How many eigenvalues of a truncated orthogonal matrix are real?

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Plan

Motivation and setup; matrix measure for truncations

Joint probability distribution of EVs

Density of complex EVs

How many EVs are real? Density of real EVs

Correlation functions

Motivation

What truncations of unitary matrices are good for?

(1) **Quantum transport problems** (Beenakker '97) Additive stats of EVs of $TT^\dagger$ describe phys quantities of interest, i.e. $\mathrm{tr}\, TT^\dagger$ for conductance of quasi one-dimensional wires

(2) **Open chaotic sys** (Fyodorov & Sommers, '97 Życzkowski & S. '00) Eigenvalues of T are used to model resonances

(3) **Combinatorics of vicious walkers**(Novak '09) $< |\,\mathrm{tr}\, T|^N >_T$ enumerates configs of random-turn vicious walkers

Singular values of T (1); eigenvalues of T (2,3)

Why truncation of orthogonal matrices? To explore the degree of universality of EVs statistics in the complex plane (no mathematical theory known).

Setup

G is either unitary or orthogonal group, i.e. $G = U(n)$ or $G = O(n)$.

Choose a matrix from G at random and consider its top-left block T of size $m \times m$. This talk is concerned with eigenvalues of T which is a random contraction.

'Choose at random' - implicit referral to uniform distribution on the group surface, known as the Haar measure.

Thus, consider the unitary group G equipped with the normalised Haar measure. The property of invariance determines this measure uniquely. This can be used for sampling the Haar distribution via Gram-Schmidt.

The Haar measure induces a probability distribution $\mathrm{d}\rho_{n,m\times m}(T)$ on truncated unitaries (orthogonals). It does not depend on the block's position (because of the invariance of the Haar measure). Thus, may as well consider bottom-left corner, etc.

Matrix measure for truncations

Truncation: $U = \left(\begin{smallmatrix} T & S \\ Q & R \end{smallmatrix}\right) \mapsto T$, where T is $m \times m$, S is $m \times (n-m)$

Since $UU^\dagger = I$ we have $TT^\dagger + SS^\dagger = I$. Hence, (typically) $SS^\dagger$ has rank m if $n \geq 2m$ and the image of G is the entire matrix ball $TT^\dagger \leq I$ in this case.

Thm 1 *(Friedman&Mello '85, Fyodorov&Sommers '03, Forrester '06)*

For $n \geq 2m$

$$\mathrm{d}\rho_{n,m\times m}(T) = \frac{1}{C}\det(I - TT^\dagger)^{\frac{\beta}{2}(n-2m+1)-1}\chi_{TT^\dagger \leq I}(T)\,\mathrm{d}T$$

where $\beta = 2$ *for unitary matrices and* $\beta = 1$ *for orthogonal matrices.*

By changing to EVals and EVecs of $TT^\dagger$, the normalization constant is given by Selberg's integral (multivariate version of Euler's beta integral)

$$C = \int_0^1 \dots \int_0^1 \prod_{j<k} |\lambda_j - \lambda_k|^\beta \prod_{j=1}^m \lambda_j^{\frac{\beta}{2}-1}(1-\lambda_j)^{\frac{\beta}{2}(n-2m+1)-1} \prod_{j=1}^m \mathrm{d}\lambda_j$$

Matrix measure

If $n < 2m$ (e.g., deleting just one column and rows) then $\lambda = 1$ is an EV of $TT^\dagger$ of multiplicity $2m - n$. Hence the image of G is a set on the boundary of $TT^\dagger \leq I$. Useful explicit expression for $\mathrm{d}\rho_{n,m\times m}(T)$ is unknown in this case. However, for statistics depending on EVs only, e.g. trace, det, etc

Thm 2 *(Fyodorov & Kh. '07) Let $n < 2m$. Then for invariant f*

$$\int f(TT^\dagger)d\rho_{n,m\times m}(T) = \text{const.}\times$$

$$\int f\begin{pmatrix} ZZ^\dagger & 0 \\ 0 & I \end{pmatrix} \det(I - ZZ^\dagger)^{\frac{\beta}{2}(2m-n+1)-1}\chi_{ZZ^\dagger \leq I}(Z)dZ$$

(matrices T are $m \times m$ and Z are $(n-m)\times(n-m)$)

SVs of truncations: with Thms 1 and 2 in hand one can study distr. of (nontrivial) eigenvalues of $TT^\dagger$.

