{L\bar{1}} & \cdots & \left[\mathbf{G}^{\mathrm{D}}\right]_{L\bar{L}} \\ \hline \left[\mathbf{G}^{\mathrm{D}}\right]_{\bar{1}1} & \cdots & \left[\mathbf{G}^{\mathrm{D}}\right]_{\bar{1}L} & \left[\mathbf{G}^{\mathrm{D}}\right]_{\overline{11}} & \cdots & \left[\mathbf{G}^{\mathrm{D}}\right]_{\overline{1L}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \left[\mathbf{G}^{\mathrm{D}}\right]_{\bar{L}1} & \cdots & \left[\mathbf{G}^{\mathrm{D}}\right]_{\bar{L}L} & \left[\mathbf{G}^{\mathrm{D}}\right]_{\overline{L1}} & \cdots & \left[\mathbf{G}^{\mathrm{D}}\right]_{\overline{LL}} \end{array}\right)$$

The eigenvalue density: sketch of the derivation (2)

- First Dyson-Schwinger equation:

$$\mathbf{G}^{\mathrm{D}} = \left(\mathbf{Z}^{\mathrm{D}} - \mathbf{\Sigma}^{\mathrm{D}}\right)^{-1}$$

- Second Dyson-Schwinger equation (depending on propagator structure):

$$\left[\mathbf{\Sigma}^{\mathrm{D}}\right]_{l\bar{l}} = \underbrace{\frac{\sigma_l^2}{\sqrt{N_l N_{l+1}}} \mathrm{Tr}\left[\mathbf{G}^{\mathrm{D}}\right]_{l+1,\overline{l+1}}}_{\doteq \alpha_l} \mathbf{1}_{N_l} \; ; \quad \left[\mathbf{\Sigma}^{\mathrm{D}}\right]_{\bar{l}l} = \underbrace{\frac{\sigma_{l-1}^2}{\sqrt{N_{l-1} N_l}} \mathrm{Tr}\left[\mathbf{G}^{\mathrm{D}}\right]_{\overline{l-1},l-1}}_{\doteq \beta_l} \mathbf{1}_{N_l}$$

$$\mathbf{Z}^{\mathrm{D}} - \mathbf{\Sigma}^{\mathrm{D}} = \left(\begin{array}{ccc|ccc} z\mathbf{1}_{N_1} & \dots & \mathbf{0} & -\alpha_1 \mathbf{1}_{N_1} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & z\mathbf{1}_{N_L} & \mathbf{0} & \dots & -\alpha_L \mathbf{1}_{N_L} \\ \hline -\beta_1 \mathbf{1}_{N_1} & \dots & \mathbf{0} & \bar{z}\mathbf{1}_{N_1} & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & -\beta_L \mathbf{1}_{N_L} & \mathbf{0} & \dots & \bar{z}\mathbf{1}_{N_L} \end{array}\right)$$

Results: equation for $M_{\mathbf{P}_L}$

- After "a little" math the Dyson-Schwinger can be solved, yielding a polynomial equation for the non-holomorphic $M-$transform

$$\prod_{l=1}^{L}\left(\frac{M_{\mathbf{P}_L}(z,\overline{z})}{R_l}+1\right)=\frac{|z|^2}{\sigma^2}$$

- Eigenvalue domain **boundary**: $|z|=\sigma=\sigma_1\sigma_2\ldots\sigma_L$
- **Rotational symmetry**:

$$M_{\mathbf{P}_L}(z,\overline{z})=\mathcal{M}_{\mathbf{P}_L}\left(|z|^2\right)$$

  This feature will also be displayed by the **eigenvalue density**
- **Singular behavior in** 0:

$$\rho_{\mathbf{P}_L}(\lambda,\overline{\lambda})\sim|\lambda|^{-2\left(1-\frac{1}{s}\right)}$$

  where $s=\#$ of R_l ratios equal to unity (matrix dimensions equal to N_1)
- **When** $L=2$ equations are easily solvable:

$$\rho_{\mathbf{P}_L}(\lambda,\overline{\lambda})=\begin{cases}\frac{1}{\pi\sigma^2}\frac{R}{\sqrt{(1-R)^2+4R\frac{|\lambda|^2}{\sigma^2}}}+f\delta^{(2)}(\lambda,\overline{\lambda}) & \text{for}\quad |\lambda|\leq\sigma\\ 0 & \text{for}\quad |\lambda|>\sigma\end{cases}$$

$f=0$ for $R\geq 1$ ($R\doteq R_2=N_2/N_1$)

