# Theory of stochastic canonical equations and its application in physics

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Our studies are essentially based on the martingale differences method developed in my previous papers for resolvents of random matrices. This method possesses the self-averaging property of the entries of resolvents of random matrices and, hence, we can deduce the stochastic canonical equation. The lecture contains the most important results from numerous papers and books dealing with the theory of Unitary random matrices and functions of random matrices. We give the REFORM method of proving of all results, avoiding the method of moments. We do not try to describe here all known properties of the eigenvalues and eigenvectors for all classes of random matrices. However, our aim is rather to present the theory of stochastic canonical equations, and to give rigorous proofs of the procedures used to deduce these equations on the base of the author's **General Statistical Analysis**. Additionally, we consider some important applications for the system of linear algebraic equations with random coefficients. We consider special classes of analytic functions of random matrices. The description problem for normalized spectral functions of some analytic functions of random matrices is discussed in detail. Specifically, we present here the new theory: *L.I.F.E.,* which is the abbreviation for: *Limit Independence of Functions of Ensembles.*

Random matrix theory is a rapidly developing field and it has a great influence to fundamental and applied sciences: statistics, nuclear physics, and linear programming. Recent results in random matrix theory promoted the interest of researchers in the field of statistical physics to the methods and ideas developed for nuclear systems. One of the most intriguing applications of random matrix theory is the application to quantum mechanics.

We assume that energy levels of an atom are described by the eigenvalues of a random Hermitian operator, called the random Hamiltonian. It is very important that the eigenvalues of certain random matrices of large dimension converge to some nonrandom values, when the dimension of the matrix tends to infinity. In this manner, we can reach an agreement with the experimental observation of an atom.

Most of the areas under consideration are strongly correlated with the spectral theory of nonsymmetric random matrices. The attention of scientists in the physics of random matrices is mainly focused on the matrices with zero expectations of their entries. The actual situation in the application of random matrices to physics is quite different. As a rule, the entries of matrices have nonzero means. We continue the development of a new V-analysis for nonsymmetric random matrices from *Girko's* ensemble when the pairs of the entries of random matrices are independent. Therefore, the main aim of the present lecture is to attract physicists to the new analysis of random matrices appearing in numerous contemporary problems.

If the dimensionality of observations is large, then most statisticians would agree that the efficiency of the classical parametric approaches is doubtful. In the GSA we try to find new statistical estimators under two general assumptions. First, we do not require the existence of a density of observations. For example, we do not require that the observations have normal distributions. Second, we develop this analysis for the case where the number of parameters $m_n$ can increase together with the number of observations $n$. Using these two assumptions we can obtain on the base of developed theory of canonical equations many new results and I am sure that the general statistical analysis will be a turning point in the multidimensional statistical analysis.

## 1. CANONICAL EQUATION $K_1$. Main assertion

**Theorem 1.** *Assume that the entries* $\xi_{ij}^{(n)}; i \geq j, i, j = 1, ..., n$, *of a symmetric random matrix* $\Xi_{n\times n} = (\xi_{ij}^{(n)})_{i,j=1}^{n}$ *are independent for each* $n = 1, 2, \ldots$ *and defined on a common probability space,*

$$\mathbf{E}\,\xi_{ij}^{(n)} = a_{ij}^{(n)},\ \mathbf{Var}\,\xi_{ij}^{(n)} = \sigma_{ij}^{(n)},\ i \geq j, \quad i, j = 1\,,\ \ldots\,,\ n,$$

$$\sup_{n} \max_{i=1,\ldots,n} \sum_{j=1}^{n} \sigma_{ij}^{(n)} \ < \ \infty, \tag{1.1}$$

$$\sup_{n} \max_{i=1,\ldots,n} \sum_{j=1}^{n} \left| a_{ij}^{(n)} \right|^{2} \ < \ \infty, \tag{1.2}$$

*and Lindeberg's condition is satisfied, i.e., for any* $\tau > 0$,

$$\lim_{n\to\infty} \max_{i=1,\ldots,n} \sum_{j=1}^{n} \mathbf{E}\ [\xi_{ij}^{(n)} - a_{ij}^{(n)}]^2 \chi\{|\xi_{ij}^{(n)} - a_{ij}^{(n)}| \ > \ \tau\} = 0, \tag{1.3}$$

*where* $\chi$ *is the indicator of a random event,*

$$\mu_n\left\{x,\,\Xi_{n\times n}\right\} = n^{-1} \sum_{k=1}^{n} \chi(\omega: \ \ \lambda_k \ < \ x),$$

*and* $\lambda_1 \geq \cdots \geq \lambda_n$ *are the eigenvalues of the symmetric random matrix* $\Xi_{n\times n} = (\xi_{ij}^{(n)})_{i,j=1}^{n}$.
*Then, for almost all* $x$

$$\lim_{n\to\infty} |\mu_n\{x, \Xi_{n\times n}\} - F_n(x)| = 0, \tag{1.4}$$

*with probability one. If, in addition,*

$$\inf_{s,l=1,\dots,n} n\sigma_{sl}^{(n)} \geq c > 0, \tag{1.5}$$

*then, with probability one,*

$$\lim_{n\to\infty} \sup_x |\mu_n\{x, \Xi_{n\times n}\} - F_n(x)| = 0, \tag{1.6}$$

*where $F_n(x)$ are distribution functions whose Stieltjes transforms are equal to*

$$\int_{-\infty}^{\infty} (x-z)^{-1} \mathrm{d}F_n(x) = n^{-1} \sum_{i=1}^{n} c_i(z), \quad z = t + \mathrm{i}s, \quad s \neq 0,$$

*and the functions $c_i(z), i = 1, ..., n$, satisfy the canonical system of equations $K_1$:*

$$c_i(z) = \left\{ \left[ A_{n\times n} - zI_{n\times n} - \left( \delta_{p\,l} \sum_{s=1}^{n} c_s(z) \sigma_{s\,l}^{(n)} \right)_{p,l=1}^{n} \right]^{-1} \right\}_{ii}, \tag{1.7}$$

*where $i = 1, \dots, n$ $\delta_{pl}$ is the Kronecker symbol, $A_{n\times n} = (a_{ij}^{(n)})_{i,j=1}^{n}$, and $I_{n\times n}$ is the identity matrix of the $n$-th order. There exists a unique solution $c_i(z), i = 1, ..., n$, of the system of equations $K_1$ in a class of analytic functions*

$$L = \{z : \operatorname{Im} z \operatorname{Im} c_i(z) > 0, \ \operatorname{Im} z \neq 0, i = 1, \dots, n\}$$

*and the functions $c_i(z)$, $i = 1, \dots, n$, are the Stieltjes transforms of certain distribution functions.*

Note that, for some special cases, equation $K_1$ has been found. In the case where the matrix $A_{n\times n}$ is diagonal, the variances of the entries of a random matrix $\Xi_{n\times n}$ are equal, and Lindeberg's condition is satisfied for the components of each row vector of the matrix $\Xi_{n\times n}$, a special case of this equation was obtained by L. Pastur. In the case where $A_{n\times n}$ is a zero matrix and the variances of the entries of a random matrix $\Xi_{n\times n}$ are bounded, it was established by Berezin. The case where the matrix $A_{n\times n}$ is diagonal and the variances of the entries of a random matrix $\xi_{ij}^{(n)}$ may be different and satisfy Lindeberg's condition was studied by Girko.

