Universal Quantum Transport in Chaotic Cavities and Painlevé Transcendents

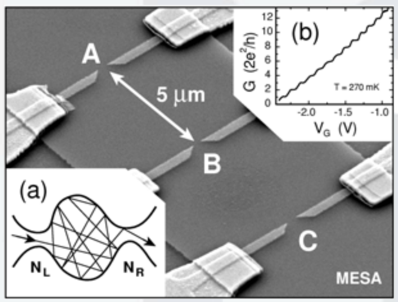
Eugene Kanzieper[1,2] & Vladimir Al Osipov[3]

Email

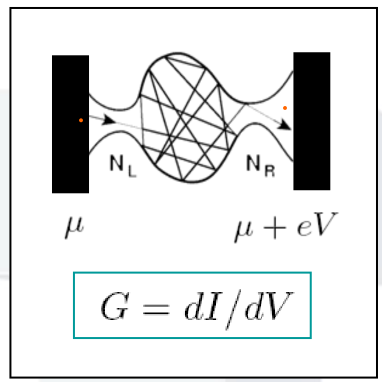
[1] Department of Applied Mathematics @ H.I.T. - Holon Institute of Technology
[2] Department of Physics of Complex Systems @ Weizmann Istitute of Science
[3] Fakultät für Physik @ Universität Duisburg-Essen

Thanks
- Satya Majumdar
- The Israel Science Foundation

- The system, the problem, and the main result



Quantum chaotic cavity



Statistics of the Landauer conductance

Paul Painlevé

- Why the result is interesting
  - Brief excursion into the Painlevé property
  - Appearance of Painlevé transcendents in statistical physics

- **The system, the problem, and the main result**
- **Why the result is interesting**
  - Brief excursion into the Painlevé property
  - Appearance of Painlevé transcendents in statistical physics
- **Derivation** (Landauer conductance)
  - Cumulants of the Landauer conductance
  - Distribution function and the end of large-N controversy
  - Relation to other works
- **Furher results** (if time permits)
  - Statistics of thermal to shot noise crossover
  - Joint cumulants of Landauer conductance and noise power
- **Conclusions**

Paul Painlevé

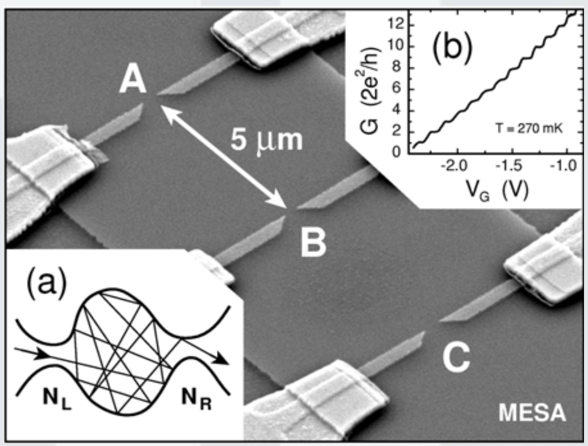
Gaston Darboux

Satya Majumdar

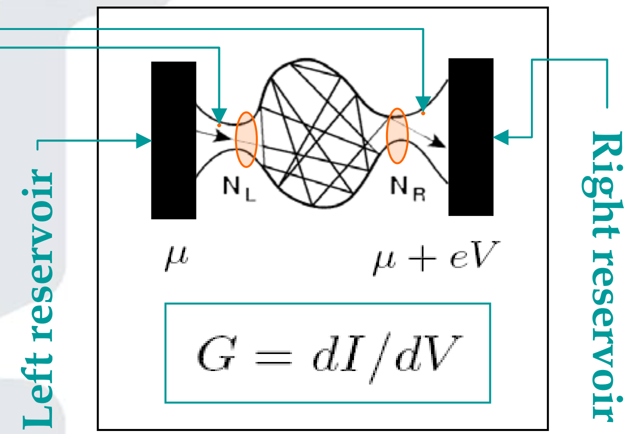
The system, the problem

$$H_{\text{tot}} = \sum_{k,\ell=1}^{M} \psi_k^\dagger \mathcal{H}_{k\ell}\psi_\ell + \sum_{\alpha=1}^{N_L+N_R} \chi_\alpha^\dagger \varepsilon_F \chi_\alpha + \sum_{k=1}^{M}\sum_{\alpha=1}^{N_L+N_R} \left(\psi_k^\dagger \mathcal{W}_{k\alpha}\chi_\alpha + \chi_\alpha^\dagger \mathcal{W}^*_{k\alpha}\psi_k\right)$$

cavity — leads — coupling



© S. Oberholzer, Universität Bazel

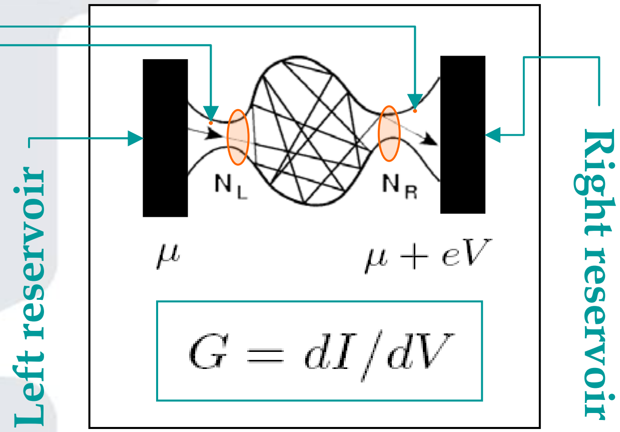


Landauer conductance

The system, the problem

$$H_{\rm tot} = \sum_{k,\ell=1}^{M} \psi_k^\dagger \mathcal{H}_{k\ell}\psi_\ell + \sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \chi_\alpha^\dagger \varepsilon_F \chi_\alpha + \sum_{k=1}^{M}\sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \left(\psi_k^\dagger W_{k\alpha}\chi_\alpha + \chi_\alpha^\dagger W_{k\alpha}^* \psi_k\right)$$

cavity — leads — coupling



Universal transport regime

- Electron dwell time $\tau_D \simeq A/(W v_F)$ much larger than the Ehrenfest time $\tau_E \simeq \lambda^{-1}\log(W/\lambda_F)$
- RMT dynamics: $\{\mathcal{H}_{k\ell}\} \in {\rm GUE}_{M\times M}$