Gaussian approximation and beyond

Borel (1906): if U is drawn at random from G then the distribution of U_{11} converges to normal distribution as $n \to \infty$.

It is easy to see from Thm 1 that in the limit when $n \to \infty$ and m is kept finite the distribution of T becomes Gaussian (entries are independent normals). This is known as the Borel Theorem (Gallardo 1982, Yor 1985).

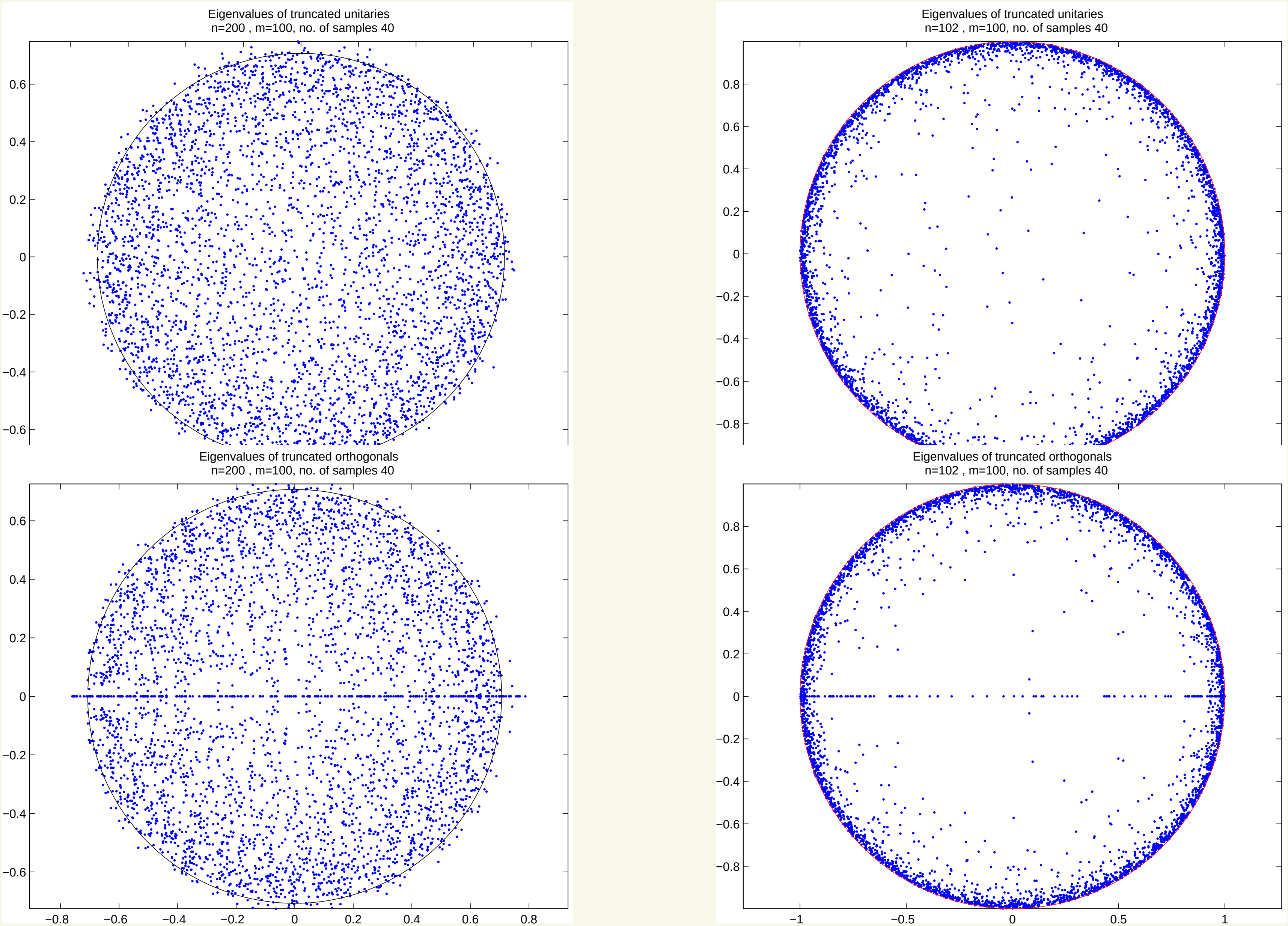
[More generally, if $m = o(\sqrt{n})$ then the distribution of T converges to standard normal. (Diaconis & collaborators, 1987, 1992, Jiang 2009.]