## 2. CANONICAL EQUATION $K_{27}$ FOR NORMALIZED SPECTRAL FUNCTIONS OF RANDOM SYMMETRIC BLOCK MATRICES

Consider random symmetric matrices $\Xi_{n\times n} = \left(\xi_{ij}^{(n)}\right)_{i,j=1}^{n}$ with asymptotically independent entries. It is proved that, for almost all $x$ and any $\varepsilon > 0$, under certain restrictions,

$$\lim_{n\to\infty} \mathbf{P}\left\{|\mu_n(x) - F_n(x)| > \varepsilon\right\} = 0,$$

where

$$\mu_n(x) = n^{-1} \sum_{k=1}^{n} \chi(\lambda_k < x),$$

$\chi(\lambda_k < x)$ is the indicator function, $\lambda_k$ are eigenvalues of the matrix $\Xi_{n\times n} = \left(\xi_{ij}^{(n)}\right)_{i,j=1}^{n}$, $F_n(x)$ is the distribution function whose Stieltjes transform is equal to

$$\int_{-\infty}^{\infty} (x-z)^{-1}\,\mathrm{d}F_n(x) = n^{-1}\sum_{k=1}^{p} \operatorname{Tr} C_{kk}(z),\ z = t + \mathrm{i}s,\ s \neq 0,$$

and the block matrices $C_{kk}(z),\ k = 1, \ldots, p$, of dimensionality $q \times q$ satisfy the system of canonical equations $K_{27}$

$$C_{kk}(z) = \Big\{ \big[A_{pq} - zI_{pq} - \big(\delta_{lj} \sum_{s=1}^{p} \mathbf{E}\, H_{js}^{(n)} C_{ss}(z) H_{js}^{(n)*}\big)_{l,j=1}^{p}\big]^{-1} \Big\}_{kk}$$

where $k = 1, \ldots, p$, $A_{pq \times pq}$ is a nonrandom matrix, $I_{pq \times pq}$ is the identity matrix, $H_{js}^{(n)}$ are random matrices of dimensionality $q \times q$, $p$ and $q$ are some integers and notation $\{A\}_{kk}$ means the $k$th diagonal block of size $q \times q$ of the matrix $A$.

## 3. *SOS*-LAWS

Recall that the first limit density for the n.s.f. of symmetric random matrices was obtained by E. Wigner, and the graph of this density is a certain semicircle(semielliptic). But this density disappointed him and other physicists. The real densities of the energy levels of atom nucleus have another form. But we are now in a position to find such limit density for the random block matrices. For the simple random block matrices, we have Block Matrix Density which, for some matrices $A_{q \times q}$ and $B_{q \times q}$, is equal to sum of the Semicircle laws ($SOS$-Laws) with different centers and radii. Therefore, it is possible to approximate any density using such $SOS$-Law and it is possible to achieve an agreement with the observed densities of energy levels of atoms and the spectral density of our random block matrix.

To obtain the simplest result, we assume that the matrices $A_{q \times q}$ and $B_{q \times q}$ commute.

**Theorem.** *If, in addition to the conditions of Section 2, we have* $A_{q \times q} = H_{q \times q} \Lambda_{q \times q}^{(1)} H_{q \times q}^{T}$, $B_{q \times q} = H_{q \times q} \Lambda_{q \times q}^{(1)} H_{q \times q}^{T}$, *where*

$$\Lambda_{q \times q}^{(1)} = (\delta_{ij} \lambda_i (A_{q \times q})),$$

$$\Lambda^{(2)}_{q\times q} = (\delta_{ij}\lambda_i(B_{q\times q})),$$

*$\lambda_1(A_{q\times q}) \leq \cdots \leq \lambda_q(A_{q\times q})$, $\lambda_1(B_{q\times q}) \leq \cdots \leq \lambda_q(B_{q\times q})$ are eigenvalues of matrices $A_{q\times q}$ and $B_{q\times q}$, and $H_{q\times q}$ is an orthogonal matrix, then, for all $x$ with probability one*

$$\lim_{p,q\to\infty} |\mu_{pq}\left(x, \Xi_{pq\times pq}\right) - F_q\left(x\right)| = 0,$$

*where $F_q(x)$ is the distribution function whose density is equal to*

$$\frac{\mathrm{d}}{\mathrm{d}x}F_q\left(x\right) = \frac{1}{q}\sum_{k=1}^{q}\frac{1}{2\pi\lambda_k^2(B_{q\times q})}\chi\left\{\left[x - \lambda_k\left(A_{q\times q}\right)\right]^2 < 4\lambda_k^2(B_{q\times q})\right\}$$
$$\times\sqrt{4\lambda_k^2(B_{q\times q}) - \left[x - \lambda_k\left(A_{q\times q}\right)\right]^2},$$

*which is equal to the sum of semicircular laws (SOS-Laws).*

## 4. THE CANONICAL EQUATION $K_{96}$ FOR GIRKO'S ENSEMBLE OF RANDOM ACE-MATRIX $\Xi_n$. ELLIPTICAL GALACTIC LAW

The structure of this Section is the following: at first we repeat the first 20 years old proof of the strong Elliptic law for random matrices $\Xi_n = \{\xi_{ij}^{(n)}\}$. Then we prove the strong Elliptic law for random matrices $\Xi_n$ of the general form, i.e. when their diagonal entries $\xi_{ii}^{(n)}$ have nonzero expectations, and when we require the existence of the probability densities of the entries of random matrices and Lyapunov condition. In this case the Elliptical Galactic law means that the support of the accompanying spectral density of eigenvalues looks like the picture of several galaxies made by telescope. If the distances between the centers of these galaxies are large enough we have several almost elliptical galaxies. These statements are based on the *VICTORIA*-transform of random matrix which is the abbreviation of the following words: *Very Important Computational Transformation Of Random Independent Arrays.*

We follow the main strategy of the theory of limit theorems of the probability theory, i.e. we try to solve the problem of description of all limits of normalized spectral functions

$$\nu_n\left(x, y, A_n \Xi_n B_n + C_n\right) \\ = \frac{1}{n}\sum_{k=1}^{n}\chi\left\{\mathrm{Re}\lambda_k(A_n\Xi_nB_n+C_n) < x, \mathrm{Im}\lambda_k(A_n\Xi_nB_n+C_n) < y\right\},$$

where $\lambda_k(A_n\Xi_nB_n+C_n)$ are eigenvalues of the matrix $A_n\Xi_nB_n+C_n$, $A_n, B_n$, and $C_n$ are nonrandom matrices, under general (as only possible) conditions on the entries $\xi_{ij}^{(n)}$ of random matrices $\Xi_n$, $\chi$ is the indicator function. We emphasize that the spectral theory of Hermitian random matrices is rather profound. For example, in 1975 in [Gir12] V. Girko proved the general stochastic canonical equation for *ACE*(Asymptotically Constant Entries)-symmetric matrices: Assume that for any $n$, the random entries $\xi_{ij}^{(n)}$, $i \geq j$, $i, j = 1, ..., n$, of a symmetric matrix $\Xi_{n\times n} = \left[\xi_{ij}^{(n)} - \alpha_{ij}^{(n)}\right]_{i,j=1}^{n}$ are independent and they are asymptotically constant entries (ACE), i.e., for any $\varepsilon > 0$,

$$\lim_{n\to\infty}\sup_{p,l=1,\ldots,n}\mathbf{P}\left\{|\xi_{pl}^{(n)}| > \varepsilon\right\} = 0, \alpha_{ij}^{(n)} = \int_{|x|<\tau} x\,\mathrm{d}\mathbf{P}\left\{\xi_{ij}^{(n)} < x\right\}$$

and $\tau > 0$ is an arbitrary constant, and that, for every $0 \leq u \leq 1$ and $0 \leq v \leq 1$,

$$K_n\left(u, v, z\right) \Rightarrow K\left(u, v, z\right), \quad -\infty < z < \infty,$$

where the symbol $\Rightarrow$ denotes the weak convergence of distribution when $n \to \infty$,

$$K_n(u,v,z) = n \int_{-\infty}^{z} y^2 \left(1+y^2\right)^{-1} \mathrm{d}\mathbf{P} \left\{ \xi_{ij}^{(n)} - \alpha_{ij}^{(n)} < y \right\},$$