Landauer conductance

Quantum Transport and Painlevé Transcendents

- The system, the problem, and the main result

$$\left(z\sigma_V''(z)\right)^2 - \left[\sigma_V(z) + 2\left(\sigma_V'(z)\right)^2 + (N_L + N_R - z)\sigma_V'(z)\right]^2 + 4\left(\sigma_V'(z)\right)^2\left(N_L + \sigma_V'(z)\right)\left(N_R + \sigma_V'(z)\right) = 0$$

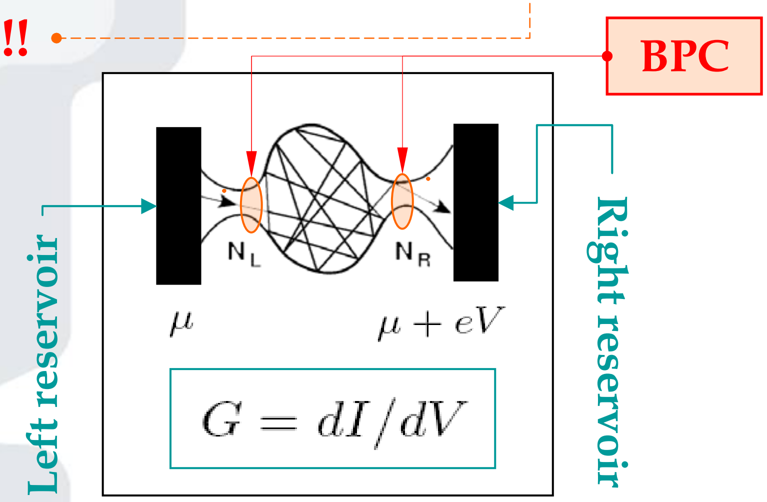
$$\sigma_V(z) = N_L N_R + \sum_{\ell=1}^{\infty} \frac{(-z)^\ell}{\Gamma(\ell)} \langle\langle (G/G_0)^\ell \rangle\rangle$$

ℓ – th cumulant of the Landauer conductance

Fifth Painlevé transcendent !!

Universal transport regime

- Electron dwell time $\tau_D \simeq A/(W v_F)$ much larger than the Ehrenfest time $\tau_E \simeq \lambda^{-1} \log(W/\lambda_F)$
- RMT dynamics: $\{\mathcal{H}_{k\ell}\} \in \mathrm{GUE}_{M\times M}$



Landauer conductance

- The system, the problem, and the main result
- **Why the result is interesting**
  - Brief excursion into the Painlevé property
  - Appearance of Painlevé transcendents in statistical physics

Paul Painlevé

- Derivation (Landauer conductance)
  - Cumulants of the Landauer conductance
  - Distribution function and the end of large-*N* controversy
  - Relation to other works
- Furher results (if time permits)
  - Statistics of thermal to shot noise crossover
  - Joint cumulants of Landauer conductance and noise power
- Conclusions

Brief excursion into Painlevé property

$$y'' = F(x, y, y')$$

Rational function of its arguments — **E. Picard** (1889)

All **movable singularities** are restricted to **poles** **(no movable branch points)**

50 → 44

(a) **linear 2nd order DEs**
(b) **Weierstrass DE**
(c) **Riccati DE**

$$(y')^2 = 4y^3 - g_2 y - g_3$$

$$y' = a(x)\,y^2 + b(x)\,y + c(x)$$

[6 **Painlevé equations P_I – P_{VI}** nonlinear special functions]

P. Painlevé (1900,1902)
B. Gambier (1905)
R. Fuchs (1910)

Brief excursion into Painlevé property

σP$_{\rm III}$

$$\left(t\sigma_{\rm III}''\right)^2 - \nu_1\nu_2\left(\sigma_{\rm III}'\right)^2 + \sigma_{\rm III}'(4\sigma_{\rm III}' - 1)(\sigma_{\rm III} - t\sigma_{\rm III}') - \frac{1}{4^3}(\nu_1 - \nu_2)^2 = 0$$

σP$_{\rm V}$

$$\left(t\sigma_{\rm V}''\right)^2 - \left[\sigma_{\rm V} - t\sigma_{\rm V}' + 2(\sigma_{\rm V}')^2 + (\nu_0 + \nu_1 + \nu_2 + \nu_3)\sigma_{\rm V}'\right]^2 + 4(\nu_0 + \sigma_{\rm V}')(\nu_1 + \sigma_{\rm V}')(\nu_2 + \sigma_{\rm V}')(\nu_3 + \sigma_{\rm V}') = 0$$

Hamiltonian system — **Bäcklund transformations**

[6 **Painlevé equations $P_{\rm I}$ – $P_{\rm VI}$** nonlinear special functions]

P. Painlevé (1900,1902)
B. Gambier (1905)
R. Fuchs (1910)

fascinating properties

P. Clarkson, Painleve equations – nonlinear special functions, J. Comp. Appl. Math. 153, 127 (2003)

Brief excursion into Painlevé property

Remark:

Painlevé property is the child of «pure reason» ...

Question:

Should Painlevé transcendents be relevant to the description of the physical world?

Answer:

Yes (seven decades after Painlevé discovery) !!

- The system, the problem, and the main result
- **Why the result is interesting**
  - Brief excursion into the Painlevé property
  - Appearance of Painlevé transcendents in statistical physics
- Derivation (Landauer conductance)
  - Cumulants of the Landauer conductance
  - Distribution function and the end of large-*N* controversy
  - Relation to other works
- Furher results (if time permits)
  - Statistics of thermal to shot noise crossover
  - Joint cumulants of Landauer conductance and noise power
- Conclusions

Paul Painlevé

Painlevé transcendents in statistical physics

- **2D Ising model**

$$H_{\rm int}^{(2D)} = -J\sum_{j,k}(\sigma_j$$

(a) R. Peierls (1936), (b)
(1941), (c) L. Onsager (19
T. Wu, B. McCoy, C. T

$$\langle \sigma_{00}\sigma_{MN}\rangle\Big|_{R=(M^2+N}^{T\to}$$

$$\tanh(J/T_c) = \sqrt{2}-1$$

2007 Wiener Prize

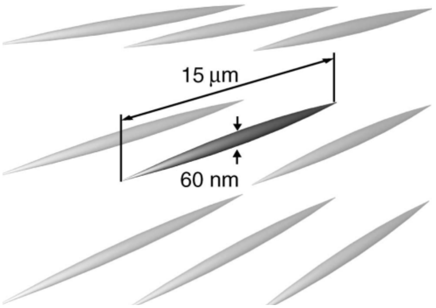
The committee also recognizes the earlier work of Craig Tracy with Wu, McCoy, and Barouch, in which Painlevé functions appeared for the first time in exactly solvable statistical mechanical models. In addition, the committee recognizes the seminal contributions of Harold Widom to the asymptotic analysis of Toeplitz determinants and their various operator theoretic generalizations.

