This talk

dedicated to

the memory of

our good friend

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3-Cycle problem in the logistic map and Sharkovskii's theorem

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What's 3-cycle?

If $f(x)$, $0 < x < 1$

$f(x_1) = x_1$ 1-cycle

x_1 fixed pt.

$f(x_1) = x_2$

$f(x_2) = x_1$ 2-cycle

or $f(f(x_1)) \equiv f^2(x_1) = x_1$

$f^2(x_2) = x_2$

$f^3(x_1) = x_1$ 3-cycle

$(x_1\ x_2\ x_3)$

Why 3-cycle?

If $f(x)$, $0 < x < 1$

the sequence of cycles:

$1, 2, 4, 8, \cdots 2^k, \cdots$

$\cdots \cdots 14, 10, 6$

$\cdots 11, 9, 7, 5, 3$

Existence of 3-cycle in f

$\longrightarrow$ all cycles : **chaos**

A. Sharkovskii (1964)

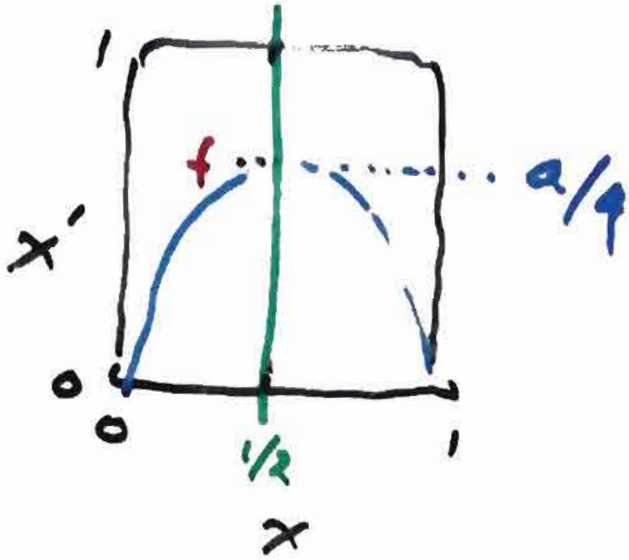
3-cycle fundamental

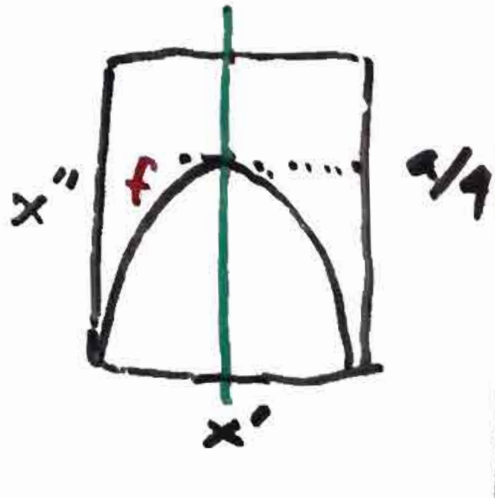
f: logistic map

$$x' = f(x)$$
$$= \underline{a}\, x(1-x)$$

$$a = (0, 4)$$

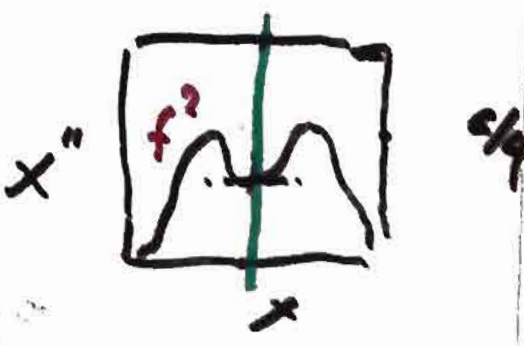
$$x, x' = (0, 1)$$





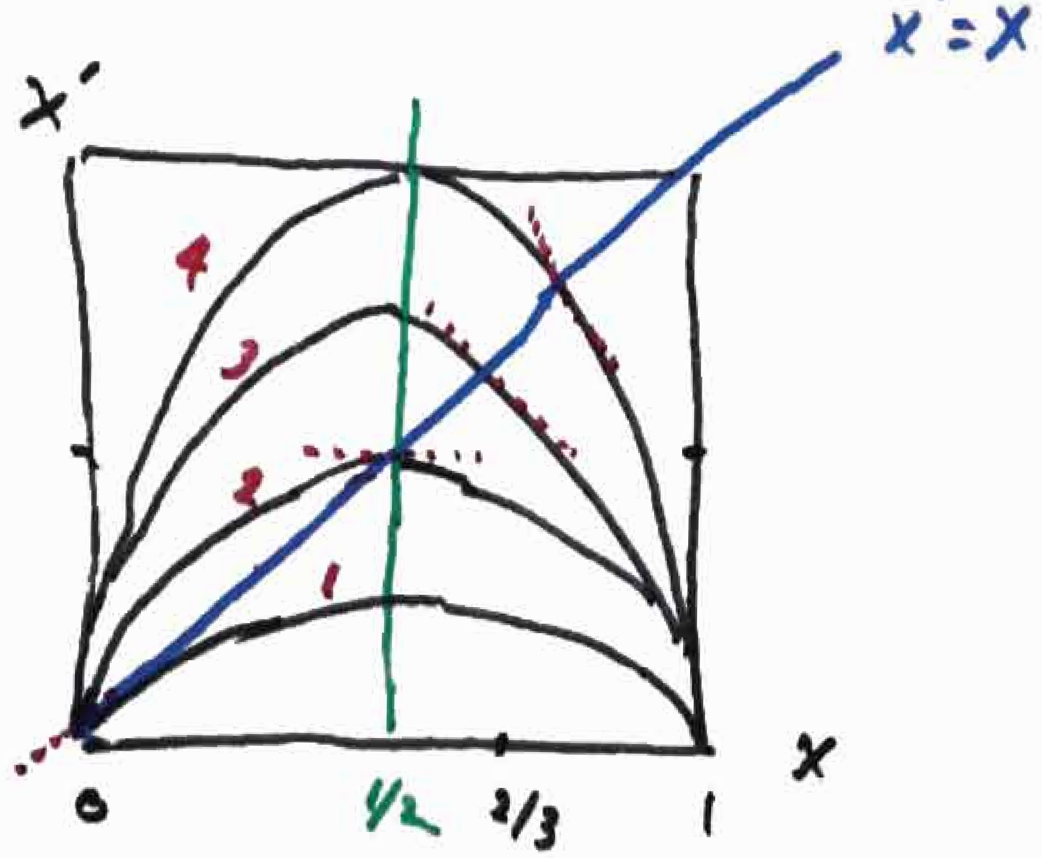
self-similar

$$x'' = f(x')$$
$$= f(f(x)) \equiv f^2(x)$$



$$x^n = f(x^{n-1}) = f^n(x)$$

fixed pt: $f(x^*) = x^*$

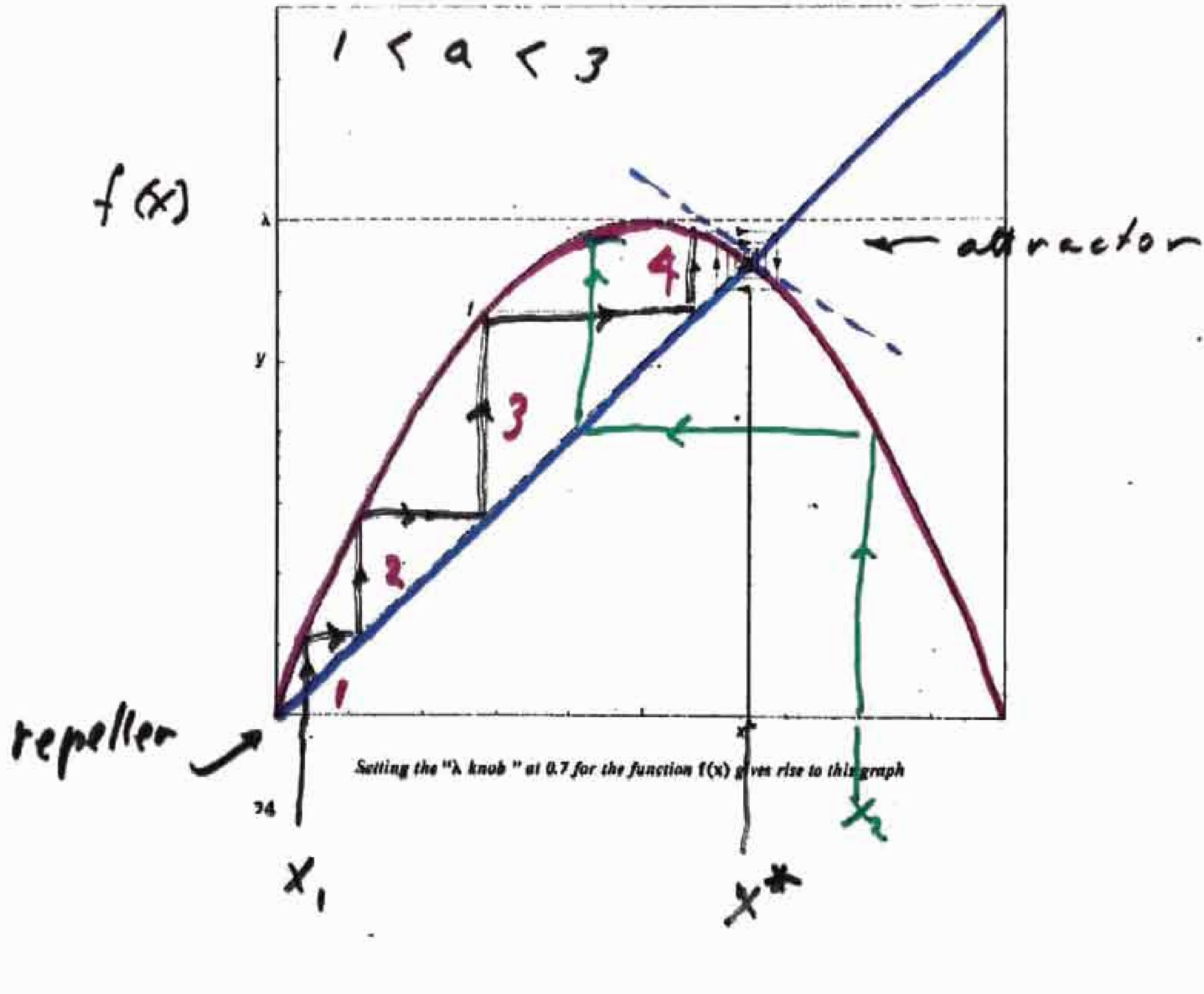


$|df/dx| < 1$: stable f.p.'s
$(1 < a < 3)$

$= 0$: super-stable
$(x = 1/2)$