$in^{-1} \le u < (i+1)\, n^{-1}$, $jn^{-1} \le v < (j+1)\, n^{-1}$, and $K(u,v,z)$ is a nondecreasing function with bounded variation in $z$ and continuous in $u$ and $v$ in the domain $0 \le u, v \le 1$. Then, with probability one, for almost all $x$,

$$\lim_{n\to\infty} \left| n^{-1} \sum_{k=1}^{n} \chi\{\lambda_k(\Xi_{n\times n}) < x\} - F(x) \right| = 0,$$

where $\lambda_k(\Xi_{n\times n})$ are eigenvalues, $F(x)$ is a distribution function whose Stieltjes transform satisfies the relation

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}F(x)}{1+\mathrm{i}tx} = \lim_{\alpha\downarrow 0} \int_0^1 \left[ \int_0^1 x \, \mathrm{d}_x G_\alpha(x,y,t) \right] \mathrm{d}y,$$

$G_\alpha(x,y,t)$, as a function of $x$, is a distribution function satisfying the regularized stochastic canonical equation $K_3$[3, 23] at the points $x$ of continuity,

$$G_\alpha(x,z,t) = \mathbf{P} \left\{ \left[ 1 + t^2 \xi_\alpha \{G_\alpha(*,*,t), z\} \right]^{-1} < x \right\}, \quad 0 \le x \le 1,$$

$\xi_\alpha \{G_\alpha(*,*,t), z\}$ is a random real functional whose Laplace transform of one-dimensional distribution is equal to

$$\mathbf{E}\, e^{\{-s\xi_\alpha[G_\alpha(*,*,t),z]\}} = \exp\left\{ \int_0^1 \int_0^1 \left[ \int_0^\infty \left[ \exp\left\{ -\frac{syx^2}{(1+\alpha|x|)^2} \right\} \right.\right.\right.$$
$$\left.\left.\left. -1 \right] \frac{1+x^2}{x^2} \mathrm{d}_x K(v,z,x) \right] \mathrm{d}_y G_\alpha(y,v,t)\, \mathrm{d}v \right\},$$

$\alpha > 0,\ s \ge 0,\ 0 \le z \le 1.$

The integrand $\left[\exp\left\{-syx^2\left(1+\alpha\left|x\right|\right)^{-2}\right\}-1\right]\left(1+x^{-2}\right)$ is defined at $x=0$ by continuity as $-sy$. There exists a unique solution of the canonical equation $K_3$ in the class $L$ of functions $G_\alpha\left(x,y,t\right)$ that are distribution functions of $x$ $(0\leq x\leq 1)$ for any fixed $0\leq y\leq 1$, $-\infty<t<\infty$, such that, for any integer $k>0$ and $z$, the function $\int_0^1 x^k \mathrm{d}_x G_\alpha\left(x,z,t\right)$ is analytic in $t$ (excluding, possibly, the origin). The solution of the canonical equation $K_3$ can be found by the method of successive approximations.

For the first time in 1980[4] and in 1990 in [G] this equation was rewritten in the following form (here we use the simplest equation, when $\alpha=0$)

$$m(s,t,z)-1=\int_0^\infty \exp\left\{\int_0^1\int_0^1\left[\int_0^\infty [m(t^2yx^2,t,v)-1]\right.\right.$$
$$\left.\left.\times\frac{1+x^2}{x^2}\mathrm{d}_xK(v,z,x)\right]\mathrm{d}v\right\}\frac{\partial}{\partial y}J_0(2\sqrt{sy})e^{-y}\mathrm{d}y,$$

where

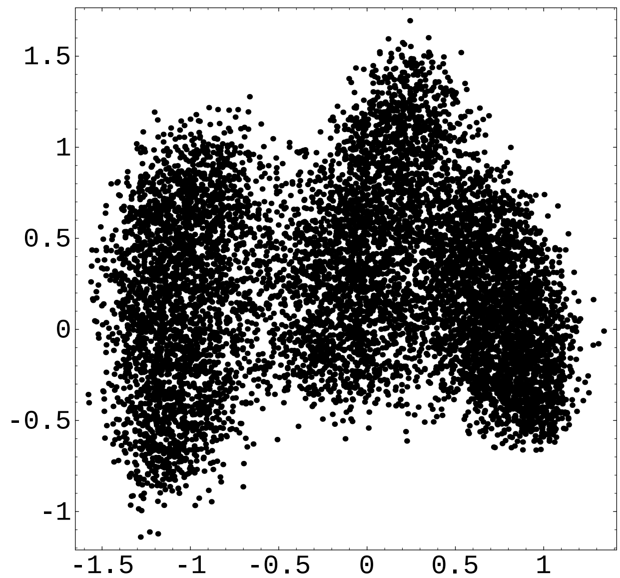
$$m(s,t,z)=\int_0^1 e^{-sx}\mathrm{d}_xG_0(x,z,t),\quad s\geq 0,$$

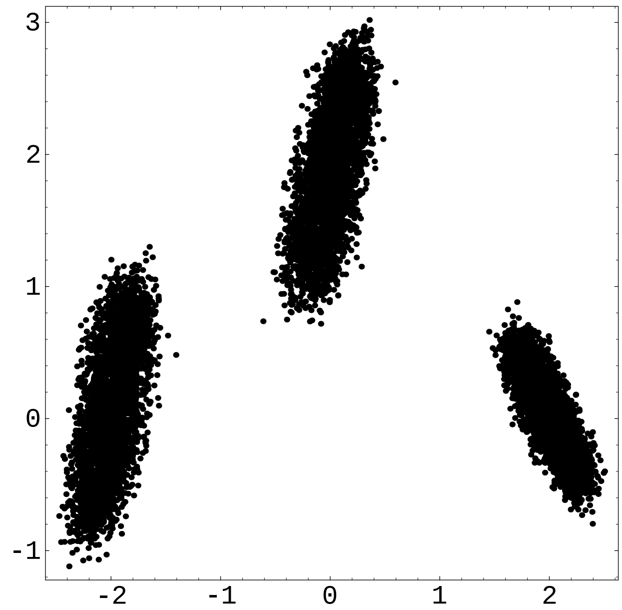
$J_0(x)$ is the Bessel function which is equal to

$$J_0(x)=\sum_{k=0}^{\infty}(-1)^k x^{2k}\frac{1}{2^{2l}k!k!}.$$

In [G] a technical improvement and a new proof of the uniqueness of solution of canonical equation $K_3$ are presented, where $m(s,t,z)$ has a unique representation in the family of integrable functions. The analytic details of the statement and of the proof are elaborate[4]. English translation: Ukrainian Math. J. 32 (1980), no. 6, 546–548 (1980).