Two other interesting regimes (non-Gaussian):

(i) $n \to \infty$, $l := n - m = const.$ **(weak non-unitarity/orthogonality)**

(ii) $n \to \infty$, $\frac{m}{n} = const.$ **(strong non-unitarity/orth.)**

Eigenvalue scatter plots



“Standing on the shoulders of giants”

Gaussian random matrices with no symmetry conditions imposed (known as the Ginibre family of ensembles)

1965 Ginibre (complex matrices)

1991 Lehmann & Sommers (jpdf of EVs for real matrices)

1993 Edelman (published ’98) (density of complex EVs)

1994 Edelman, Kostlan & Shub (density of real EVs)

2005 Kanzieper & Akemann (probabilities of k real EVs)

2007 Forrester & Nagao (paffian representation for EV corr fncs via skew-orthogonal pols)

2007 Sommers and Sommers & Wieczorek (alternative derivation of pfaff representation via Grassmann integration)

Truncations: joint distribution of eigenvalues z_j

Thm 3 *(Życzkowski & Sommers '00) For* ***truncated Haar unitaries****, the prob of having real eigvs is zero, and (*$|z_j| < 1$*)*

$$d\mu(z_1, \ldots, z_m) \propto \prod_{1 \leq j < k \leq m} |z_j - z_k|^2 \prod_{j=1}^{m} (1 - |z_j|^2)^{l-1} \prod_{j=1}^{m} dz_j \wedge dz_j^*$$

Thm 4 *(Kh. Sommers & Życzkowski '10) For* ***truncated Haar orthogonals****, the prob of having real eigvs is not zero, and (*$|z_j| < 1$*)*

$$d\mu(z_1, \ldots, z_m) \propto \prod_{1 \leq j < k \leq m} (z_j - z_k) \prod_{j=1}^{m} f(z_j) \prod_{j=1}^{m} dz_j$$

with

$$f^2(z) = 2|1 - z^2|^{l-2} \int_{\frac{2|\operatorname{Im} z|}{|1-z^2|}}^{1} (1 - t^2)^{\frac{l-3}{2}} \, \mathrm{d}t.$$

Caveats: pairs of complex conjugated eigvs, ordering, conditioning by no of real eigvs, $f^2(z) = (2\pi|1 - z^2|)^{-1}$ for $l = 1$. Similarity to the Ginibre ens.

(Mean) EV densities

Density of complex eigenvalues $\rho_m^{(C)}(z)$: For D in $\mathbf{C}\backslash\mathbf{R}$:

average number of EVs in D given by $\int_D \rho_m^{(C)}(z)\mathrm{d}^2z$

Density of real eigenvalues $\rho_m^{(R)}(x)$:

average number of EVs in (a,b) given by $\int_a^b \rho_m^{(R)}(x)\mathrm{d}x$

Normalisation:

$$\int \rho_m^{(C)}(z)\mathrm{d}^2z + \int \rho_m^{(R)}(x)\mathrm{d}x = m$$

where m is the matrix dimension.

EV densities of truncated Haar unitaries

Let l be the no. of columns (rows) removed, i.e. $l = n - m$. The following is a corollary of Thm 3:

Thm 5 *Consider* ***truncated random unitary matrices****. Then* $\rho_m^{(R)}(x) = 0$ *and*

$$\rho_m^{(C)}(z) = \frac{l}{\pi} \frac{1}{(1-|z|^2)^{l-1}} \sum_{j=0}^{m-1} \binom{l+m}{m} |z|^{2j}, \quad |z| \le 1.$$

The truncated binomial series on the rhs can be expressed in terms of the **incomplete Beta function** $I_x(a,b) = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1}\, \mathrm{d}t$ leading to

$$\rho_m^{(C)}(z) = \frac{l}{\pi} \frac{1}{(1-|z|^2)^2}\, I_{1-|z|^2}(l+1, m), \quad |z| \le 1.$$

This comes in handy for asymptotic analysis (large truncation size).