$\lambda = \log|df/dx|$ Lyapunov

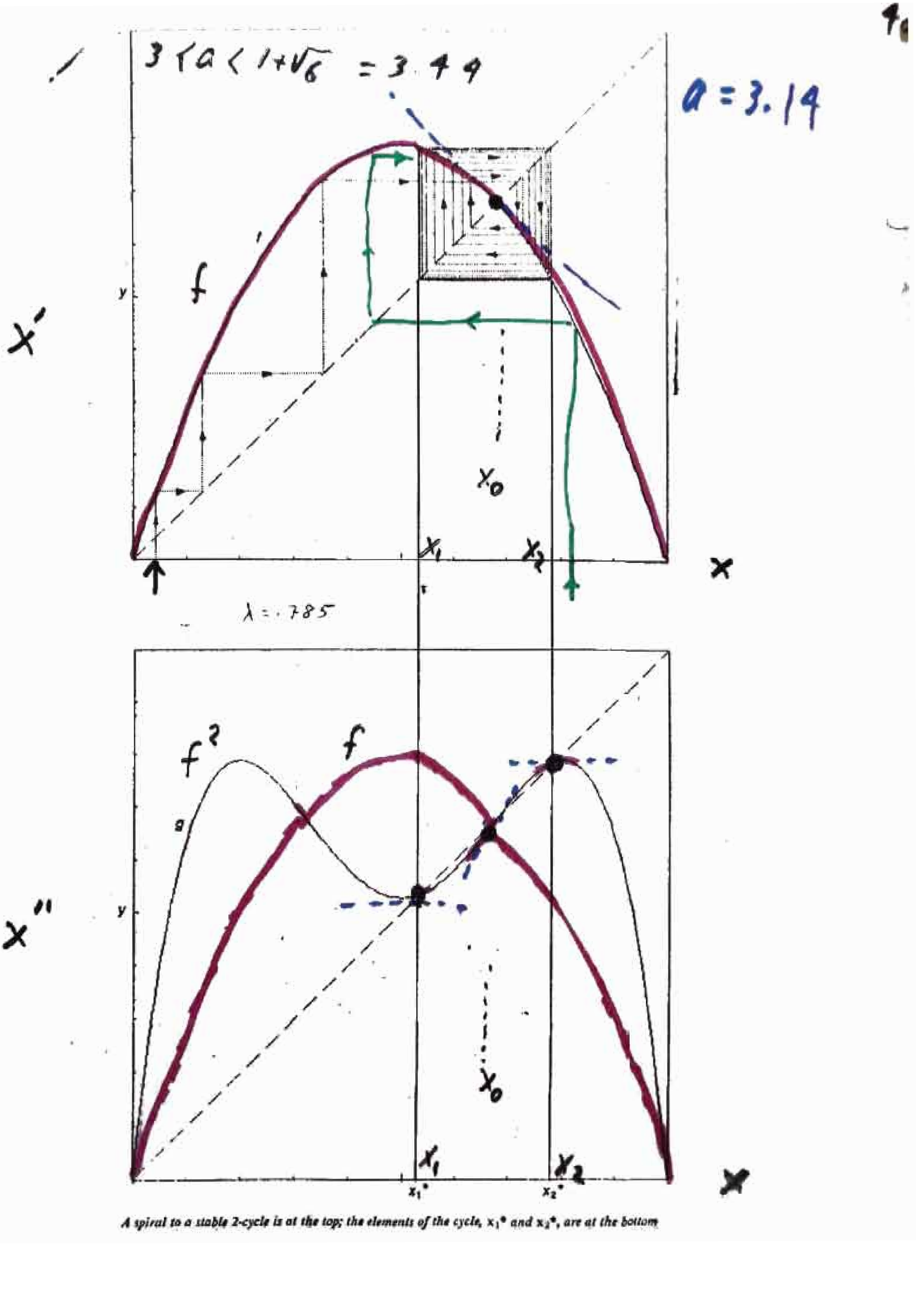
Iteration if $|df(x^*)/dx| < 1$



Setting the "λ knob" at 0.7 for the function f(x) gives rise to this graph

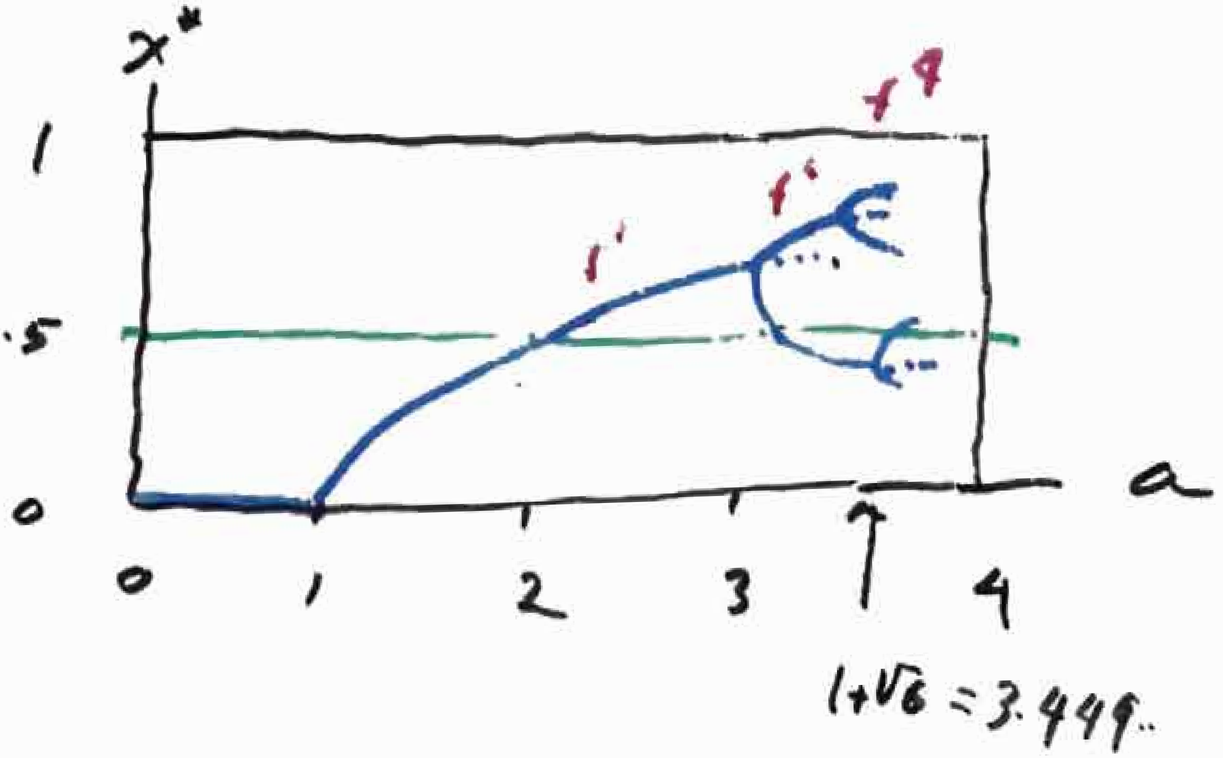
$$x^* = \lim_{n \to \infty} x_{1,2}$$

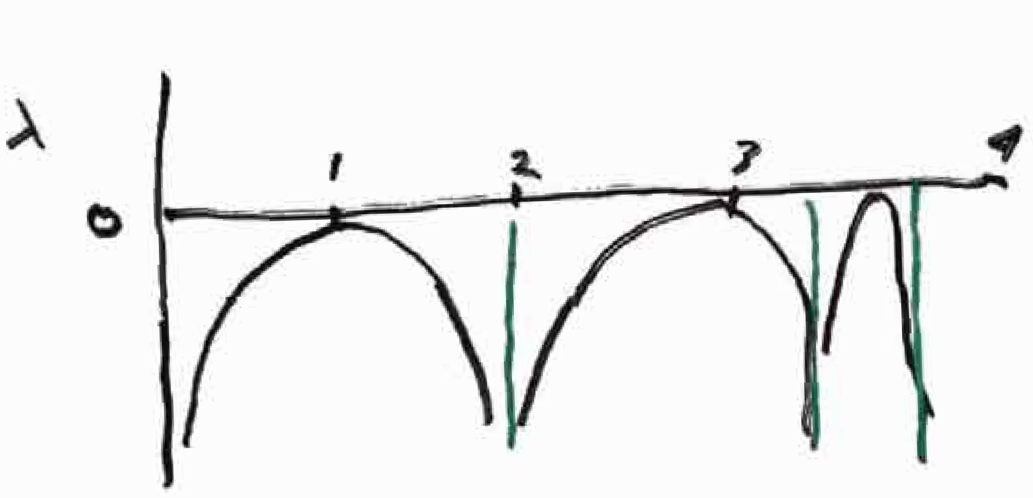
(Asymptotic Value)



A spiral to a stable 2-cycle is at the top; the elements of the cycle, x_1^* and x_2^*, are at the bottom

x^* vs a : Bifurcation





periodic

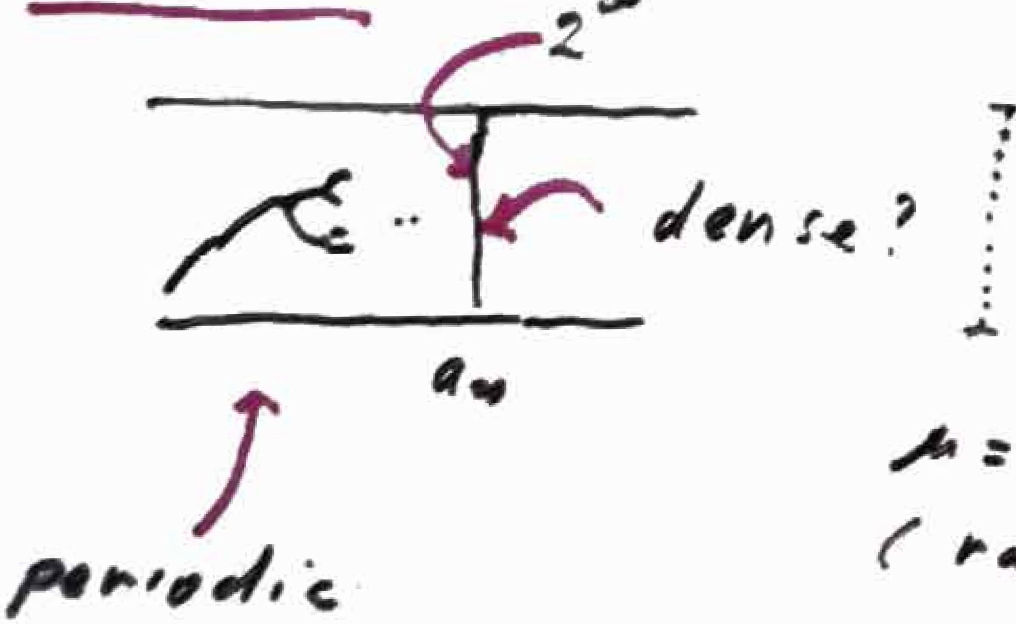
Stable Bifurcation

$$1 \to 2 \to 4 \to 8 \to \qquad \to 2^{\infty}$$

a_1	a_2	a_3	a_∞
1	3	$1+\sqrt{6} = 3.4494$	?