σP_{III}

Painlevé transcendents in statistical physics

- **Impenetrable Bose gas** $g \to \infty$

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial z_j^2} + g \sum_{i<j} \delta(z_i - z_j)$$



© T. Esslinger ETH Zürich

FIG. 1. The geometry and size of trapped 1D gases in a two-dimensional optical lattice. The spacing between the 1D tubes in the horizontal and vertical direction is 413 nm.

M. Girardeau (1960)
A. Lenard (1964)

M. Jimbo, T. Miwa, Y. Môri, and M. Sato (1980)

$$\varrho_N(x) = N \int_0^L dz_2 \cdots dz_N \, \Psi^*(x, z_2, \cdots, z_N) \, \Psi(0, z_2, \cdots, z_N)$$

$$\varrho_\infty(x) = \lim_{N\to\infty} \varrho_N(x)\Big|_{L=N} = \exp\left(\int_0^{\pi x} \frac{dt}{t}\, \sigma_{\rm V}(t)\right)$$ ⬅ σP_V

Painlevé transcendents in statistical physics

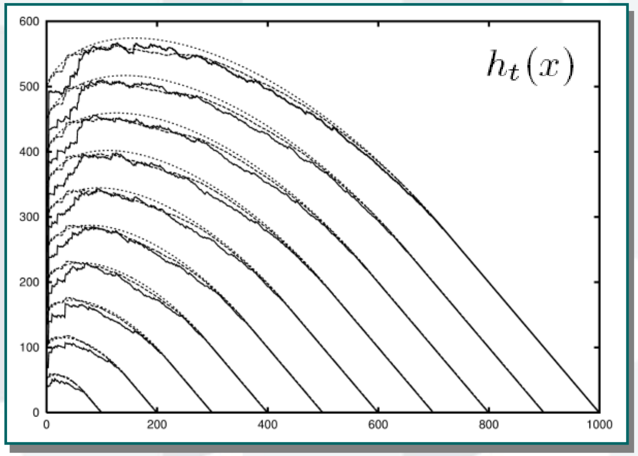
- **Growth models in (1+1)D / oriented digital boiling**

$$h_{t+1}(x) = \max\left\{h_t(x-1),\, h_t(x) + \varepsilon_{x,t}\right\}$$

$$h_0(x) = \begin{cases} 0 & \text{if } x = 0 \\ -\infty & \text{otherwise} \end{cases}$$

$$\varepsilon_{x,t} \sim \mathrm{Ber}(p)$$

J. Gravner, C. Tracy, and H. Widom (2001)



Universal regime of shape fluctuations

$$x \to \infty,\; t \to \infty,\; p < p_c = 1 - x/t < 1$$

$$\mathrm{Prob}\left(\frac{h_t(x) - c_1 t}{c_2 t^{1/3}} < s\right) = \boldsymbol{F}_2\left([\sigma_{\mathrm{II}}]\,;\, s\right)$$

σP_{II}

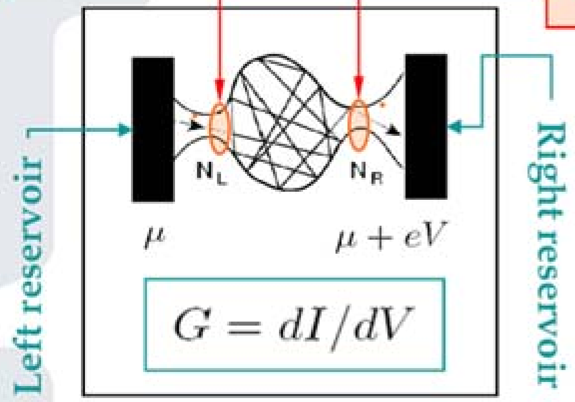
The system, the problem, and the main result

$$\sigma_V(z) = N_L N_R + \sum_{\ell=1}^{\infty} \frac{(-z)^\ell}{\Gamma(\ell)} \langle\langle (G/G_0)^\ell \rangle\rangle$$

ℓ – th cumulant of the Landauer conductance

Fifth Painlevé transcendent !!

BPC



Landauer conductance

Universal transport regime

- ✓ Electron dwell time $\tau_D \simeq A/(W v_F)$ much larger than the Ehrenfest time $\tau_E \simeq \lambda^{-1} \log(W/\lambda_F)$
- ✓ RMT dynamics: $\{\mathcal{H}_{k\ell}\} \in \mathrm{GUE}_{M\times M}$

- The system, the problem, and the main result
- Why the result is interesting
  - Brief excursion into the Painlevé property
  - Appearance of Painlevé transcendents in statistical physics
- Derivation (Landauer conductance)
  - Cumulants of the Landauer conductance
  - Distribution function and the end of large-N controversy
  - Relation to other works
- Furher results (if time permits)
  - Statistics of thermal to shot noise crossover
  - Joint cumulants of Landauer conductance and noise power
- Conclusions

Paul Painlevé

Cumulants

$$\mathcal{S}(\varepsilon_F) = \mathbb{1} - 2i\pi\mathcal{W}^\dagger(\varepsilon_F - \mathcal{H} + i\pi\mathcal{W}\mathcal{W}^\dagger)^{-1}\mathcal{W}$$

$$\mathcal{H} \in \mathrm{GUE}_{M\times M}$$

P. Brouwer (1995)

$$P(\mathcal{S}) \propto \left[\det\left(\mathbb{1} - \bar{\mathcal{S}}\mathcal{S}^\dagger\right)\det\left(\mathbb{1} - \mathcal{S}\bar{\mathcal{S}}^\dagger\right)\right]^{-(N_{\mathrm{L}}+N_{\mathrm{R}})}$$

$$\bar{\mathcal{S}} = V^\dagger \,\mathrm{diag}(\sqrt{1-\Gamma_j})\, V, \quad V \in G(N)/G(N_{\mathrm{L}})\times G(N_{\mathrm{R}})$$