EV densities of truncated Haar orthogonals

Let l be the no. of columns (rows) removed, i.e. $l = n - m$. The following is a corollary of Thm 4 (involves **evaluating** $\langle |\det(z-T)|^2 \rangle_T$):

Thm 6 *Consider* ***truncated random orthogonal matrices****. Assume m is even (technical). Then ($|x| \leq 1$)*

$$\rho_m^{(R)}(x) = \frac{I_{1-x^2}(l+1, m-1)}{B(\frac{l}{2}, \frac{1}{2})(1-x^2)} + \frac{(1-x^2)^{\frac{l}{2}-1}|x|^{m-1} I_{x^2}(\frac{m-1}{2}, \frac{l+2}{2})}{B(\frac{m}{2}, \frac{l}{2})}$$

and ($|z| \leq 1$)

$$\rho_m^{(C)}(z = x + \mathrm{i}y) = \frac{l(l-1)}{\pi} \, \frac{I_{1-|z|^2}(l+1, m-1)}{(1-|z|^2)^{l+1}} \, |y| \, f^2(z)$$

with $f^2(z) = 2|1-z^2|^{l-2} \int_{\frac{2|y|}{|1-z^2|}}^{1} (1-t^2)^{\frac{l-3}{2}} \, \mathrm{d}t$

These expressions become rather simple for m large!

Strong non-unitarity/orthogonality - density of complex EVs

Consider $m, l \to \infty$, $m \propto n$. For **truncated unitaries**:

$$\rho_m^{(C)}(z) \simeq \frac{l}{\pi} \frac{1}{(1-|z|^2)^2} \, \Theta\left(\frac{m}{n} - |z|^2\right)$$

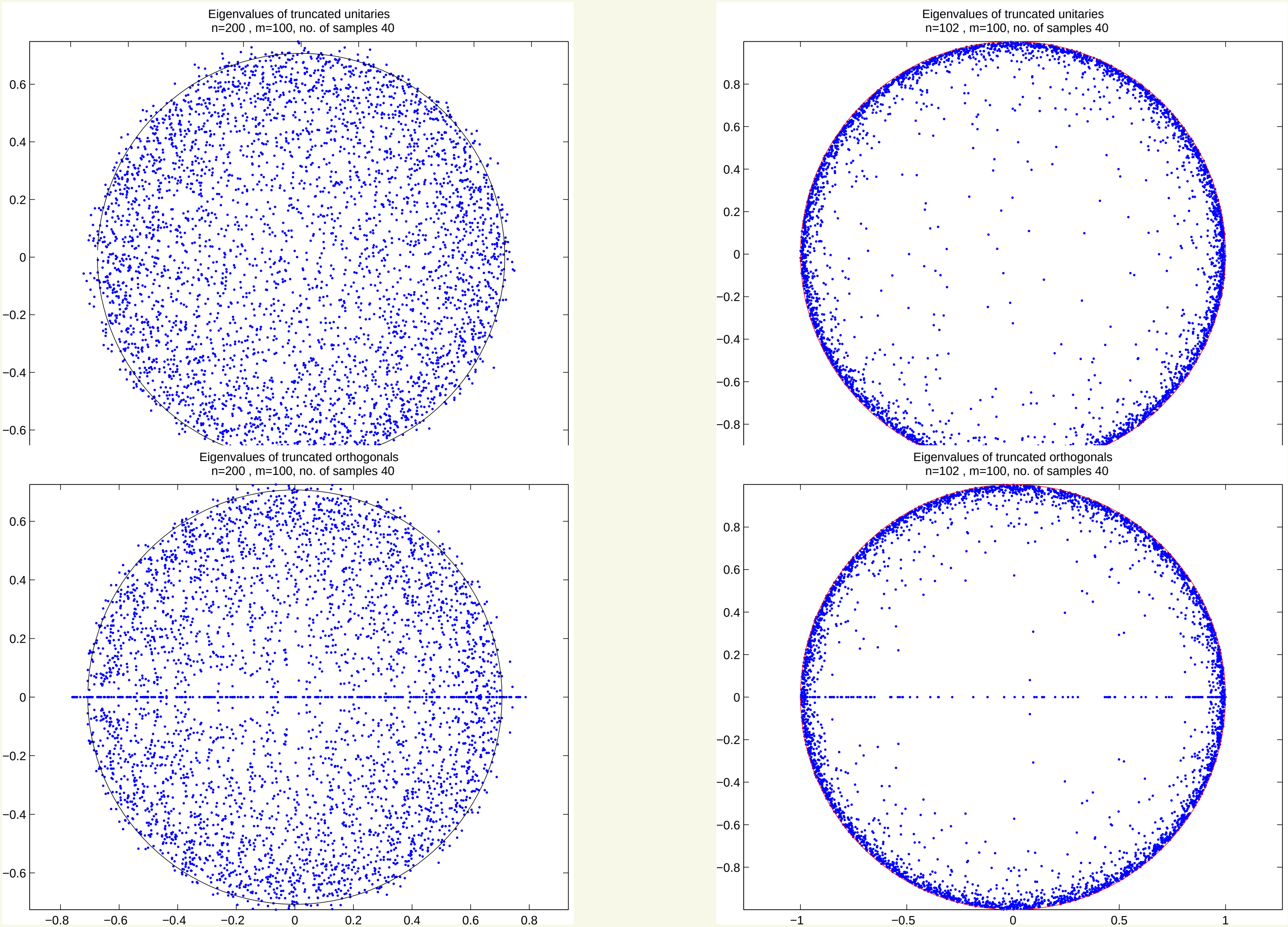
Same limiting form of $\rho_m^{(C)}(z)$ for **truncated orthogonals** away from the real axis (recall that $\rho_m^{(C)}$ vanishes on the real axis for m finite, hence finite size corrections to $\rho_m^{(C)}$ differ).