$a_\infty = 3.5699$ Feigenbaum (1979)

$a \to a_\infty$

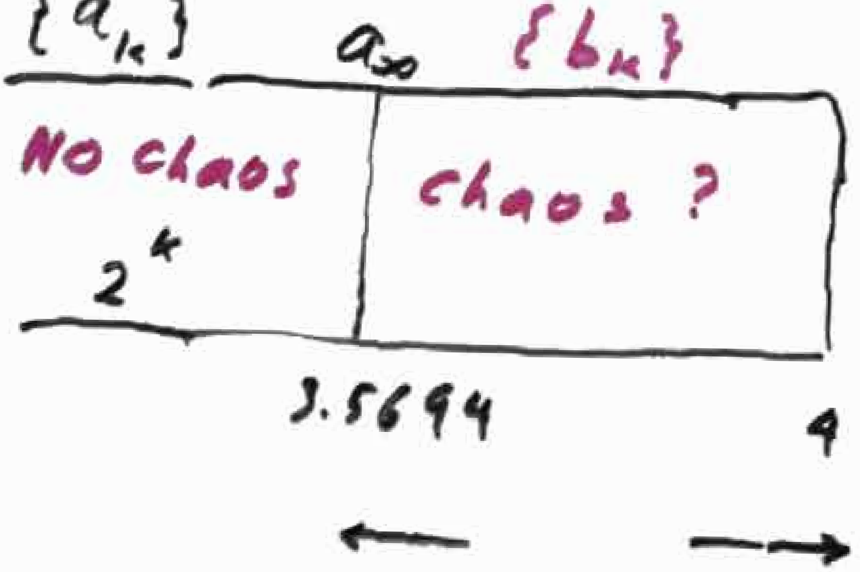


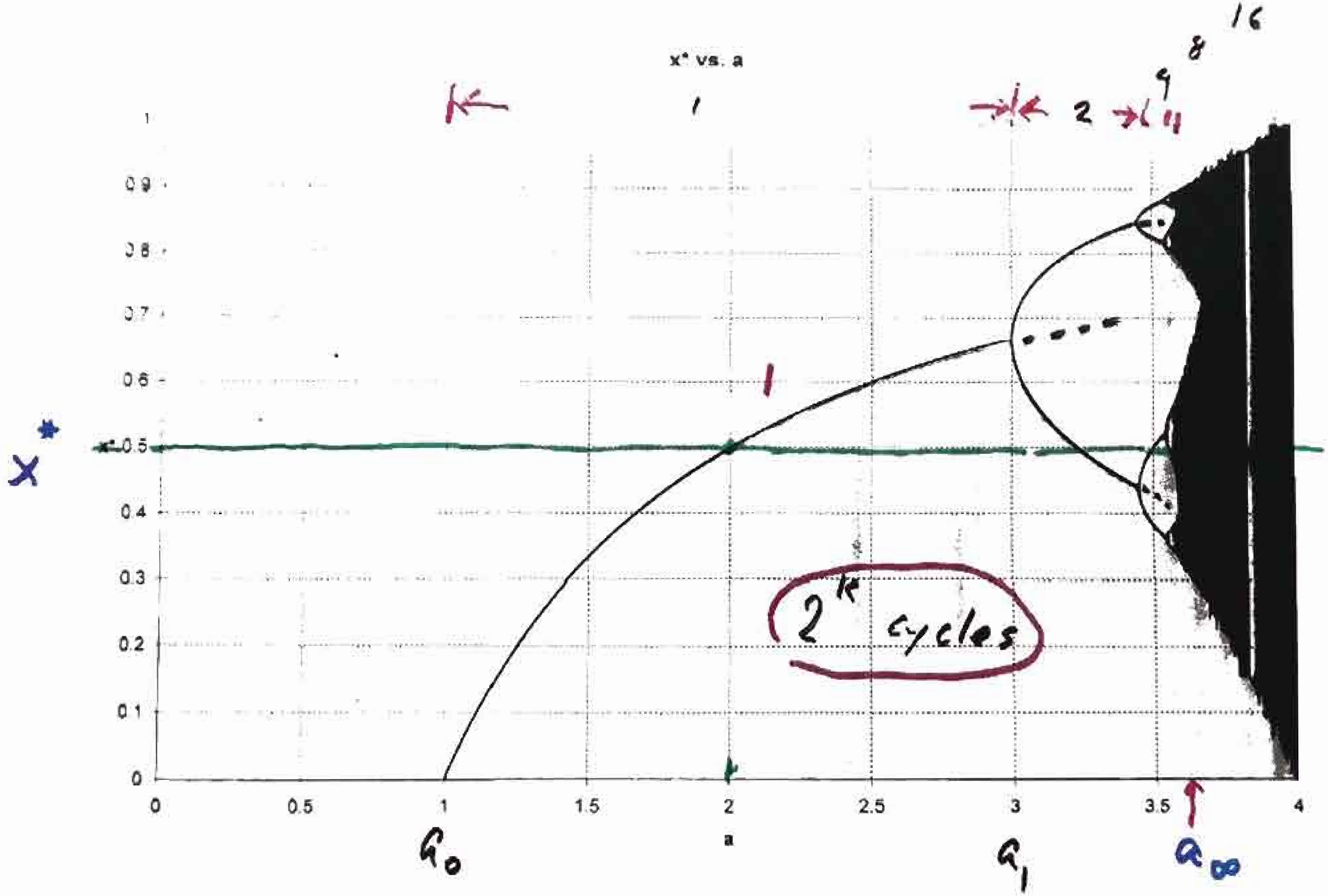
$2^{N\to\infty}$

$\mu = 0$

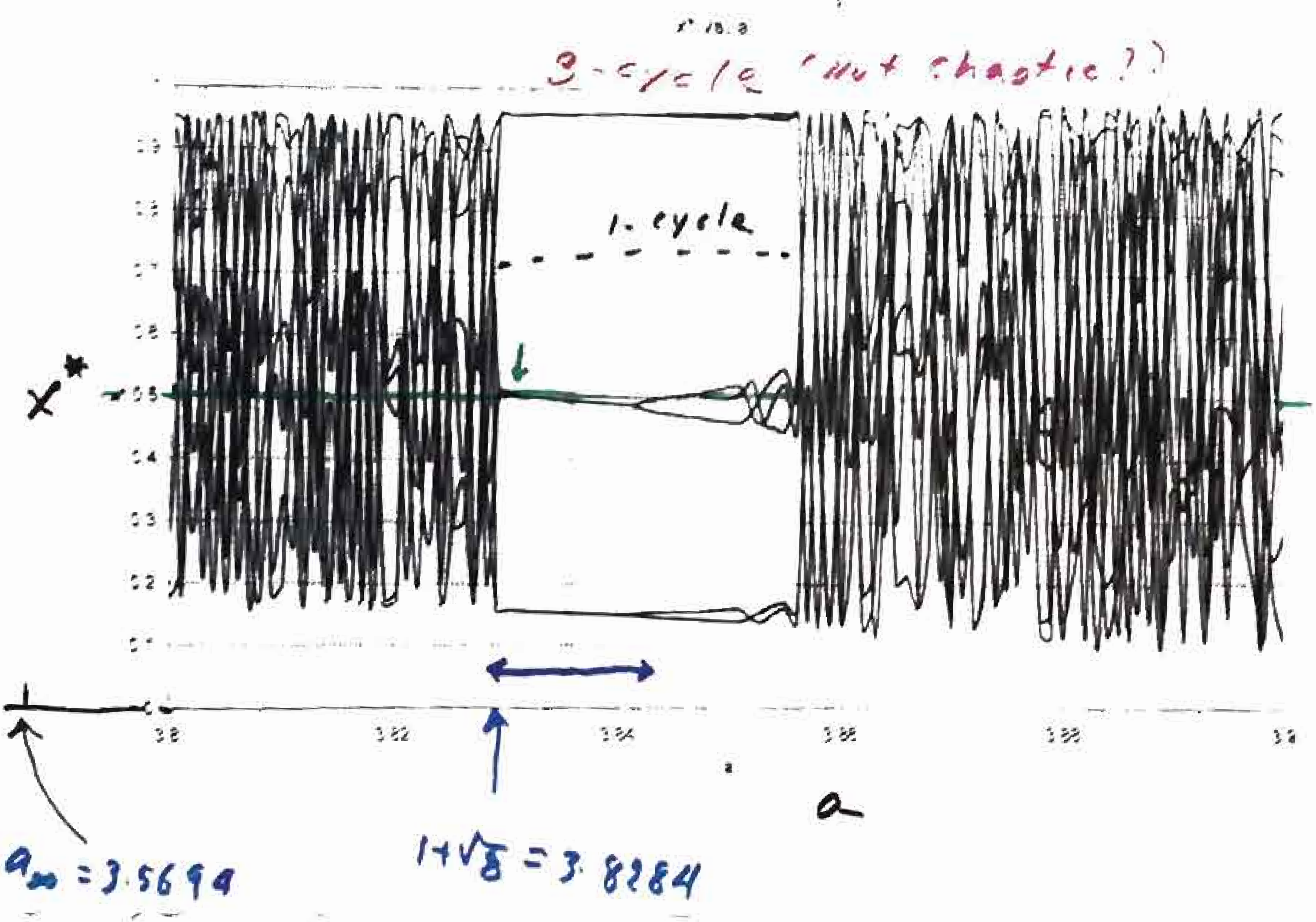
(rat. #'s)

What lies betw a_∞ and 4?





x* vs. a
1
2
4
8
16
1
2^k cycles
x^*
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.5