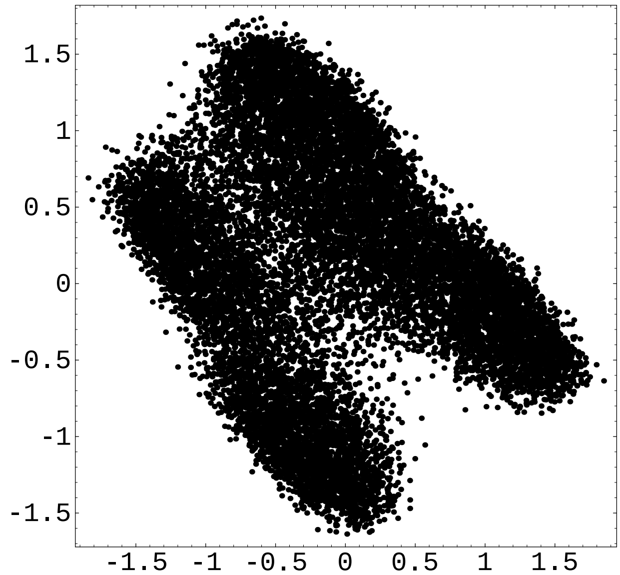
We prove the strong Elliptical Galactic law for random matrices $\Xi_n$ of the general form, i.e. their diagonal entries $\xi_{ij}^{(n)}$ have nonzero expectations and the pairs of the entries $(\xi_{ij}^{(n)}, \xi_{ji}^{(n)})$ have nonzero covariances. In this case the Elliptical Galactic law means that the support of the accompanying spectral density of eigenvalues of matrix $\Xi_n$ looks like the picture of several galaxies made by telescope:

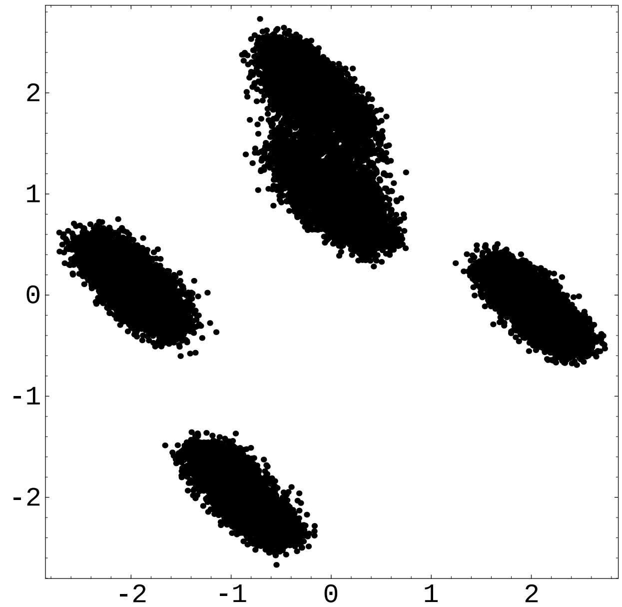





Figures 1 and 2

The picture 1 shows the *collision* of elliptic supports of the limit spectral density of n.s.f. of random matrix $A_n + \Lambda_n \Xi_n$, where $A_n$ is a diagonal complex matrix with diagonal entries $(0.7, 0), (-1, 0)$, $(0, 0.7\mathrm{i})$ for corresponding three equal parts of the main diagonal, and random matrix $\Xi_n$ has equal covariances $\rho(\sqrt{\rho} = 0.2 + \mathrm{i}0.8)$ of independent pairs of entries $(\xi_{ij}^{(n)}, \xi_{ji}^{(n)})$ with zero mean and is multiplied by diagonal matrix $\Lambda_n$ with diagonal entries $(1, 0), (0.5, 0.5\mathrm{i}), (-1, 0)$ for corresponding three equal parts of the main diagonal. We have chosen in picture 1 three different diagonal entries of the matrix $A_n$ at a short distance. In picture 2, we consider the diagonal matrix $A_n$ with diagonal entries $(2, 0), (-2, 0), (0, 2\mathrm{i})$ at a large distant for corresponding three equal parts of the main diagonal. In the letter case we have several domains-supports like ellipses. For the exposition of the Elliptic Law we have chosen the random matrix $\Xi_n$ of dimension 30 and 300 its Monte-Carlo simulation. If the distances between the centers of these *galaxies* are large enough we have several almost *elliptical galaxies.*



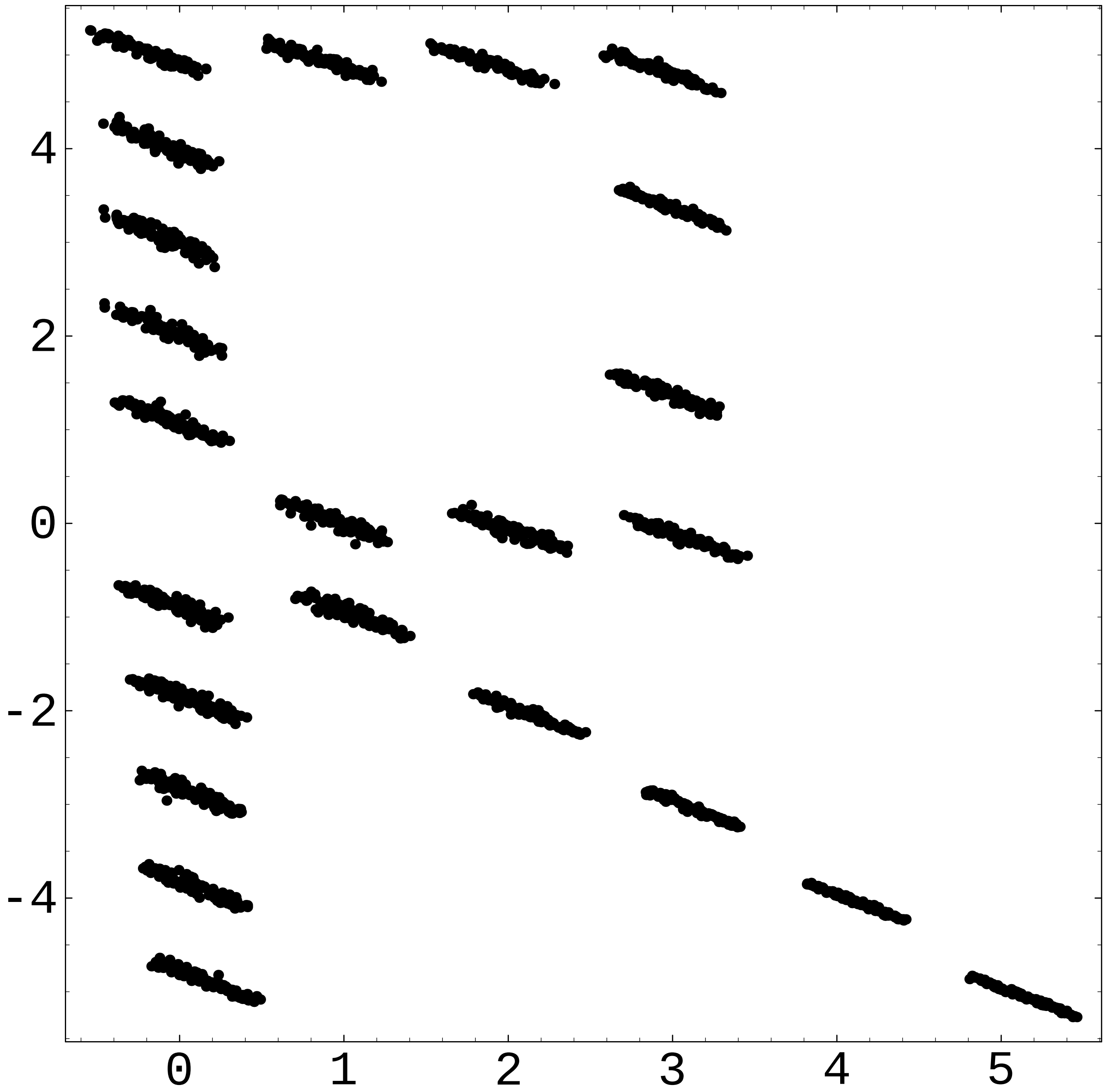



Figures 3 and 4.