Finite size corrections

- **Jump** on the boundary in the thermodynamic limit

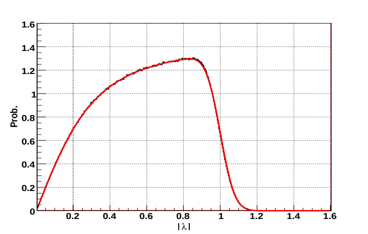
$$\rho_{\mathbf{P}_L}(\lambda,\overline{\lambda})\Big|_{|\lambda|=\sigma} = \frac{1}{\pi\sigma^2} R_h \ , \quad \frac{1}{R_h} = \sum_{l=1}^{L} \frac{1}{R_l}$$

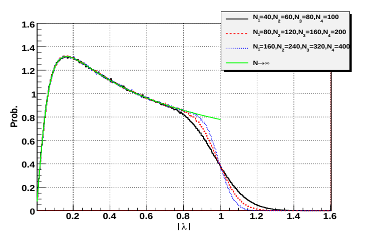
- **Conjecture** for finite size corrections $(\rho^{\text{rad.}}_{\mathbf{P}_L}(r) = 2\pi r \rho_{\mathbf{P}_L}(\lambda,\overline{\lambda})\big|_{|\lambda|=r})$

B. A. Khoruzhenko and H. J. Sommers, *arXiv:0911.5645*; E. Kanzieper, *Frontiers in Field Theory*, Nova Science Publ. (2005)

$$\rho^{\text{eff.}}_{\mathbf{P}_L}(r) \doteq \rho^{\text{rad.}}_{\mathbf{P}_L}(r)\frac{1}{2}\text{erfc}\left(q(r-1)\sqrt{N}\right) \ ; \quad q \sim \mathcal{O}(1) \ ; \quad N \sim \text{matrix dimension}$$

The eigenvalue density: numerical confirmation





LEFT: 2 matrices, $N_1 = 100$, $N_2 = 200$, 10^7 eigenvalues. RIGHT: eigenvalue density in the thermodynamical limit.

The singular value density: sketch of the derivation (1)

- $\mathbf{Q}_L \doteq \mathbf{P}_L^\dagger \mathbf{P}_L$, with $\mathbf{P}_L = \mathbf{A}_1 \mathbf{A}_2 \dots \mathbf{A}_L$ (squared singular values)
- **Idea**: write $\mathbf{Q}_L$ as a product of **Hermitian matrices** in order to apply the **Free Random Variables (FRV) multiplication law**
- Auxiliary square matrices (for generic $l = 1, 2, \dots, L$)

$$\mathbf{Q}_l \doteq (\mathbf{A}_1 \mathbf{A}_2 \dots \mathbf{A}_{l-1} \mathbf{A}_l)^\dagger (\mathbf{A}_1 \mathbf{A}_2 \dots \mathbf{A}_{l-1} \mathbf{A}_l) = \mathbf{A}_l^\dagger \mathbf{A}_{l-1}^\dagger \dots \mathbf{A}_2^\dagger \mathbf{A}_1^\dagger \mathbf{A}_1 \mathbf{A}_2 \dots \mathbf{A}_{l-1} \mathbf{A}_l$$

$$\widetilde{\mathbf{Q}_l} \doteq \mathbf{A}_l \mathbf{A}_l^\dagger \mathbf{A}_{l-1}^\dagger \dots \mathbf{A}_2^\dagger \mathbf{A}_1^\dagger \mathbf{A}_1 \mathbf{A}_2 \dots \mathbf{A}_{l-1} = \left(\mathbf{A}_l \mathbf{A}_l^\dagger \right) \mathbf{Q}_{l-1}$$

- Relation between $M-$transforms:

$$M_{\mathbf{Q}_l}(z) = \frac{R_l}{R_{l+1}} M_{\widetilde{\mathbf{Q}_l}}(z)$$

- Relations between $N-$transforms:

$$M_{\mathbf{Q}}(N_{\mathbf{Q}}(z)) = N_{\mathbf{Q}}(M_{\mathbf{Q}}(z)) = z \quad \Longrightarrow \quad N_{\mathbf{Q}_l}(z) = N_{\widetilde{\mathbf{Q}}_l}\left(\frac{R_{l+1}}{R_l} z \right)$$

FRV multiplication law $\Longrightarrow$ $N_{\widetilde{\mathbf{Q}}_l}(z) = \frac{z}{z+1} N_{\mathbf{A}_l \mathbf{A}_l^\dagger}(z) N_{\mathbf{Q}_{l-1}}(z)$

The singular value density: sketch of the derivation (2)

- From the relations on the $N-$transforms we have the recurrence

$$N_{\mathbf{Q}_l}(z) = \frac{z}{z + \frac{R_l}{R_{l+1}}} N_{\mathbf{A}_l \mathbf{A}_l^\dagger}\left(\frac{R_{l+1}}{R_l} z\right) N_{\mathbf{Q}_{l-1}}\left(\frac{R_{l+1}}{R_l} z\right) \; ; \quad N_{\mathbf{Q}_1}(z) = N_{\mathbf{A}_l \mathbf{A}_l^\dagger}\left(\frac{R_2}{R_1} z\right)$$