1
1.5
2
2.5
3
3.5
4
a
a_0
a_1
a_∞



3-cycle (not chaotic?)
1-cycle
x*
3.8
3.82
3.84
3.86
3.88
3.9
a
a∞ = 3.5694
1+√8 = 3.8284

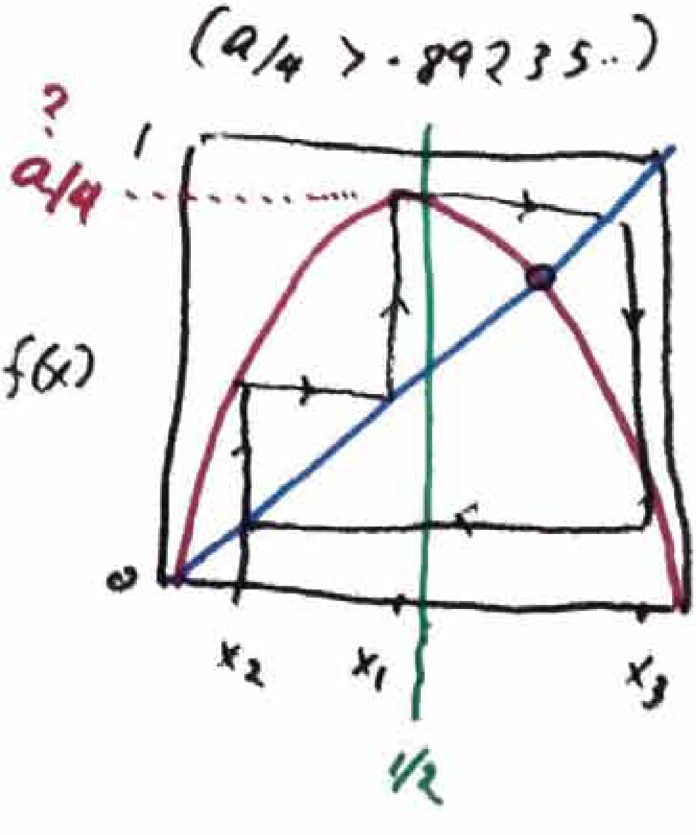
3-cycle and Chaos in logistic map and Sharkovskii theorem.

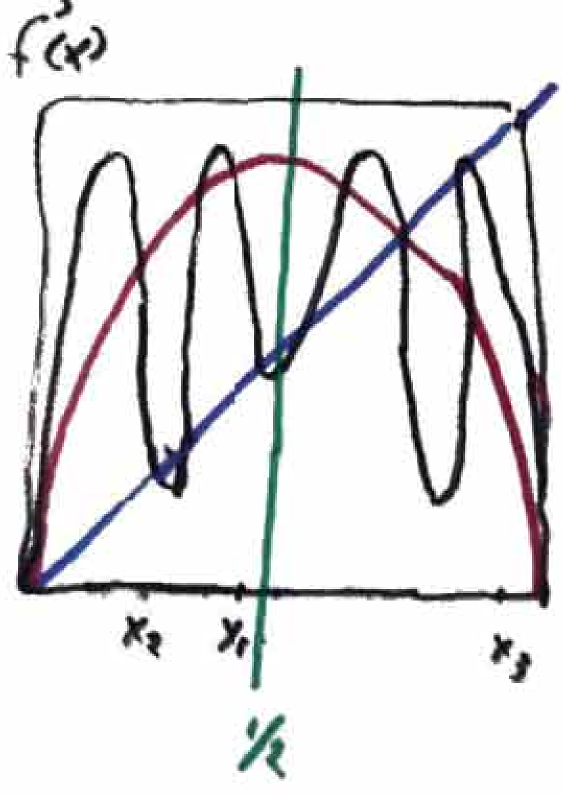
Can one prove a 3-cyle exists analytically?