Close to the real line ($y \propto \frac{1}{\sqrt{m}}$) the density of complex EVs of **truncated orthogonals** is described by the **scaling law**

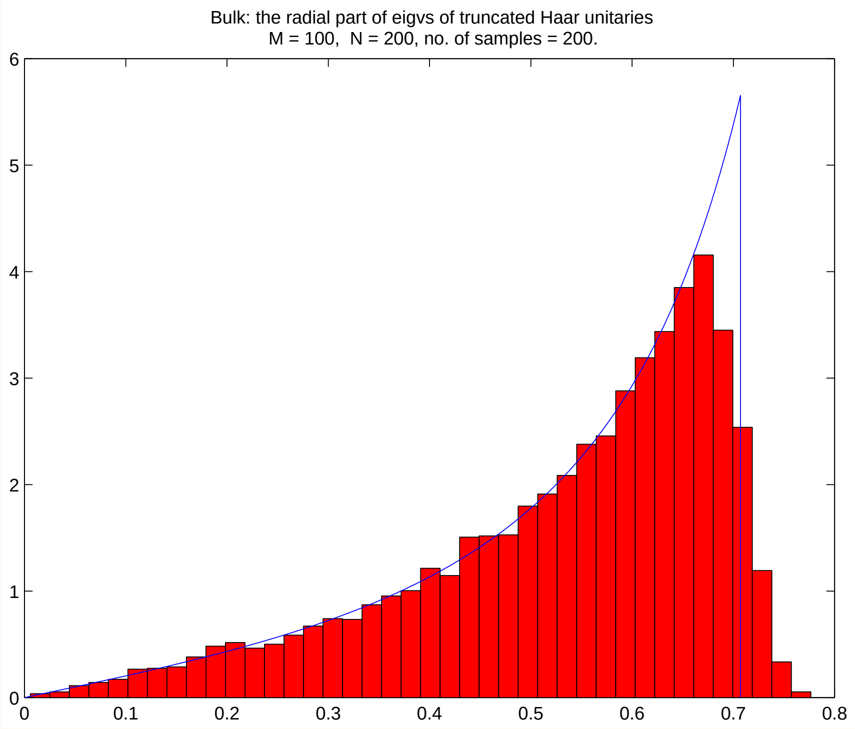
$$\rho_m^{(C)}(z) \simeq \rho_m^{(R)}(x)^2 h(y \rho_m^{(R)}(x)), \; h(y) = 4\pi|y| \mathrm{e}^{4\pi y^2} \operatorname{erfc}(\sqrt{4\pi}|y|)$$

where $\rho_m^{(R)}(x) = \sqrt{\frac{l}{2\pi}\frac{1}{1-x^2}}$ is the density of real EVs (more on this later). Same form as for the real Ginibre except $\rho_m^{(R)}$ is different. **Universality?**

Eigenvalue scatter plots



Strong non-unitarity: transition at the boundary of EV distribution



When one traverses the boundary of the support of EV distribution, the density vanishes at a Gaussian rate. In the transitional region

$$\rho_m^{(C)}\left(\sqrt{\frac{m}{n}}+\frac{x}{\sqrt{m}}\right)\simeq\frac{m\alpha}{2\pi}\,\mathrm{erfc}\left(\sqrt{2\alpha}\,x\right),\quad \alpha=\frac{n^2}{lm}$$