These pictures show the elliptic support of the limit spectral density of n.s.f. of random matrix $A_n + \Xi_n$, where $A_n$ is a diagonal matrix with 5 different diagonal entries $(1, 0)$; $(-1, 0)$; $(-0.5, -\mathrm{i})$; $(0, 0.5\mathrm{i})$; $(0, \mathrm{i})$ and random matrix $\Xi_n$ has equal covariances $\rho(\sqrt{\rho} = 0.5 + \mathrm{i}0.5)$of the entries $(\xi_{ij}, \xi_{ji})$. We have chosen five different diagonal entries of the matrix $A_n$ at a short distance in picture 1 and at a large $(2, 0); (-2, 0); (-1, -2\mathrm{i}); (0, \mathrm{i}); (0, 2\mathrm{i})$ in picture 2. In the letter case we have several domains-supports like ellipses. For the exposition of the Elliptic Law we have chosen the random matrix $\Xi_n$ of dimension 50 and 300 its Monte-Carlo simulation

If the distances between the centers of these galaxies are large enough we have several almost elliptical galaxies.

Maybe the reader remembers the Monte Carlo simulations of eigenvalues of matrices $\Xi_n + A_n$, where $\Xi_n$ belongs to the domain of attraction of Circular law and $A_n$ is the diagonal matrix whose diagonal entries forms letter **R** on a complex plain[25]–[27]. For the case when the matrix $\Xi_n$ belongs to the domain of attraction of Elliptic law the simulation of eigenvalues of the matrix $\Xi_n + A_n$ looks like the following picture:




There are essentially three methods of the proof of Elliptic Laws that have been proposed: the REFORM method and Berry-Esseen inequality[11], the method of perpendiculars[15,16], the method of the central limit theorem and limit theorems for eigenvalues of random matrices[23]. The main advantage of REFOEM approach is that it enables the results of the previous version of Elliptic law to be extended to the case under consideration. The REFORM-method(or $G$-martingale approach) enables us to suggest a new method for construction of stochastic canonical equations.

We prove the following *Elliptical Galactic Law* which generalizes the *Strong Circular Law* and *Weak Circular Law*(see the sketch of the proof of this law in the paper *V-transform*, Dopovidi Akademii nauk Ukrainskoi RSR, Seria A Fizyko-Matematychni ta technichni nauky, 1982, N3, pp.5-6.): For every $n$, let the pairs of random entries $(\xi_{ij}^{(n)}, \xi_{ji}^{(n)}); i = 1, ..., n,\ j = 1, ..., n$, of the complex matrix $\Xi_{n\times n} = (\xi_{ij}^{(n)})_{i=1,\dots,n}^{j=1,\dots n}$ be independent and given on a common probability space, $\mathbf{E}\,\xi_{ij}^{(n)} = 0, \mathbf{E}\left|\xi_{ij}^{(n)}\right|^2 = \sigma_{ij}^{(n)} n^{-1}, 0 < r_1 < \sigma_{ij}^{(n)} < r_2 < \infty, \mathbf{E}\,\xi_{ij}^{(n)}\xi_{ji}^{(n)} = \rho_{ij}^{(n)} n^{-1}, i \neq j,\ i, j = 1, ..., n$, and

$$\sup_n \max_{\substack{i=1,\dots,n,\\ j=1,\dots,n}} \left\{ \sum_{j=1}^n \left|(A_n^{-1} C_n B_n^{-1})_{ij}\right|^2 + \sum_{i=1}^n \left|(A_n^{-1} B_n^{-1})_{ij}\right|^2 \right.$$
$$\left. + \sum_{j=1}^n \left|(A_n^{-1} C_n B_n^{-1})_{ji}\right|^2 + \sum_{i=1}^n \left|(A_n^{-1} B_n^{-1})_{ji}\right|^2 \right\} < \infty,$$

where $A_n = \left\{a_{ij}^{(n)}\right\}_{i,j=1,\dots,n}$, $B_n = \left\{b_{ij}^{(n)}\right\}_{i,j=1,\dots,n}$ and $C_n = \left\{c_{ij}^{(n)}\right\}_{i,j=1,\dots,n}$ are nonrandom matrices, $\det A_n \neq 0, \det B_n \neq 0$, and the real and imaginary parts of entries $\sqrt{n}\xi_{ij}^{(n)},\ \sqrt{n}\xi_{ji}^{(n)}, i > j$

have the densities

$$p_{ij}^{(n)}(x_1, x_2, y_1, y_2)$$
$$= \frac{\partial^4}{\partial x_1\, \partial x_2\, \partial y_1\, \partial y_2} \mathbf{P}\left\{ \mathrm{Re}\sqrt{n}\xi_{ij}^{(n)} < x_1,\ \mathrm{Re}\sqrt{n}\xi_{ji}^{(n)} < x_2, \right.$$
$$\left. \mathrm{Im}\sqrt{n}\xi_{ij}^{(n)} < y_1,\ \mathrm{Im}\sqrt{n}\xi_{ji}^{(n)} < y_2 \right\}$$

satisfying the corrected *Elliptic condition*: for some $\beta > 1$

$$\sup_n \max_{\substack{l=1,\dots,n \\ k \neq l}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \max_{k=1,..,n}$$

$$\times \left[ \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} p_{kl}^{(n)}(x, y, u, v) \mathrm{d}y \right]^{\beta} \mathrm{d}x \right]^{1/\beta} \mathrm{d}u\, \mathrm{d}v < \infty,$$

or

$$\sup_n \max_{\substack{l=1,\dots,n \\ k \neq l}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \max_{k=1,..,n}$$

$$\times \left[ \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} p_{kl}^{(n)}(x, y, u, v) \mathrm{d}x \right]^{\beta} \mathrm{d}y \right]^{1/\beta} \mathrm{d}u\, \mathrm{d}v < \infty,$$

and there exist the densities $p_{ii}^{(n)}(x)$ of the entries $\sqrt{n}\mathrm{Re}\xi_{ii}^{(n)}$, or the densities $q_{ii}^{(n)}(x)$ of the entries $\sqrt{n}\mathrm{Im}\,\xi_{ii}^{(n)}$, satisfying the condition: for some $\beta_1 > 1$

$$\sup_n \max_{k=1,\dots,n} \int_{-\infty}^{\infty} \left[ p_{kk}^{(n)}(x) \right]^{\beta_1} \mathrm{d}x < \infty,$$

or

$$\sup_n \max_{k=1,\dots,n} \int_{-\infty}^{\infty} \left[q_{kk}^{(n)}(x)\right]^{\beta_1} \mathrm{d}x < \infty,$$

the Lyapunov condition is fulfilled: for some $\delta > 0$,

$$\max_{p,l=1,\dots,n} \mathbf{E} \left|\xi_{pl}^{(n)}\sqrt{n}\right|^{2+\delta} \leq c < \infty.$$

Then, with probability one, for almost all $x$ and $y$

$$\lim_{\alpha\downarrow 0} \lim_{n\to\infty} \left|\nu_n(x,y,A_n\Xi_nB_n+C_n) - F_{n,\alpha}(x,y)\right| = 0,$$

where

$$\nu_n(x,y,A_n\Xi_nB_n+C_n) = n^{-1}\sum_{k=1}^{n}\chi\{\operatorname{Re}\lambda_k < x, \operatorname{Im}\lambda_k < y\},$$