$a^{(\mathrm{L})} = \begin{pmatrix} a_1^{(\mathrm{L} \\ \vdots \\ a_{N_\mathrm{L}}^{(\mathrm{L} \end{pmatrix}$

$b^{(\mathrm{L})} = \begin{pmatrix} b_1^{(\mathrm{L} \\ \vdots \\ b_{N}^{(\mathrm{L} \end{pmatrix}$

0_{N_R} $\mathbb{1}_{N_\mathrm{R}}$

Landauer conductance

Cumulants of the Landauer conductance (derivation)

$$H_{\rm tot} = \sum_{k,\ell=1}^{M} \psi_k^\dagger \mathcal{H}_{k\ell} \psi_\ell + \sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \chi_\alpha^\dagger \varepsilon_F \chi_\alpha + \sum_{k=1}^{M} \sum_{\alpha=1}^{N_{\rm L}+N_{\rm R}} \left(\psi_k^\dagger \mathcal{W}_{k\alpha} \chi_\alpha + \chi_\alpha^\dagger \mathcal{W}_{k\alpha}^* \psi_k \right)$$

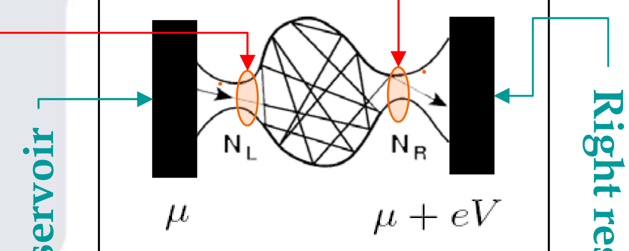
$$\mathcal{S} \in {\rm CUE}(N_{\rm L} + N_{\rm R})$$

BPC

Scattering matrix approach

$$G/G_0 = {\rm tr}\, \left(\mathcal{C}_1 \mathcal{S} \mathcal{C}_2 \mathcal{S}^\dagger \right)$$

$$\mathcal{C}_1 = \begin{pmatrix} \mathbb{1}_{N_{\rm L}} & \\ & 0_{N_{\rm R}} \end{pmatrix} \quad \mathcal{C}_2 = \begin{pmatrix} 0_{N_{\rm L}} & \\ & \mathbb{1}_{N_{\rm R}} \end{pmatrix}$$



$$G = dI/dV$$

Landauer conductance

Cumulants of the Landauer conductance (derivation)

$$\mathcal{F}_n(z) = \left\langle e^{-z \,\mathrm{tr}\,(\mathcal{S}\mathcal{C}_1\mathcal{S}^\dagger\mathcal{C}_2)} \right\rangle_{\mathcal{S}\in \mathrm{CUE}(N_\mathrm{L}+N_\mathrm{R})}$$

Cumulant generating function

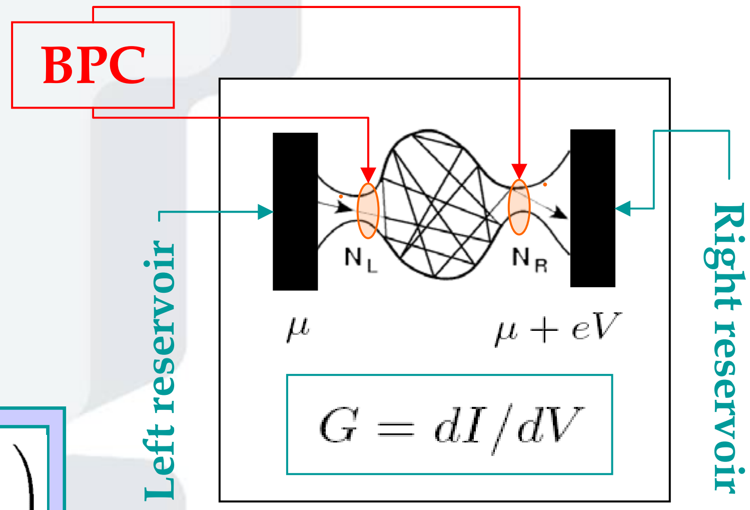
$n = \min(N_\mathrm{L}, N_\mathrm{R})$

Itzykson-Zuber formula, but: high degeneracy of C-matrices

$$\mathcal{S} = \begin{pmatrix} r_{N_\mathrm{L}\times N_\mathrm{L}} & t'_{N_\mathrm{L}\times N_\mathrm{R}} \\ t_{N_\mathrm{R}\times N_\mathrm{L}} & r'_{N_\mathrm{R}\times N_\mathrm{R}} \end{pmatrix}$$

$$\mathcal{C}_1 = \begin{pmatrix} \mathbb{1}_{N_\mathrm{L}} & \\ & 0_{N_\mathrm{R}} \end{pmatrix} \quad \mathcal{C}_2 = \begin{pmatrix} 0_{N_\mathrm{L}} & \\ & \mathbb{1}_{N_\mathrm{R}} \end{pmatrix}$$

Truncate! Hua (1963)
Zyczkowski & Sommers (2000)



Landauer conductance

Cumulants of the Landauer conductance (derivation)

$$\mathcal{F}_n(z) = \left\langle e^{-z\,\mathrm{tr}\,(\mathcal{S}\mathcal{C}_1\mathcal{S}^\dagger\mathcal{C}_2)} \right\rangle_{\mathcal{S}\in\mathrm{CUE}(N_\mathrm{L}+N_\mathrm{R})}$$

Cumulant generating function

$$n = \min(N_\mathrm{L}, N_\mathrm{R})$$

$$\nu = |N_\mathrm{L} - N_\mathrm{R}|$$

Itzykson-Zuber formula, but: high degeneracy of C-matrices

$$\mathcal{S} = \begin{pmatrix} r_{N_\mathrm{L}\times N_\mathrm{L}} & t'_{N_\mathrm{L}\times N_\mathrm{R}} \\ t_{N_\mathrm{R}\times N_\mathrm{L}} & r'_{N_\mathrm{R}\times N_\mathrm{R}} \end{pmatrix}$$

$$\mathcal{F}_n(z) = \left\langle e^{-z\,\mathrm{tr}\,(\boldsymbol{t}\boldsymbol{t}^\dagger)} \right\rangle_{\boldsymbol{t}_{N_\mathrm{R}\times N_\mathrm{L}}}$$