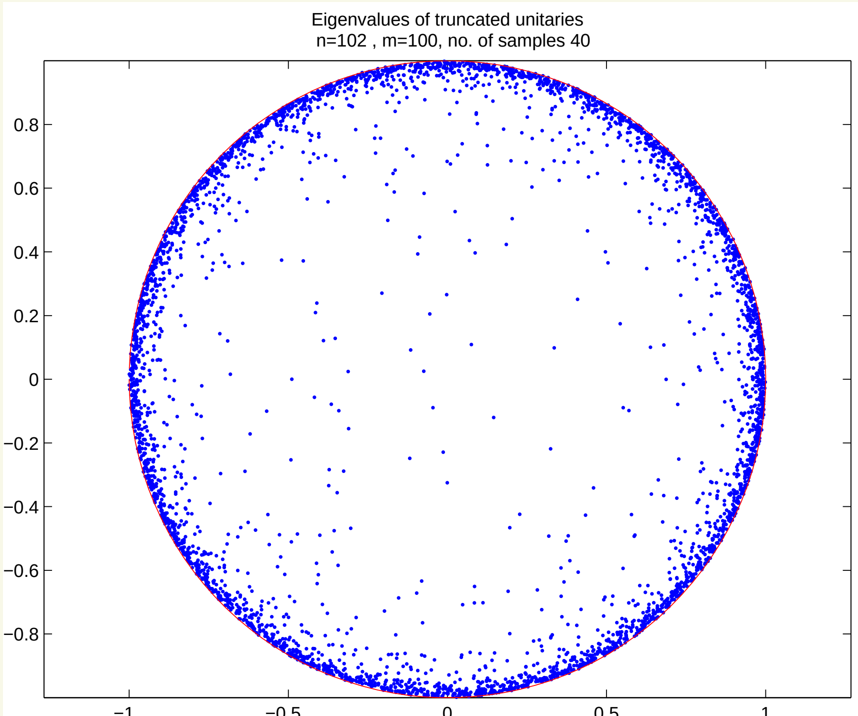
Same Erfc Law as for the Ginibre ensemble (Forrester ’99, Kanzieper ’03)

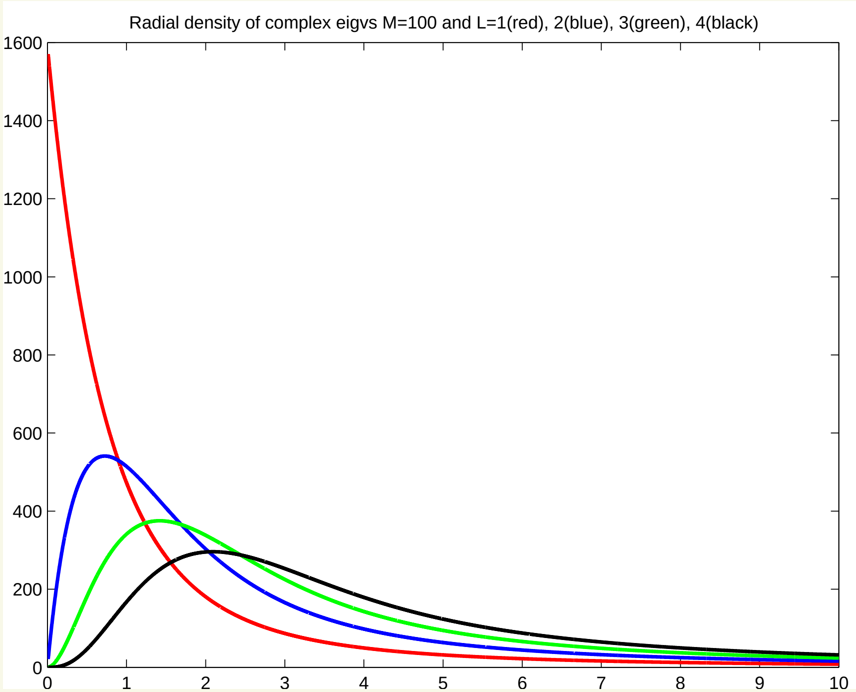
Weak non-unitarity/orthogonality: density of complex EVs