$\lambda_k$ are eigenvalues of the matrix $A_n\Xi_nB_n+C_n$, the Global probability density $p_{n,\alpha}(t,s) = (\partial^2/\partial t\,\partial s)F_{n,\alpha}(t,s)$ is equal to

$$p_{n,\alpha}(t,s) = \begin{cases} -\frac{1}{4\pi}\int_{\alpha}^{\infty}\left[\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial s^2}\right] b_n(y,t,s)\,\mathrm{d}y \;\; for \;\; (t,s)\notin G_n, \\ 0 \qquad\qquad \text{for} \;\; (t,s)\in G_n, \end{cases}$$

where $\alpha > 0$,

$$b_n\Big(y,t,s\Big) = \frac{\mathrm{i}}{2\sqrt{y}} n^{-1}\mathrm{Tr}\left[I_{2n}\mathrm{i}\sqrt{y} - Q_{2n}(y,t,s) + C_{2n}(y,t,s)\right]^{-1},$$

$$Q_{2n}(y,t,s) = \Big(\delta_{ij}Q_{2\times 2}^{(ii)}(y,t,s)\Big)_{i,j=1,\dots,n}, C_{2n}(t,s) = \{C_{2\times 2}^{(ij)}(t,s)\},$$

where $C_{2n}(t,s) = \left(c^{(ij)}_{2\times 2}(t,s)\right)_{i,j=1,\ldots,n}$ is a block matrix, $C^{(ij)}_{2\times 2} = \left\{\begin{matrix} 0 & s^{(n)}_{ij} \\ \bar{s}^{(n)}_{ji} & 0 \end{matrix}\right\}$, $s^{(n)}_{ij}(t,s)$ are entries of the matrix

$$S_n(t,s) = A_n^{-1}(C_n - I_n\tau)B_n^{-1} = \{s^{(n)}_{ij}(t,s)\},$$

and $Q_{2n}(y,t,s) = \left(\delta_{ij}Q^{(ii)}_{2\times 2}(y,t,s)\right)_{i,j=1,\ldots,n}$ is the block diagonal matrices, whose diagonal block $Q^{(ii)}_{2\times 2}(t,s)$ satisfy the system of canonical equations $K_{97}$

$$Q^{(jj)}_{2\times 2}(y,t,s) = \{\mathrm{i}I_{2n}\sqrt{y} + C_{2n}(t,s)$$
$$-\left[\delta_{ij}\sum_{i=1}^{n}\mathbf{E}\left\{\begin{matrix} 0 & \xi_{ij} \\ \xi^*_{ji} & 0 \end{matrix}\right\}Q^{(ii)}_{2\times 2}(y,t,s)\left\{\begin{matrix} 0 & \xi_{ij} \\ \xi^*_{ji} & 0 \end{matrix}\right\}^*\right]_{i,j=1,\ldots,n}\Bigg\}^{-1}_{jj}$$

$j = 1, ..., n$, and $\bar{G}$ is a support of the Global probability density, where

$$G = \{(t,s) : \lim_{\alpha\downarrow 0}\lim\sup_{n\to\infty}\alpha\frac{\partial}{\partial\alpha}b_n(\alpha,t,s) = 0.\}$$

There exists a unique solution of canonical equation $K_{97}$ in the class of positive definite block matrices $Q^{(ii)}_{2\times 2}(y,t,s) > 0, y > 0, i = 1, ..., n$ of the order $2\times 2$, analytic in $y > 0, t, s$.

## 5. The border of the support of limit spectral density $p(x,y)$ for pure G-ensemble when only two constant of diagonal matrix $A_n$ are pure imaginary numbers. Sand clock density

The next example is the simplest case of matrices from G-ensemble, when only two diagonal complex entries of diagonal matrix $A_n$ are different and random matrix $\Xi_n$ is Hermitian matrix. We can find the border support of accompanying spectral density, but even in this simple case the solution is not simple and the mathematical equation for the curve of the border of accompanying support of limit spectral density occupies almost the half of a page.

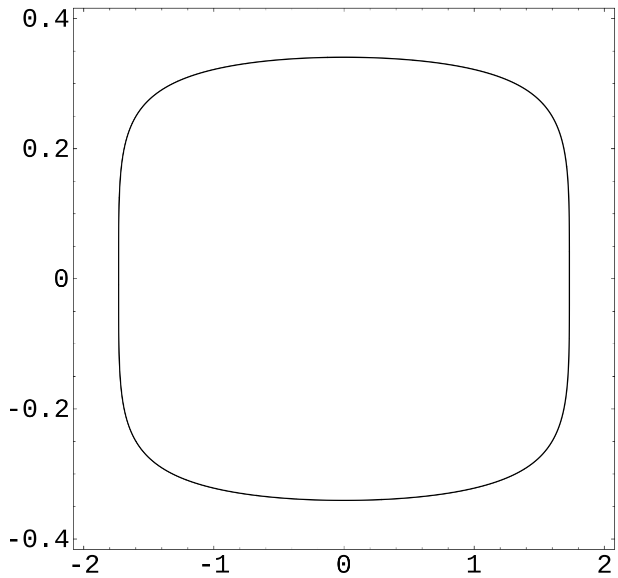
**Theorem.** *If additionally to the conditions of Theorem 97.1 $\rho^{(n)} = 1$, $a_k = \mathrm{i}r, k = 1, ..., [n/2]; a_l = -\mathrm{i}r, l = [n/2]+1, ..., n$ then the border of the support of accompanying probability spectral density is given by the following equation*

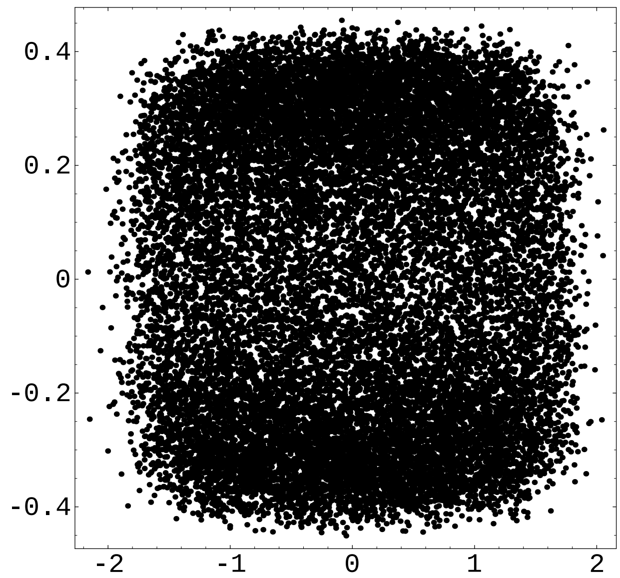
$(d^4k^4 + d^3k^2(((-2)bkl + 2al^2 - 4akq)) + d^2((c^2k^3l + b^2k^2l^2 - 2abkl^3 + a^2l^4 + 3bck^3p - 5ack^2lp - 3abk^2p^2 + 4a^2klp^2 + 2b^2k^3q + 2abk^2lq - 4a^2kl^2q + 6a^2k^2q^2)) + d(((-c^3)k^3p - b^2ck^2lp + 2abckl^2p - a^2cl^3p + b^3k^2p^2 + 3ac^2k^2p^2 - 2ab^2klp^2 + a^2bl^2p^2 - 3a^2ckp^3 + a^3p^4 - 4bc^2k^3q - 2b^3k^2lq2ac^2k^2lq + 4ab^2kl^2q - 2a^2bl^3q + 2abck^2pq+$

$2a^2cklpq + 2a^2bkp^2q - 4a^3lp^2q - 4ab^2k^2q^2 + 2a^2bklq^2 + 2a^3l^2q^2 - 4a^3kq^3)) - q(((-c^4)k^3 - b^2c^2k^2l + 2abc^2kl^2 - a^2c^2l^3 + b^3ck^2p + 3ac^3k^2p - 2ab^2cklp + a^2bcl^2p - 3a^2c^2kp^2 + a^3cp^3 - b^4k^2q - 4abc^2k^2q + 2ab^3klq + 3a^2c^2klq - a^2b^2l^2q + 5a^2bckpq - 3a^3clpq - a^3bp^2q - 2a^2b^2kq^2 + 2a^3blq^2 - a^4q^3))) = 0$, *where* $a = 1, k = 1, p = 0, b = \frac{t^2}{2} - 2r^2, c = -\frac{2r^2}{s}, d = \frac{t^4}{16} + \frac{t^2r^2}{2} + r^4$, $l = \frac{t^2}{2} - 1 - 2r^2$, $q = \frac{t^4}{16} + \frac{t^2r^2}{2} + r^4 - \frac{t^2}{4} - r^2$.