Truncate! Hua (1963)
Zyczkowski & Sommers (2000)

$$\mathcal{F}_n(z) \propto \int_{(0,1)^n} \prod_{j=1}^{n} dT_j\, T_j^\nu \exp(-zT_j) \cdot \Delta_n^2(\boldsymbol{T})$$

$$\mathcal{S} \in \mathrm{CUE}(N_\mathrm{L} + N_\mathrm{R})$$

M. Adler, T. Shiota, P. van Moerbeke, Phys. Lett. A 208 (1995) 67.
M. Adler, P. van Moerbeke, Ann. Math. 153 (2001) 149.

Correlations of RMT characteristic polynomials and integrability: Hermitean matrices

Vladimir Al. Osipov[a], Eugene Kanzieper[b,c,*]

[a] Fakultät für Physik, Universität Duisburg-Essen, D-47057 Duisburg, Germany
[b] Department of Applied Mathematics, H.I.T. – Holon Institute of Technology, Holon 58102, Israel
[c] Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel

Annals of Physics 325 (2010) 2251–2306

- Cumulants of the ... rating

$$\mathcal{F}_n(z) = \Big\langle e^{-z\,\mathrm{tr}} \ldots$$

$N_{\rm R})$

$\ldots |N_{\rm L} - N_{\rm R}|$

- Gap formation probability in LUE
- Fruitful ideas of integrability

How can it be evaluated **nonperturbatively ?!**

$$\mathcal{F}_n(z) \propto \int_{(0,1)^n} \prod_{j=1}^{n} dT_j\, T_j^{\nu} \exp(-zT_j) \cdot \Delta_n^2(\boldsymbol{T})$$

Quantum Transport and Painlevé Transcendents

Cumulants of the Landauer conductance

$$\mathcal{J}_n(z) = \int_{(0,1)^n} \prod_{j=1}^{n} dT_j\, T_j^{\nu}\, e^{-zT_j}\, \Delta_n^2(\boldsymbol{T})$$

$$\tau_n(\boldsymbol{t}; z) = \frac{1}{n!} \int_{(0,1)^n} \prod_{j=1}^{n} dT_j\, T_j^{\nu}\, e^{-zT_j + V(\boldsymbol{t}; T_j)}\, \Delta_n^2(\boldsymbol{T})$$

$$V(\boldsymbol{t}; T) = \sum_{k=1}^{\infty} t_k T^k$$

- **Kadomtsev-Petviashvili equation** (the whole hierarchy)

$$\left(\frac{\partial^4}{\partial t_1^4} + 3\frac{\partial^2}{\partial t_2^2} - 4\frac{\partial^2}{\partial t_1 \partial t_3} \right) \log \tau_n(\boldsymbol{t}; z) + 6 \left(\frac{\partial^2}{\partial t_1^2} \log \tau_n(\boldsymbol{t}; z) \right)^2 = 0$$

- **Virasoro constraints** $q \geq 0$

$$\left[\hat{\mathcal{L}}_{q+1}(\boldsymbol{t}) - \hat{\mathcal{L}}_q(\boldsymbol{t}) - z\frac{\partial}{\partial t_{q+2}} + (z+\nu)\frac{\partial}{\partial t_{q+1}} - \nu \frac{\partial}{\partial t_q} \right] \tau_n(\boldsymbol{t}; z) = 0$$

$$\hat{\mathcal{L}}_q(\boldsymbol{t}) = \sum_{j=1}^{\infty} j t_j \frac{\partial}{\partial t_{q+j}} + \sum_{j=0}^{q} \frac{\partial^2}{\partial t_j \partial t_{q-j}}$$

$$[\hat{\mathcal{L}}_p, \hat{\mathcal{L}}_q] = (p-q)\hat{\mathcal{L}}_{p+q}$$

Virasoro algebra

- **Relation**

$$\mathcal{J}_n(z) = n!\, \tau_n(\boldsymbol{t}, z)\Big|_{\boldsymbol{t}=0}$$

$$\mathcal{I}_n(z) = \int_{(0,1)^n} \prod_{j=1}^{n} dT_j\, T_j^{\nu}\, e^{-zT_j}\, \Delta_n^2(\boldsymbol{T})$$

Solve the two together at $\boldsymbol{t} = 0$!!

- **Kadomtsev-Petviashvili equation** (the whole hierarchy)

$$\left(\frac{\partial^4}{\partial t_1^4} + 3\frac{\partial^2}{\partial t_2^2} - 4\frac{\partial^2}{\partial t_1 \partial t_3}\right) \log \tau_n(\boldsymbol{t}; z) + 6\left(\frac{\partial^2}{\partial t_1^2} \log \tau_n(\boldsymbol{t}; z)\right)^2 = 0$$

- **Virasoro constraints**

$$\left[\hat{\mathcal{L}}_{q+1}(\boldsymbol{t}) - \hat{\mathcal{L}}_q(\boldsymbol{t}) - z\frac{\partial}{\partial t_{q+2}} + (z+\nu)\frac{\partial}{\partial t_{q+1}} - \nu\frac{\partial}{\partial t_q}\right] \tau_n(\boldsymbol{t}; z) = 0$$

$$\hat{\mathcal{L}}_q(\boldsymbol{t}) = \sum_{j=1}^{\infty} j t_j \frac{\partial}{\partial t_{q+j}} + \sum_{j=0}^{q} \frac{\partial^2}{\partial t_j \partial t_{q-j}}$$

$q = 0,\ q = 1$

- **Relation**

$$\mathcal{I}_n(z) = n!\, \tau_n(\boldsymbol{t}, z)\Big|_{\boldsymbol{t}=0}$$

$$\mathcal{I}_n(z) = \int_{(0,1)^n} \prod_{j=1}^{n} dT_j\, T_j^{\nu}\, e^{-zT_j}\, \Delta_n^2(\boldsymbol{T})$$

Solve the two together at $t = 0$!!