If $a > a_\infty = 3.5694\ldots$

3-cycle possible

$$x_1 = f(x_2)$$
$$x_2 = f(x_3)$$
$$x_3 = f(x_1)$$





3-cycle problem: $f^3(x) - x = 0$

$$Q = x^6 - a^{-1}(3a+1)x^5$$
$$+ a^{-2}(3a^2+4a+1)x^4$$
$$- a^{-3}(a^3+5a^2+3a+1)x^3$$
$$+ a^{-4}(2a^3+3a^2+3a+1)x^2$$
$$- a^{-5}(a^3+2a^2+2a+1)x$$
$$+ a^{-6}(a^2+a+1) = 0 \qquad [1]$$

$$Q = (x-x_1)(x-x_2)(x-x_3)\cdot(x-x_1')(x-x_2')(x-x_3')$$
$$= Q^+ \cdot Q^- .$$

sextic Eq. (polyn. 6^{th})

Solvable ?

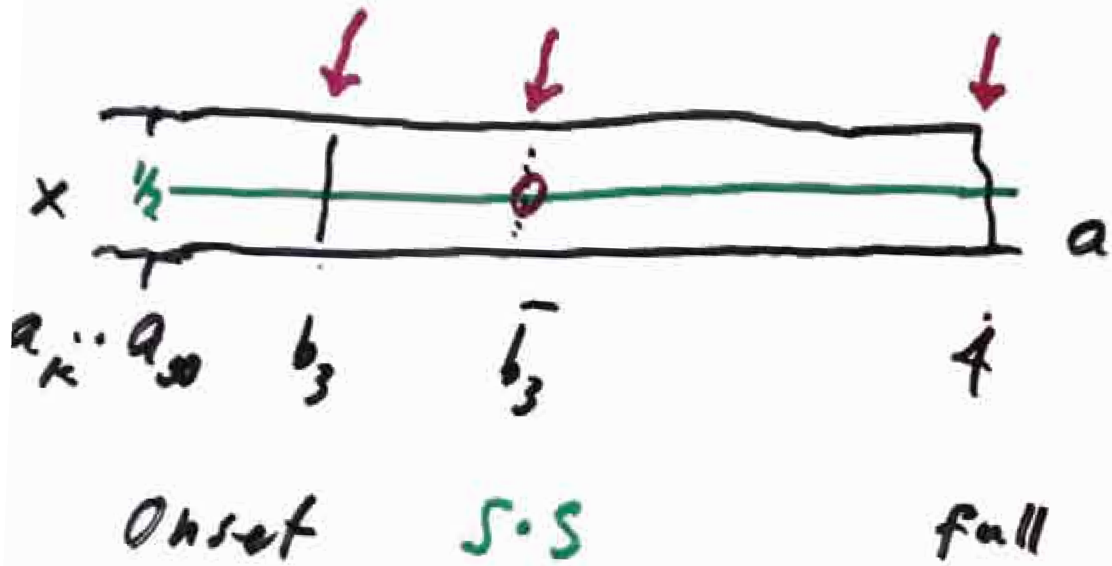
If $6^{th} \longrightarrow \{3^{rd}\}$,

by

Cardano - Tartaglia

(Cubic Eq)

Special solns of $\varphi(x, a)$



$a = \bar{b}_3$ $\quad$ $x = 1/2$: $\quad \varphi(a)$

$a = 4$ $\quad$ $\varphi(x)$

$a = b_3$ $\quad$ $\varphi = \varphi_+ \cdot \varphi_- = \varphi_+^2$

$\varphi = \varphi(x, a)$ general soln

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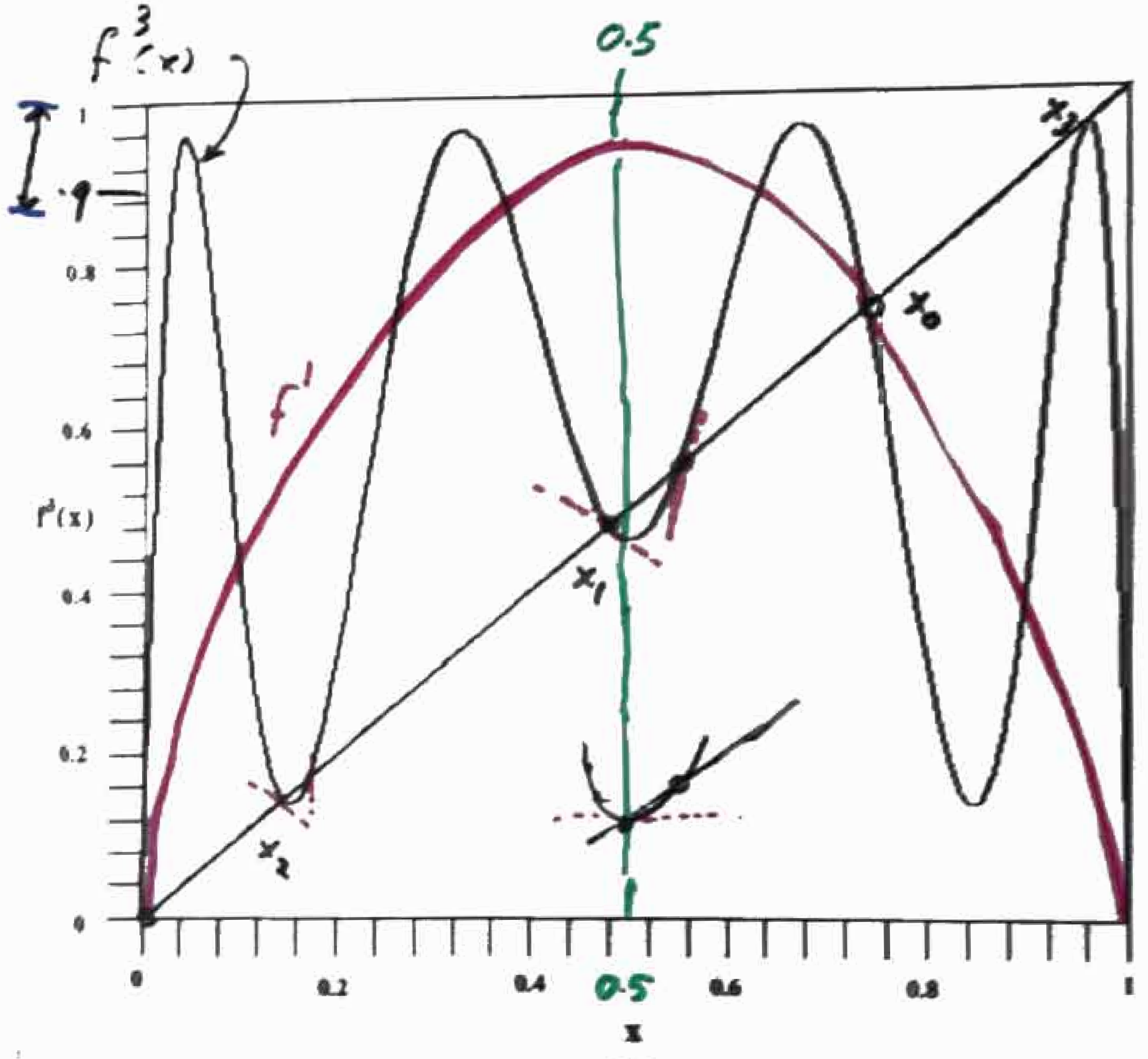