## 6. Several examples of the border support and Monte Carlo simulations performed by Mathematica 5 for pure Girko's ensemble. Sand clock density

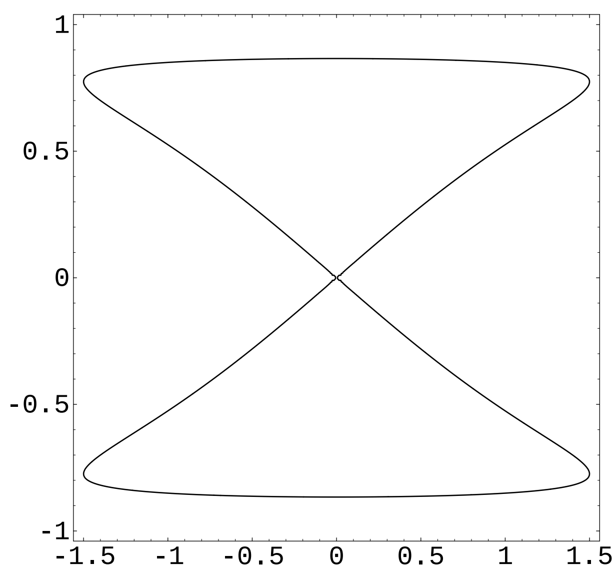
We give here several examples. For the reader conveniences we provide them by corresponding program of Mathematica 5. Enjoy considering different cases of random matrices.

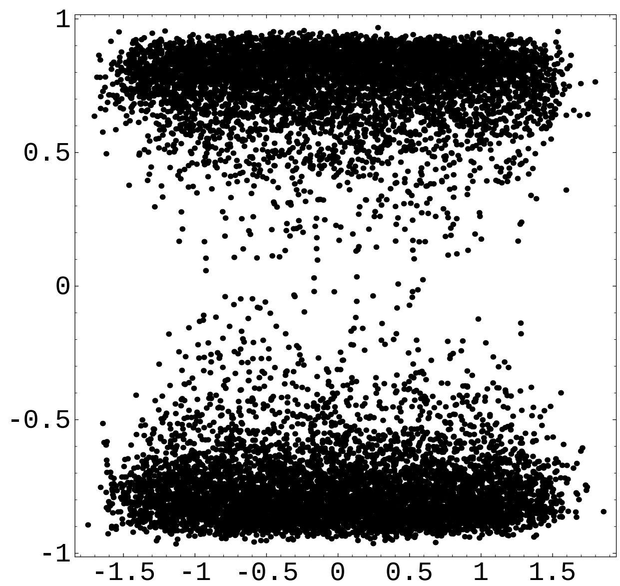





Figures 5 and 6

This picture shows the elliptic support of the limit spectral density of n.s.f. of random matrix $A_n + \Xi_n$. We have chosen the constant $r = 0.5, \rho = 1$. In this case we have one domain like ellipse. For the exposition of the Elliptic Law we have chosen the random Hermitian matrix $\Xi_n$ of dimension 20 and 500 its Monte-Carlo simulation.

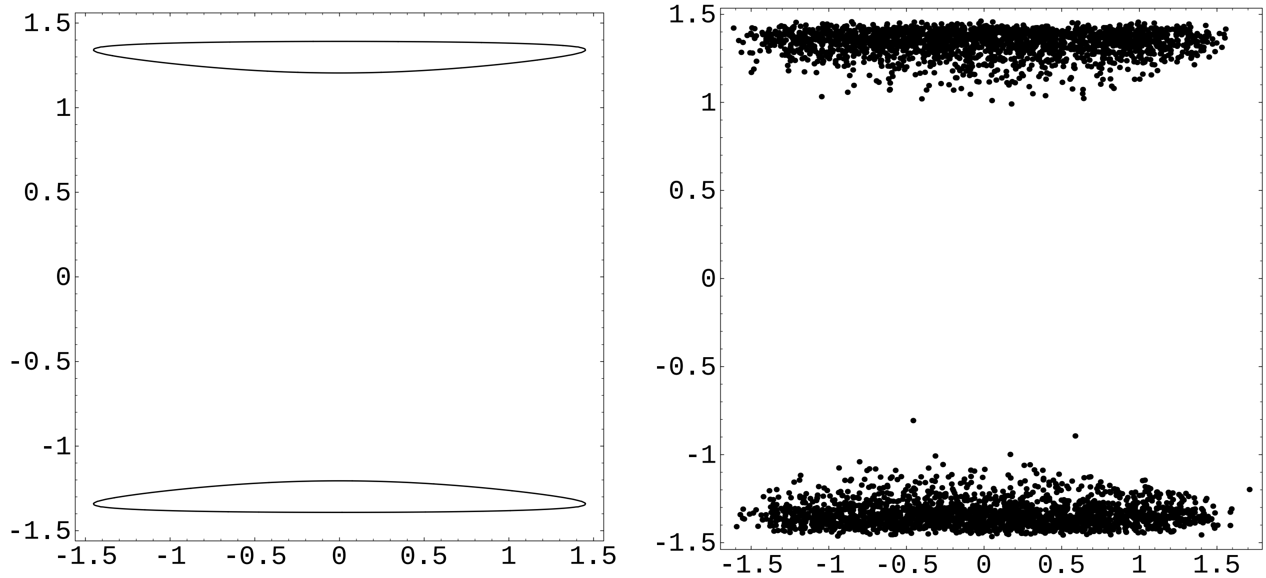





Figures 7 and 8

This picture shows the elliptic support of the limit spectral density of n.s.f. of random matrix $A_n + \Xi_n$ considered in Theorem

19.1. We have chosen the constant $r = 1, \rho = 1$. In this case we have two domain like ellipses with one common point which look like *Sand clock* . For the exposition of the Elliptic Law we have chosen the random Hermitian matrix $\Xi_n$ of dimension 20 and 300 its Monte-Carlo simulation.




Figures 9 and 10

This picture shows the elliptic support of the limit *Sand Clock* spectral density of n.s.f. of random matrix $A_n + \Xi_n$. We have chosen the constant $r = 1.5, \rho = 1$. In this case we have two separated domain like ellipses. For the exposition of the Elliptic Law we have chosen the random matrix $\Xi_n$ of dimension 20 and 100 its Monte-Carlo simulation.