- **Final result**

  - **Introduce** $$\sigma_{\rm V}(z) = N_{\rm L}N_{\rm R} + z\frac{\partial}{\partial z}\log \mathcal{I}_n(z)$$

    cumulant generating function (for $\log \mathcal{I}_n(z)$)

  - **Observe the fifth Painleve transcendent**

$$\left(z\frac{\partial^2}{\partial z^2}\sigma_{\rm V}(z)\right)^2 - \left[\sigma_{\rm V}(z) + 2\left(\frac{\partial}{\partial z}\sigma_{\rm V}(z)\right)^2 + (N_{\rm L}+N_{\rm R}-z)\frac{\partial}{\partial z}\sigma_{\rm V}(z)\right]^2 + 4\left(\frac{\partial}{\partial z}\sigma_{\rm V}(z)\right)^2\left(N_{\rm L}+\frac{\partial}{\partial z}\sigma_{\rm V}(z)\right)\left(N_{\rm R}+\frac{\partial}{\partial z}\sigma_{\rm V}(z)\right) = 0$$

$$\mathcal{J}_n(z) = \int_{(0,1)^n} \prod_{j=1}^{n} dT_j \, T_j^{\nu} \, e^{-zT_j} \, \Delta_n^2(\boldsymbol{T})$$

Solve the two together at $t = 0$!!

- **Final result**

cumulant generating function

$$\left(z\frac{\partial^2}{\partial z^2}\sigma_{\rm V}(z)\right)^2 - \left[\sigma_{\rm V}(z) + 2\left(\frac{\partial}{\partial z}\sigma_{\rm V}(z)\right)^2 + (N_{\rm L}+N_{\rm R}-z)\frac{\partial}{\partial z}\sigma_{\rm V}(z)\right]^2$$
$$+4\left(\frac{\partial}{\partial z}\sigma_{\rm V}(z)\right)^2\left(N_{\rm L}+\frac{\partial}{\partial z}\sigma_{\rm V}(z)\right)\left(N_{\rm R}+\frac{\partial}{\partial z}\sigma_{\rm V}(z)\right) = 0$$

$$\sigma_{\rm V}(z) = N_{\rm L}N_{\rm R} + \sum_{\ell=1}^{\infty}\frac{(-z)^{\ell}}{\Gamma(\ell)}\langle\langle (G/G_0)^{\ell}\rangle\rangle$$

ℓ – th cumulant of the Landauer conductance

Fifth Painleve transcendent ← Taylor expansion

Cumulants of the Landauer conductance: Recurrence relation

$$\kappa_\ell = \langle\langle (G/G_0)^\ell \rangle\rangle$$

$$\kappa_1(g) = \frac{N_L N_R}{N_L + N_R}$$

$$\kappa_2(g) = \frac{\kappa_1^2(g)}{(N_L + N_R)^2 - 1}$$

$$\begin{aligned} [(N_L + N_R)^2 - j^2](j+1)\kappa_{j+1}(g) &= 2\sum_{\ell=0}^{j-1} (3\ell+1)(j-\ell)^2 \binom{j}{\ell} \kappa_{\ell+1}(g)\kappa_{j-\ell}(g) \\ &- (N_L + N_R)(2j-1)\, j\kappa_j(g) - j(j-1)(j-2)\,\kappa_{j-1}(g) \end{aligned}$$

!

$$\sigma_V(z) = N_L N_R + \sum_{\ell=1}^{\infty} \frac{(-z)^\ell}{\Gamma(\ell)} \langle\langle (G/G_0)^\ell \rangle\rangle$$

ℓ – th cumulant of the Landauer conductance

Fifth Painleve transcendent ← Taylor expansion

!

Cumulants of the Landauer conductance: $1/n$ expansion

$$\kappa_\ell(g) = \frac{n}{2}\delta_{\ell,1} + \frac{1}{16}\delta_{\ell,2} + \frac{1+(-1)^\ell}{8}\frac{(\ell-1)!}{(4n)^\ell}$$

$$[(N_{\rm L}+N_{\rm R})^2 - j^2](j+1)\kappa_{j+1}(g) = 2\sum_{\ell=0}^{j-1}(3\ell+1)(j-\ell)^2\binom{j}{\ell}\kappa_{\ell+1}(g)\kappa_{j-\ell}(g) - (N_{\rm L}+N_{\rm R})(2j-1)\,j\kappa_j(g) - j(j-1)(j-2)\,\kappa_{j-1}(g)$$

!

$$\sigma_{\rm V}(z) = N_{\rm L}N_{\rm R} + \sum_{\ell=1}^{\infty}\frac{(-z)^\ell}{\Gamma(\ell)}\langle\langle (G/G_0)^\ell\rangle\rangle$$

ℓ – th cumulant of the Landauer conductance

Fifth Painleve transcendent ← Taylor expansion

!

The system, the problem, and the main result

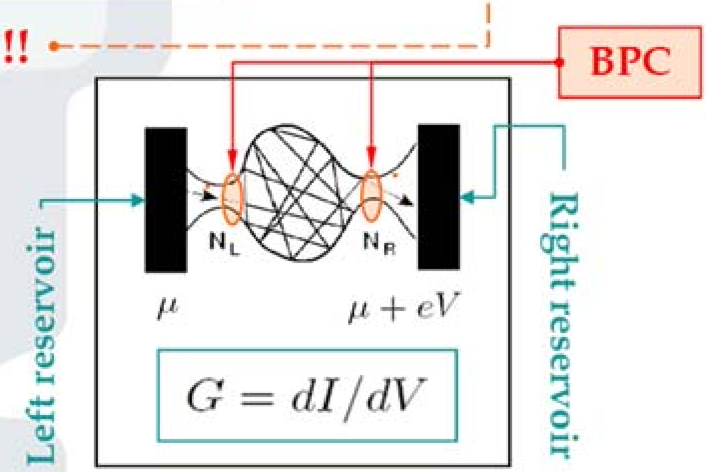
$$\sigma_{\mathrm{V}}(z) = N_{\mathrm{L}} N_{\mathrm{R}} + \sum_{\ell=1}^{\infty} \frac{(-z)^\ell}{\Gamma(\ell)} \langle\langle (G/G_0)^\ell \rangle\rangle$$

ℓ – th cumulant of the Landauer conductance

Fifth Painlevé transcendent !!