- Solving the recurrence:

$$N_{\mathbf{Q}_L}(z) = \frac{z^{L-1}}{(z+R_2)(z+R_3)\dots(z+R_L)} N_{\mathbf{A}_1 \mathbf{A}_1^\dagger}\left(\frac{z}{R_1}\right) \dots N_{\mathbf{A}_L \mathbf{A}_L^\dagger}\left(\frac{z}{R_L}\right)$$

- Using the Wishart ensemble $N-$transform

J. Feinberg and A. Zee, *J. Stat. Phys.* **83** (1997)

$$N_{\mathbf{A}_l \mathbf{A}_l^\dagger}(z) = \sigma_l^2 \frac{(z+1)\left(\sqrt{\frac{N_l}{N_{l+1}}} z + \sqrt{\frac{N_{l+1}}{N_l}}\right)}{z}$$

- Equation for $M_{\mathbf{Q}_L}$:

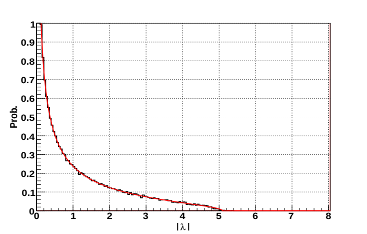
$$\frac{M_{\mathbf{Q}_L}(z)+1}{M_{\mathbf{Q}_L}(z)} \prod_{l=1}^{L} \left(\frac{M_{\mathbf{Q}_L}(z)}{R_l} + 1\right) = \frac{z}{\sigma^2}$$

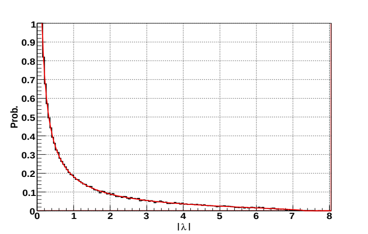
The singular value density

- Equation for $M_{\mathbf{Q}_L}$ converted into an equation for $G_{\mathbf{Q}_L}$
- Density derived numerically from: $\rho_{\mathbf{Q}_L}(\lambda) = -\frac{1}{\pi}\lim_{\epsilon\to 0^+} \mathrm{Im}\, G_{\mathbf{Q}_L}(\lambda + i\epsilon)$
- **Singular behavior**:

$$\rho_{\mathbf{Q}_L}(\lambda) \sim \lambda^{-\frac{s}{s+1}} \qquad \text{as} \qquad \lambda \to 0$$

Numerical confirmation





LEFT: 2 matrices, $N_1 = 50$, $N_2 = 150$. RIGHT: 4 matrices, $N_1 = 50$, $N_2 = 100$, $N_3 = 150$, $N_4 = 200$. In both cases: 10^6 eigenvalues.

Conclusions and perspectives (1)

$$\prod_{l=1}^{L}\left(\frac{M_{\mathbf{P}_L}(z,\overline{z})}{R_l}+1\right)=\frac{|z|^2}{\sigma^2} \qquad \longleftrightarrow \qquad \frac{M_{\mathbf{Q}_L}(z)+1}{M_{\mathbf{Q}_L}(z)}\prod_{l=1}^{L}\left(\frac{M_{\mathbf{Q}_L}(z)}{R_l}+1\right)=\frac{z}{\sigma^2}$$

- **Conjecture**: $\mathbf{X} \to$ non-Hermitian random matrix model whose mean eigenvalue spectrum displays **rotational symmetry**
- The non-holomorphic $M-$transform also displays rotational symmetry

$$M_{\mathbf{X}}(z,\overline{z})=\mathcal{M}_{\mathbf{X}}(|z|^2)$$

- Functional inversion: **"rotationally symmetric non-holomorphic $N-$transform"**

$$\mathcal{M}_{\mathbf{X}}(\mathcal{N}_{\mathbf{X}}(z))=z$$

- $\mathcal{N}_{\mathbf{X}}$ conjectured to be simply related to the ordinary $N-$transform of the $\mathbf{X}^\dagger\mathbf{X}$ Hermitian ensemble
- Typical situation (as in the present case): the ordinary $N-$transform is much more easily obtained: FRV multiplication law would avoid the use of complicated diagrammatics (or other methods)
- Use of FRV calculus to compute the mean spectral density of non-Hermitian "rotationally symmetric" products of random matrices?

Conclusions and perspectives (2)

- **Eigenvalue and singular value density**
  - **Excellent match** between theoretical predictions and numerical data
  - Conjecture for **finite size corrections**
  - **Singularities** at the origin characterized analytically
- **Possible applications** to
  - Wireless telecommunications
  - Entanglement physics
  - Quantitative finance / Econophysics

Thank you