Assume $m \to \infty$, l is finite (delete l rows& columns). In this limit the EVs of **truncated Haar unitaries** lie close to the unit circle, typically at distance $\propto \frac{1}{m}$. Scaling z accordingly, one finds the EV density

$$\rho_m^{(C)}\left(\left(1-\frac{r}{m}\right)\mathrm{e}^{\mathrm{i}\varphi}\right) \simeq \frac{m^2}{\pi}\frac{(2r)^{l-1}}{(l-1)!}\int_0^1 \mathrm{e}^{-2rt}t^l\,\mathrm{d}t.$$

Away from the real axis same result for **truncated orthogonals**.





Truncated orthogonal matrices - average no. of real EVs

$N_m^{(R)} = \int_{-1}^{1} \rho_m^{(R)}(x)\mathrm{d}x$ is the average number of real EVs

Finite matrix dimension:

$$N_m^{(R)} = 1 + \frac{l}{2}\int_0^1 \frac{\mathrm{d}s}{s^{l+1}}\ I_{s^2}\left(l/2, 1/2\right)\ I_{\frac{2s}{1+s}}\left(l+1, m-1\right)$$

In the limit of **strong non-orthogonality,** $m, l \to \infty$**,** $l \propto m$:

$$N_m^{(R)} \simeq \sqrt{\frac{l}{2\pi}} \ln \frac{\sqrt{n}+\sqrt{m}}{\sqrt{n}-\sqrt{m}} \propto \sqrt{m}$$

In the limit of **weak non-orthogonality,** $m \to \infty$**,** l **is finite**:

$$N_m^{(R)} \simeq \frac{\log m}{B\left(\frac{l}{2}, \frac{1}{2}\right)}$$

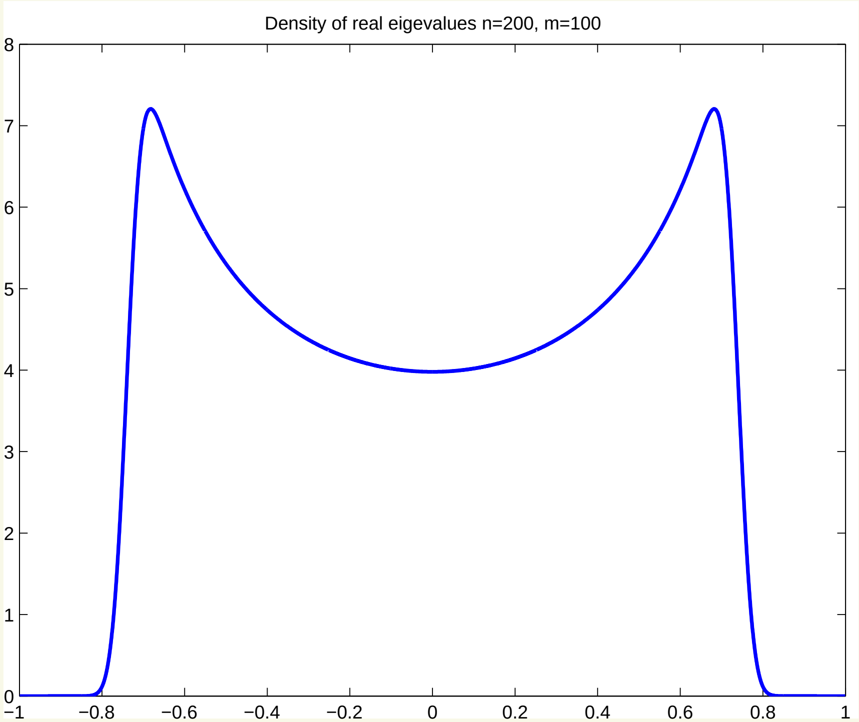
Cf.: $N_m^{(R)} \propto \sqrt{m}$ in the real Ginibre (Edelman,Kostlan & Shub, 1994);

$N_m^{(R)} \propto \log m$ for random real polynomials (Kac, 1948).