Universal transport regime

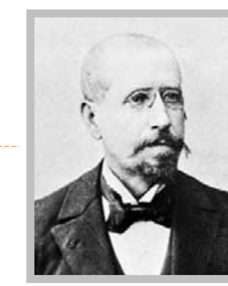
- ✓ Electron dwell time $\tau_D \simeq A/(W v_F)$ much larger than the Ehrenfest time $\tau_E \simeq \lambda^{-1} \log(W/\lambda_F)$
- ✓ RMT dynamics: $\{\mathcal{H}_{k\ell}\} \in \mathrm{GUE}_{M\times M}$



Landauer conductance

- The system, the problem, and the main result
- Why the result is interesting
    - Brief excursion into the Painlevé property
    - Appearance of Painlevé transcendents in statistical physics
- Derivation (Landauer conductance)
    - Cumulants of the Landauer conductance
    - Distribution function
    - Relation to other works
- Furher results (if time permits)
    - Statistics of thermal to shot noise crossover
    - Joint cumulants of Landauer conductance and noise power
- Conclusions

Paul Painlevé



Gaston Darboux

Distribution of the Landauer conductance (derivation)

$$\mathcal{F}_n(z) = c_n^{-1} \int_{(0,1)^n} \prod_{j=1}^{n} dT_j \, T_j^{\nu} \exp(-zT_j) \cdot \Delta_n^2(\boldsymbol{T}) \quad = \frac{n!}{c_n} \det{}_{\ell,k} \left[\int_0^1 dT \, T^{\ell+k+\nu} e^{-zT} \right]$$

Laplace transform of conductance probability density function

$$\Delta_n(\boldsymbol{T}) = \det \left[T_j^{\ell-1} \right]$$

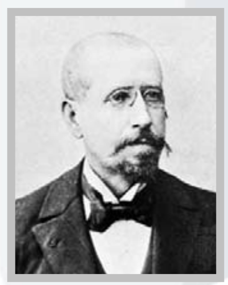
Hankel determinant

$$\mathcal{F}_n(z) = \frac{n!}{c_n} \det \left[(-\partial_z)^{\ell+k} \, \mathcal{F}_1(z) \right]$$

$$\mathcal{F}_1(z) = \frac{(\nu+1)!}{z^{\nu+1}} \left(1 - e^{-z} \sum_{\ell=0}^{\nu} \frac{z^\ell}{\ell!} \right)$$

$$\mathrm{var}_n(G) = \frac{n+1}{n} \frac{c_{n-1} c_{n+1}}{c_n^2}$$

$$\frac{N_{\mathrm{L}}^2 N_{\mathrm{R}}^2}{(N_{\mathrm{L}} + N_{\mathrm{R}})^2 [(N_{\mathrm{L}} + N_{\mathrm{R}})^2 - 1]}$$



Gaston Darboux

$$\mathcal{F}_n(z) \, \mathcal{F}_n''(z) - \left(\mathcal{F}_n'(z) \right)^2 = \mathrm{var}_n(G) \, \mathcal{F}_{n-1}(z) \, \mathcal{F}_{n+1}(z)$$

$$\mathcal{F}_0(z) = 1$$

The Toda Lattice equation → recursive procedure ← exact solution

Applied Mathematics

מכון טכנולוגי חולון
Holon Institute of Technology

Qua

D

π^{08}

Correlations of RMT characteristic polynomials and integrability: Hermitean matrices

Vladimir Al. Osipov[a], Eugene Kanzieper[b,c,*]

[a] *Fakultät für Physik, Universität Duisburg-Essen, D-47057 Duisburg, Germany*
[b] *Department of Applied Mathematics, H.I.T. – Holon Institute of Technology, Holon 58102, Israel*
[c] *Department of Physics of Complex Systems, Weizmann Institute of Science, Rehovot 76100, Israel*

Annals of Physics 325 (2010) 2251–2306

Distribution of the Landauer conductance (derivation)

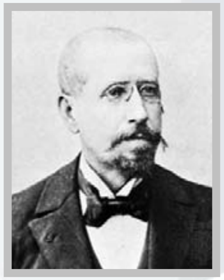
- **Recurrence brings**

$$\mathcal{F}_n(z) = \sum_{\ell=0}^{n} e^{-\ell z} \mathcal{A}_\ell \left(\frac{1}{z}\right)$$

- **Inverse Laplace transform yields the conductance p. d. f.**

$$\mathrm{var}_n(G) = \frac{n+1}{n} \frac{c_{n-1} c_{n+1}}{c_n^2}$$

$$\frac{N_\mathrm{L}^2 N_\mathrm{R}^2}{(N_\mathrm{L}+N_\mathrm{R})^2[(N_\mathrm{L}+N_\mathrm{R})^2-1]}$$



Gaston Darboux

$$\mathcal{F}_n(z) = \frac{n!}{c_n} \det\left[(-\partial_z)^{\ell+k} \mathcal{F}_1(z)\right]$$

$$\mathcal{F}_1(z) = \frac{(\nu+1)!}{z^{\nu+1}} \left(1 - e^{-z} \sum_{\ell=0}^{\nu} \frac{z^\ell}{\ell!}\right)$$

$$\mathcal{F}_n(z)\,\mathcal{F}_n''(z) - \left(\mathcal{F}_n'(z)\right)^2 = \mathrm{var}_n(G)\,\mathcal{F}_{n-1}(z)\,\mathcal{F}_{n+1}(z)$$

$$\mathcal{F}_0(z) = 1$$

The Toda Lattice equation → recursive procedure ← exact solution

Quantum Transport and Painlevé Transcendents

Outline

π^{07}

- The system, the problem, and the main result
- Why the result is interesting
  - Brief excursion into the Painlevé property
  - Appearance of Painlevé transcendents in statistical physics
- **Derivation** (Landauer conductance)
  - Cumulants of the Landauer conductance
  - Distribution function and the end of large-n controversy
  - Relation to other works
- Furher results (if time permits)
  - Statistics of thermal to shot noise crossover
  - Joint cumulants of Landauer conductance and noise power
- Conclusions