$f^3(x)$ versus x a=3.85

$\left(\frac{a}{4} = .9625\right)$

$\underline{x = 1/2}$

$$Q(a) = a^6 - 6a^5 + 4a^4 + 24a^3 - \cdots$$
$$-16a^2 - 32a^1 - 64 = 0$$

$Q(-a) \neq 0$

$a \to a'$

$$Q(a') = Q(-a') = 0$$

$a = a' + 1,$

$$Q(a') = a'^6 - 11a'^4 + 35a'^2 - 89$$

cubic Eq in a'^2

Solvable by radicals

$$a \equiv \bar{b}_3$$

$2f \quad \underline{a = 4}, \quad x = t/4$

$Q(x) \rightarrow Q(t)$

$= t^6 - 13t^5 + 65t^4 - 157t^3$

$+ 189t^2 - 105t + 21$

$= (t^3 - 7t^2 + 14t - 7)$

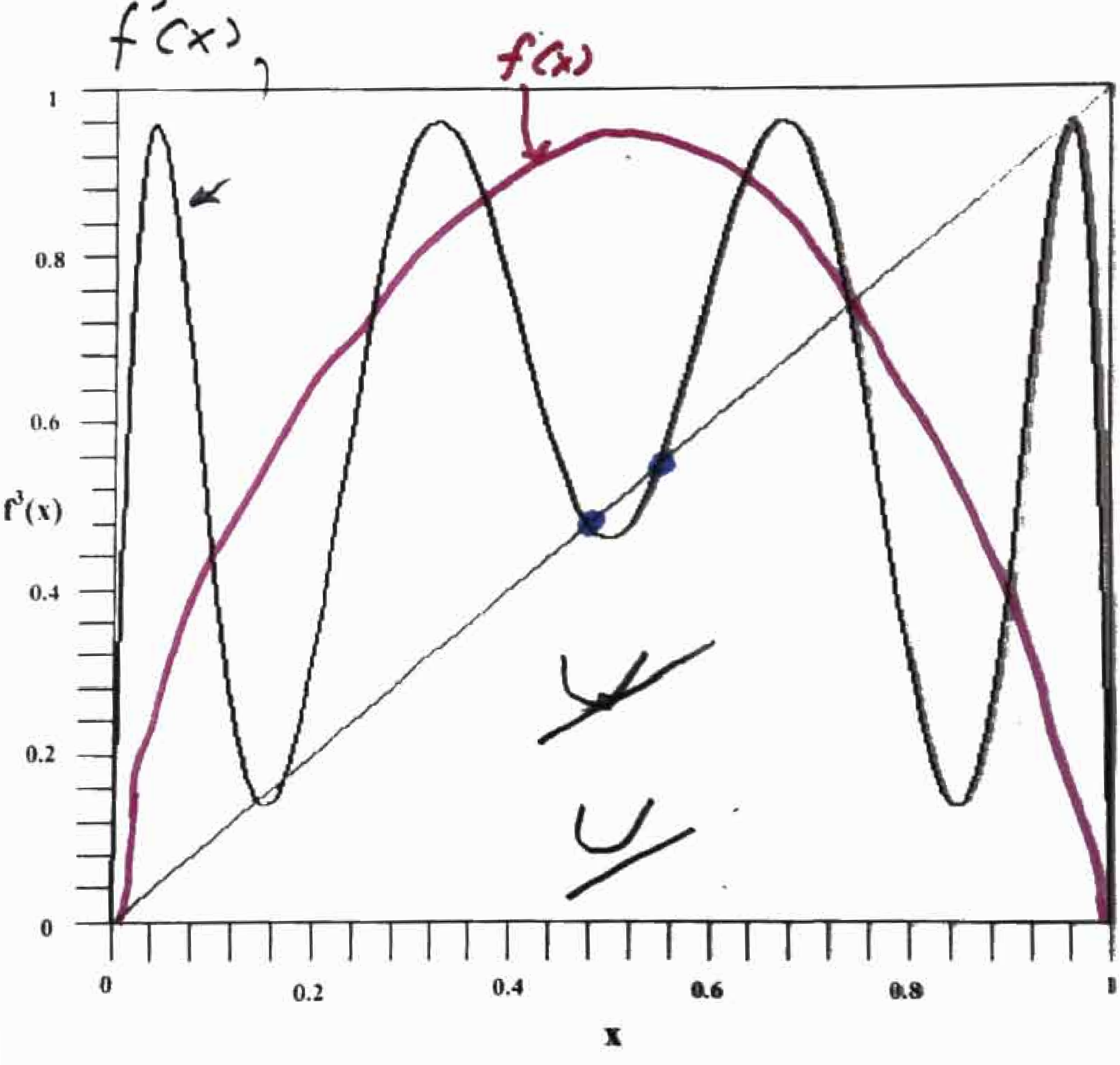
$\times (t^3 - 6t^2 + 9t - 3)$

$\equiv Q_+ \cdot Q_-$

$Q_+ : \quad \sin^2 \underline{\pi/7}, \quad \sin^2 \underline{2\pi/7}, \quad \sin^2 \underline{3\pi/7}$

$Q_- : \quad \sin^2 \underline{\pi/9}, \quad \sin^2 \underline{2\pi/9}, \quad \sin^2 \underline{4\pi/9}$