Strong non-orthogonality - density of real EVs,

Consider $m, l \to \infty$, $m \propto n$. In this limit the distribution of the real EVs of **truncated Haar orthogonals** is described by the **'Artanh Law'**

$$\rho_m^{(R)}(x) \simeq \sqrt{\frac{l}{2\pi}} \frac{1}{1-x^2} \Theta\left(\frac{m}{n} - x^2\right).$$

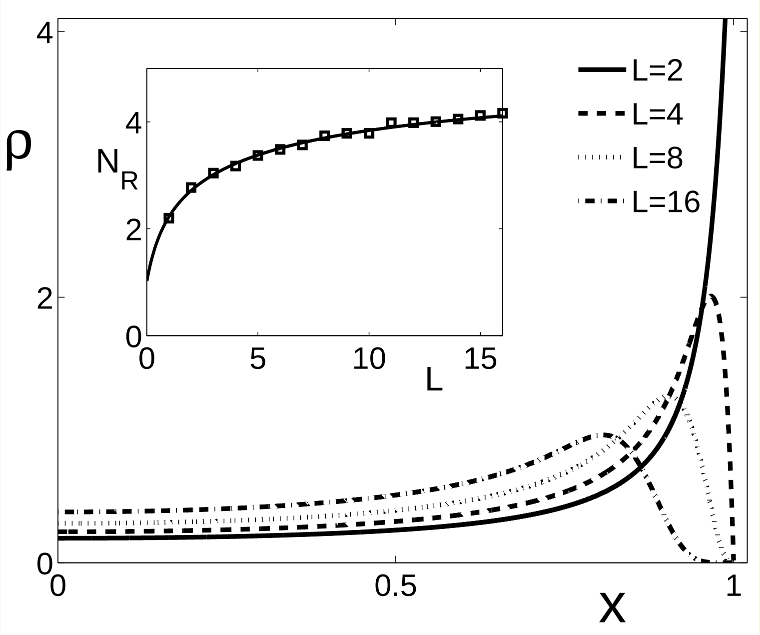


Weak non-orthogonality - density of real EVs

Consider $m \to \infty$, l is finite. In this limit the bulk of the real EVs lie in the vicinity of ± 1. The (average) number of real EVs away from these accumulation points is finite and they are distributed there with density (compare with strong non-orthogonality!)

$$\rho_m^{(R)}(x) \simeq \sqrt{\frac{l}{2\pi}} \frac{1}{1-x^2}, \quad x \in (-1+\varepsilon, 1-\varepsilon).$$

Interesting behaviour near the acc. pnts, below $\rho := \rho_{32}^{(R)}(x)/N_{32}^{(R)}$



Weak non-orthogonality - density of real EVs near the acc. pnts

Real EVs are accumulating near $x \pm 1$. Need to rescale $x = 1 - \frac{u}{m}$ and $p_m(u) = \frac{1}{m}\rho_m^{(R)}(1 - \frac{u}{m})$ to see the shape of distribution. We have

$$p_m(u) \simeq p(u) = \frac{1}{2u}\frac{1}{B(\frac{l}{2},\frac{1}{2})}\frac{\int\limits_0^{2u} t^l \mathrm{e}^{-t}\,\mathrm{d}t}{\Gamma(l+1)} + \frac{u^{\frac{l}{2}-1}\mathrm{e}^{-u}}{2\Gamma(\frac{l}{2})}\frac{\int\limits_u^{\infty} t^{\frac{l}{2}}\mathrm{e}^{-t}\,\mathrm{d}t}{\Gamma(\frac{l}{2}+1)}.$$

The 2nd term describes density **for small u; have $p(u) \simeq \frac{u^{\frac{l-2}{2}}}{2\Gamma(\frac{l}{2})}$**

Note different behaviour when $l = 1$, $l = 2$, $l = 3$ and $l \geq 4$. In the latter case the real EVs are ‘repelled’ from $x = 1$.

The 1st term describes behaviour **for large u; have $p(u) \propto \frac{1}{u}$, a heavy tail** leading to the $\log m$ growth of the number of real eigenvalues.

EV correlations

- **Away from the real axis**

  Same (scaling limit) for complex Ginibre, real Ginibre, truncated Haar unitaries and orthogonals in the regime of strong non-orthogonality.

  Same for truncated Haar unitaries and orthogonals in the regime of weak non-orthogonality.

- **Near the real axis**

  Same (scaling limit) real Ginibre and truncated Haar orthogonals in the regime of strong non-orthogonality.

  New correlations in the regime of weak non-orthogonality.

- **On the real axis**

  Same (scaling limit) real Ginibre and truncated Haar orthogonals

  New correlations in the regime of weak non-orthogonality.

Outlook/Open Problems