Paul Painlevé

Gaston Darboux

Satya Majumdar

Applied Mathematics

מכון טכנולוגי חולון
Holon Institute of Technology

Quantum Transport and Painlevé Transcendents

- Gram-Charlier expansion
- *1/n* cumulant expansion

$$\kappa_\ell(g) = \frac{n}{2}\delta_{\ell,1} + \frac{1}{16}\delta_{\ell,2} + \frac{1+(-1)^\ell}{8}\frac{(\ell-1)!}{(4n)^\ell}$$

uncontrolled combination

The end of

- **Osipov-Kanzieper result** [Phys. Rev. Lett. 101, 176804 (2008)]

$$\log f_n^{\mathrm{OK}}(g) \sim \begin{cases} -2n^2\eta^2, & \text{Gaussian} \quad \text{for } |\eta| < \frac{1}{2} \rightsquigarrow \text{(bulk)} \\ -2n^2\left(|\eta| - \frac{1}{4}\right) - \frac{3}{4}\log n, & \text{Exponential} \quad \text{for } \frac{1}{2} < |\eta| < 1 \rightsquigarrow \text{(tails)} \\ (n^2-1)\log(2|\eta-\eta_*|) - \frac{n^2}{2} + \frac{1}{12}\log n, & \text{for } |\eta - \eta_*| \le \frac{2}{n} \rightsquigarrow \text{(edges)} \end{cases}$$

Power -law (edges)

- **Bohigas-Majumdar-Vivo result** [Phys. Rev. Lett. 101, 216809 (2008)]

$$\log f_n^{\mathrm{BMV}}(g) \sim \begin{cases} -2n^2\eta^2 + o(n^2), & \text{Gaussian} \quad \text{for } |\eta| < \frac{1}{2} \rightsquigarrow \text{(bulk)} \\ n^2\log(2|\eta-\eta_*|) - \frac{n^2}{2} + o(n^2), & \text{for } \frac{1}{2} < |\eta| < 1 \rightsquigarrow \text{(tails)} \end{cases}$$

Power -law (tails)

$$\eta = 2\frac{g}{n} - 1, \qquad \eta_* = \pm 1$$

The end of the large-n controversy

- **Painlevé analysis**

$$\mathcal{F}_n(z) = \exp\left(\int_0^z dt\, \frac{\sigma_V(t) - n^2}{t}\right)$$

$$\log \mathcal{F}_n(z = 4np) = n^2 \begin{cases} -4p + \frac{3}{2} - \log(-p), & p < -1 \\ \frac{p^2}{2} - 2p, & |p| < 1 \\ \frac{3}{2} - \log p, & p > +1 \end{cases}$$

- **In the global regime $\frac{g}{n}$ fixed:**

$$\log\, f_n^{\mathrm{BMV}}(g) \sim \begin{cases} -2n^2\eta^2 + o(n^2), & \text{for } |\eta| < \frac{1}{2} \quad \rightsquigarrow \text{(bulk)} \\ n^2 \log(2|\eta - \eta_*|) - \frac{n^2}{2} + o(n^2), & \text{for } \frac{1}{2} < |\eta| < 1 \quad \rightsquigarrow \text{(tails)} \end{cases}$$

Gaussian (bulk); Power-law (tails)

$\sigma_n(z = 4nt) = n^2\Lambda(t) + n\Gamma(t) + \Omega(t) + O(n^{-1})$

No exponential tails !!

- The system, the problem, and the main result
- Why the result is interesting
  - Brief excursion into the Painlevé property
  - Appearance of Painlevé transcendents in statistical physics
- Derivation (Landauer conductance)
  - Cumulants of the Landauer conductance
  - Distribution function and the end of large-n controversy
  - Relation to other works
- Furher results (if time permits)
  - Statistics of thermal to shot noise crossover
  - Joint cumulants of Landauer conductance and noise power
- Conclusions

Paul Painlevé

Gaston Darboux

Satya Majumdar

Relation to other works

- Savin, Sommers 2006
- Savin, Sommers, Wieczorek 2007, 2008
- Novaes 2008
- Osipov, Kanzieper 2008, 2009
- Vivo, Majumdar, Bohigas 2008, 2010
- Khoruzhenko, Savin, Sommers 2009

Quantum Transport and Painlevé Transcendents

Outline

- The system, the problem, and the main result
- Why the result is interesting
  - Brief excursion into the Painlevé property
  - Appearance of Painlevé transcendents in statistical physics
- Derivation (Landauer conductance)
  - Cumulants of the Landauer conductance
  - Distribution function and the end of large-n controversy
  - Relation to other works
- Furher results (if time permits)
  - Statistics of thermal to shot noise crossover
  - Joint cumulants of Landauer conductance and noise power
- Conclusions

Paul Painlevé

Gaston Darboux

Satya Majumdar

The system, the problem, and the main result

$$\sigma_{\mathrm{V}}(z) = N_{\mathrm{L}} N_{\mathrm{R}} + \sum_{\ell=1}^{\infty} \frac{(-z)^{\ell}}{\Gamma(\ell)} \langle\langle (G/G_0)^{\ell} \rangle\rangle$$

$\ell-$th cumulant of the Landauer conductance

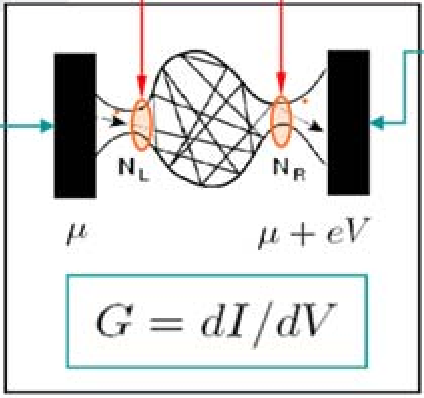
Fifth Painlevé transcendent !!

Universal transport regime

- ✓ Electron dwell time $\tau_D \simeq A/(W v_F)$ much larger than the Ehrenfest time $\tau_E \simeq \lambda^{-1} \log(W/\lambda_F)$
- ✓ RMT dynamics: $\{\mathcal{H}_{k\ell}\} \in \mathrm{GUE}_{M \times M}$

BPC

Left reservoir



Right reservoir

Landauer conductance

Applied Mathematics

מכון טכנולוגי חולון
Holon Institute of Technology

[Quantum Transport and Painlevé Transcendents

- Phys. Rev. Lett. 101, 176804 (2008) & JPA 42,475101 (2009)]

π^{00}

Universal Quantum Transport in Chaotic Cavities and Painlevé Transcendents

Eugene Kanzieper[1,2] & Vladimir Al Osipov[3]

[1] Department of Applied Mathematics @ H.I.T. - Holon Institute of Technology
[2] Department of Physics of Complex Systems @ Weizmann Istitute of Science
[3] Fakultät für Physik @ Universität Duisburg-Essen

מקרן הלאומית למדע • ISRAEL SCIENCE FOUNDATION •