$a > b_3$



$f^3(x)$ versus x a=3.85 $> 1+\sqrt{8} = 3.82$

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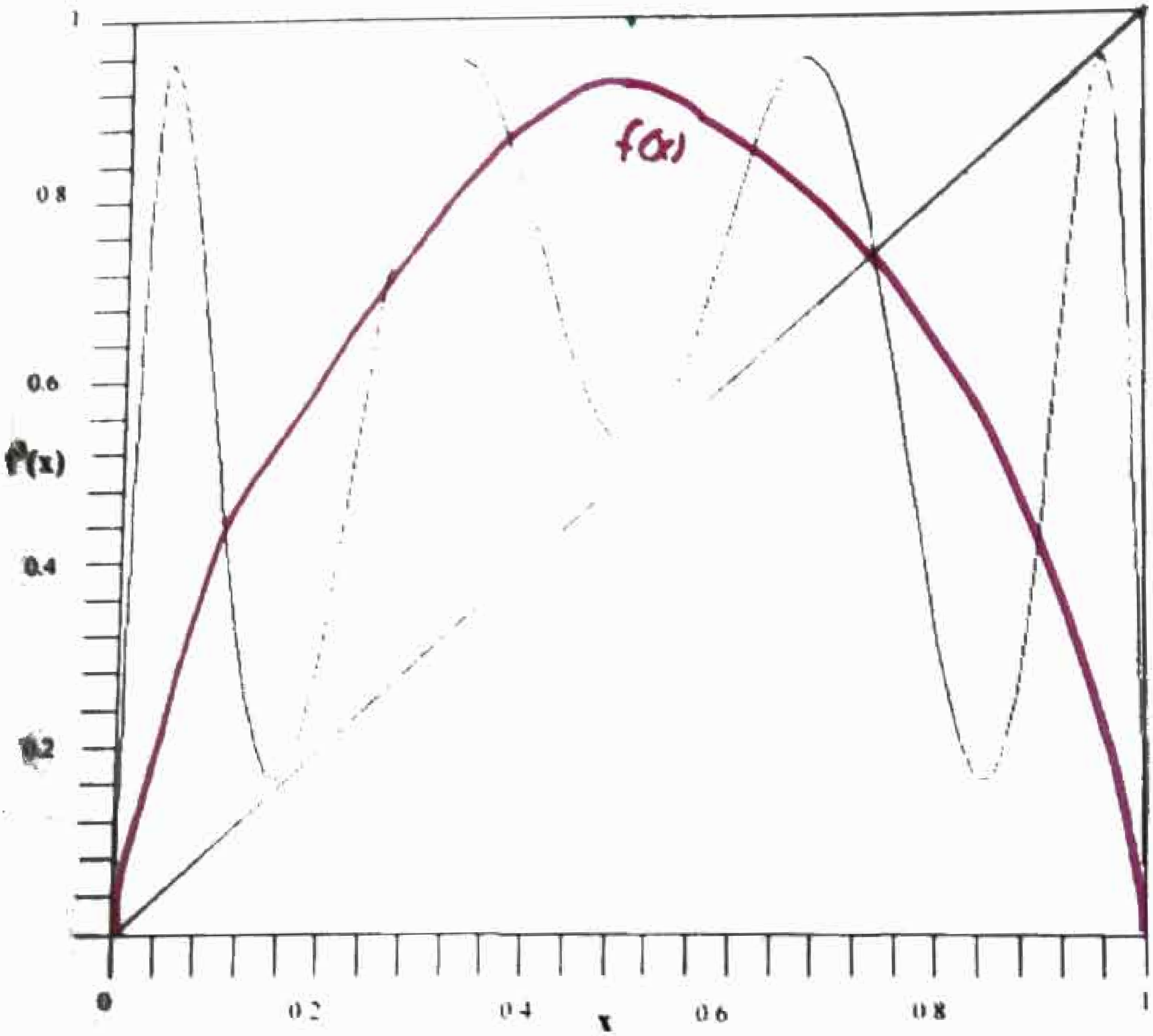
$f^3(x)$ $a < b_3$



$f^3(x)$ versus x $a=1+(8)^{(1/2)}-0.008$ $= 3.820$

$b_3 = 1+\sqrt{8} = 3.828$

if $\underline{a < b_3}$,

x_i, x_i' complex

if $\underline{a = b_3}$, $x_i = x_i'$ real.

$$Q = Q_+ \cdot Q_- = Q_+^2$$

$$Q_+ = (x - x_1)(x - x_2)(x - x_3)$$

$$= x^3 - \alpha x^2 + \beta x - \gamma$$

$$\alpha = x_1 + x_2 + x_3 \qquad (\mathrm{Tr}\, \mathcal{X})$$

$$\beta = x_1 x_2 + x_2 x_3 + x_3 x_1$$

$$\gamma = x_1 x_2 x_3 \qquad (\det \mathcal{X})$$

$$\mathcal{X} = \begin{bmatrix} x_1 & \cdot & \cdot \\ \cdot & x_2 & \cdot \\ \cdot & \cdot & x_3 \end{bmatrix}$$

$$Q = Q_+^2 = (x^3 - \alpha x^2 + \beta x - \gamma)^2$$

$$= x^6 - \underline{2\alpha}\, x^5 + \underline{(2\beta + \alpha^2)}\, x^4$$

$$- \underline{2(\gamma + \alpha\beta)}\, x^3 + (2\alpha\gamma + \beta^2)\, x^2$$

$$- 2\beta\gamma . x + \gamma^2 \qquad [2]$$

Compare [1] and [2] term by term

$$Q = x^6 - \underline{a^{-1}(3a+1)}\, x^5 + \underline{a^{-2}(3a^2}$$

$$\underline{+ 4a + 1)}\, x^4 - \underline{a^{-3}(a^3 + 5a^2 + 3a + 1)}\, x^3$$

$$+ a^{-4}(2a^3 + 3a^2 + 3a + 1)\, x^2 - a^{-5}(a^3$$

$$+ 2a^2 + 2a + 1)\, x + a^{-6}(a^2 + a + 1) \qquad [1]$$

$x^5 : \quad 2\underline{\alpha} = 3 + a^{-1}$

$x^4 : \quad 2\underline{\beta} = 3/4 + 5/2\, a^{-1} + 3/4\, a^{-2}$

$x^3 : \quad 2\underline{\gamma} = -1/8 + 7/8\, a^{-1} + 5/8\, a^{-2}$
$\qquad + 5/8\, a^{-3}$

$\overline{x^2, x^1,}\ x^0 :$

$S_2 = 3a^4 - 12a^3 - 141a^2 + 52a$
$\qquad + 35 = 0$

$S_1 = 3a^5 - 11a^4 - 18a^3 + 42a^2$
$\qquad + 63a + 49 = 0$

$S_0 = a^6 - 14a^5 + 39a^4 + 60a^3$
$\qquad - 161a^2 - 206a - 231 = 0$

$$S_2 = B_2 \cdot A$$

$$S_1 = B_1 \cdot A$$

$$S_0 = B_0 \cdot A$$

if $A = 0$, $S_2 = S_1 = S_0 = 0$

$$A = a^2 - 2a - 7$$

$$a \equiv b_3 = 1 + \sqrt{8} \quad (!)$$

(conjectured)

$$B_2 = 3a^2 - 6a - 5$$

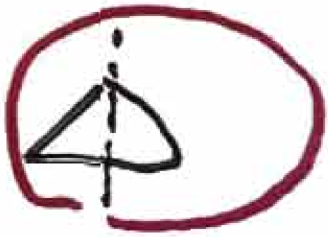
$$B_1 = 3a^3 - 5a^2 - 7a - 7$$

$$B_0 = a^4 - 12a^3 + 22a^2 + 20a + 33$$

2.6329.. ; 2.7984.. ; 3.5822..

$$\bar{b}_3 = 1 + \left\{ \frac{11}{3} + \frac{2}{3}\left[(100+\sqrt{9936})^{1/3} \right.\right.$$

$$\left.\left. + (100-\sqrt{9936})^{1/3} \right\}^{1/2}$$



$$= 3.831874025528315\,56841\,03627\ldots$$

Galois

$$x_3 = \frac{1}{4}\bar{b}_3 = .95797\ldots$$

$$x_2 = \frac{1}{4}\bar{b}_3^{\,2}\left(1-\frac{1}{4}\bar{b}_3\right) = .15429\ldots$$

$$x_1 = .5 \quad (\text{given})$$

J Math phys (Dec 09)