## 7. THE CANONICAL EQUATION $K_{91}$ FOR GROWING MATRIZANT $\prod_{i=1}^{m}\{I_n + I_n\frac{f(i/n)}{m} + \frac{g(i/n)}{\sqrt{m}}\Xi_n^{(i)}\}$ OF INDEPENDENT RANDOM ACE-MATRICES $\Xi_n^{(i)}$ WITH DIFFERENT VARIANCES OF THEIR ENTREES

We consider the random matrizant

$$\prod_{i=1}^{m}\left[I_n + I_n\frac{f(\frac{i}{m})}{m} + \frac{g(\frac{i}{m})}{\sqrt{m}}\Xi_n^{(i)}\right]$$

of random ACE(asymptotical constant entries) matrices $\Xi_n^{(i)}$ whose entrees may have different variances.

## 8. *V.I.C.T.O.R.I.A.*-transform for the matriciant of the growing dimension

We give a new method of deriving general canonical equation for the *V.I.C.T.O.R.I.A.*-transform of normalized spectral functions (n.s.f.)

$$\nu_n(u,v) = n^{-1}\sum_{k=1}^{n}\chi\{\Im\lambda_k(Z_{n\times n}^{(m)}) < u, \Re\lambda_k(Z_{n\times n}^{(m)}) < u\}$$

of the product of random matrices (matriciant)

$$Z_{n\times n}^{(m)} = \prod_{i=1}^{m}\left[I_{n\times n} + I_{n\times n}\frac{f(\frac{i}{m})}{m} + \frac{g(\frac{i}{m})}{\sqrt{m}}\Xi_{n\times n}^{(i)}\right]$$

of the independent matrices $\Xi_n^{(i)}$, which was recently obtained in for some particular cases on the base of free probability theory. Here $\lambda_k(Z_{n\times n}^{(m)}), k = 1, ..., n$ mean the eigenvalues of matrix, $f(x)$ and $g(x)$ are certain functions, $I_{n\times n}$ is identity matrix and the product of random matrices is taken from the left to the right. We will use for quadratic matrices two notations: $A_{n\times n}$ and $A_n$. We apply the REFORM-method and Girko's theory of the proof of the Circular law ([22-24]) for the deduction of the system of canonical equations $K_{91}$ for normalized spectral functions $\nu_n(u,v)$ of this matriciant $Z_{n\times n}^{(m)}$. The probability distributions of random matrices $\Xi^{(i)}_{n\times n}, i = 1, 2, ...$ belong to the domain of attraction of the Circular law.

## 9. $G$-method

In this section we show the power of our $G$-method with comparison to *replica trick, the supersymmetry approach and free probability theory* , on the example of the product of two random matrices. Other examples when we can consider more matrices will easily follow from this example.

*Step 1.* We can establish the self averaging property of n.s.f. of Permutation matrices $A_{2n\times 2n}A^*_{2n\times 2n}$ due to the presence the logarithmic function in the $V$-transform .

*Step 2.* We can make the $V$-regularization choosing very small parameter of regularization like $\alpha = n^{-q_2}$. Where $q_2$ is a number and for our theory the value of this number is not important. For our purposes it is enough that this number is fixed and does not depend on $n$.

*Step 3.* Then we extend $\det[\alpha_n I_{2n\times 2n} + A_{2n\times 2n}A^*_{2n\times 2n}]$ once again:

$$\det[\alpha_n I_{2n\times 2n} + A_{2n\times 2n}A^*_{2n\times 2n}]$$

$$= (-1)^{2n} \det \begin{bmatrix} \mathrm{i}\sqrt{\alpha}_n I_{2n\times 2n} & A^*_{2n\times 2n} \\ A_{2n\times 2n} & \mathrm{i}\sqrt{\alpha}_n I_{2n\times 2n} \end{bmatrix}$$

and now we consider the n.s.f. of Hermitian matrices

$$G_{4n\times 4n} = \begin{bmatrix} 0 & A^*_{2n\times 2n} \\ A_{2n\times 2n} & 0 \end{bmatrix}.$$

*Step 4.* We find a canonical equation for the Stieltjes transform of the non-random accompanying n.s.f. $\mu_n(t, s, x)$.

*Step 5.* Then we can use the rough estimator of convergence of the n.s.f. of the corresponding random permutation matrix with the speed of convergency like $n^{-q_3}$. All calculations are almost the same which were used in the $G$-theory.

## 10. $G$-method, the Berry-Esseen inequality

We are using the Berry-Esseen inequality, the random variable $\gamma$ and we do not pursue the precise order of convergency of n.s.f. $\nu_n(G_{4n\times 4n}, t, s, x, )$ to the accompanying n.s.f. $\mu_n(t, s, x)$ :

$$\sup_x |\nu_n(G_{4n\times 4n}, t, s, x, ) - \mu_n(t, s, x)| \leq cn^{-q_3},\ c > 0.$$

Then we can perform the limit procedure in $V$-transform.

## 11. $G$-method. Canonical equation $K_{91}$ for the product of two independent matrices with independent entries

**Theorem .** *If the real matrices $\Xi_n^{(j)} = \{\xi_{pl}^{(n,j)}\}, j = 1, 2; p, l = 1, ..., n$ be independent for every $n = 1, 2, ...$ and their entries satisfy the conditions of Circular law, Then for every $t$ and $s$*

$$\lim_{\alpha\downarrow 0} p \lim_{n\to\infty} \left\{ \nu_n \left[ t, s, \prod_{j=1}^{2} (I_n + \varepsilon_j \Xi_n^{(j)}) \right] \right.$$

$$\left. + \frac{1}{4\pi} \int_\alpha^\infty \left( \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial t^2} \right) \frac{1}{-2\mathrm{i}\sqrt{y}} b_n \Big( y, t, s \Big) \mathrm{d}y \right\} = 0,$$

*where $b_n\Big(y, t, s\Big) = \frac{1}{n} \sum_{p=1}^{4n} r_{pp}^{(n)} \Big(y, t, s\Big)$ and $r_{pp}^{(n)}(y, t, s), p = 1, ..., 4n$ satisfy the following system of canonical equations $K_{91}$ :*
$$r_{kk}^{(n)} = \left( \left[ \mathrm{i}y^{(1/2)} I_{4n\times 4n} + B_{4n\times 4n} - G_{4n\times 4n} \Big( y, t, s \Big) \right]^{-1} \right)_{kk}$$

$k = 1, ..., 4n,$ *where*

$$B_{4n\times 4n}\left(t,s\right) = \left(b_{ij}^{(n)}\left(t,s\right)\right)_{i=1,\ldots,4n}^{j=1,\ldots,4n}$$
$$= \left\{\begin{array}{cccc} 0 & 0 & I_n\tau^{(1/2)} & L_1 \\ 0 & 0 & L_2 & I_n\tau^{(1/2)} \\ I_n\bar{\tau}^{(1/2)} & L_2^* & 0 & 0 \\ L_2^* & I_n\bar{\tau}^{(1/2)} & 0 & 0 \end{array}\right\},$$

$$L_i = I_n + \varepsilon_i \mathbf{E}\,\Xi_n^{(i)}$$

$$G_{4n\times 4n}\left(y,t,s\right) = \left(g_{ij}^{(n)}\left(y,t,s\right)\right)_{i=1,\ldots,4n}^{j=1,\ldots,4n}$$
$$= \left\{\begin{array}{cccc} \varepsilon_1^2 G_n^{(1)} & 0 & 0 & 0 \\ 0 & \varepsilon_2^2 G_n^{(2)} & 0 & 0 \\ 0 & 0 & \varepsilon_1^2 G_n^{(3)} & 0 \\ 0 & 0 & 0 & \varepsilon_2^2 G_n^{(4)} \end{array}\right\},$$

$$G_n^{(1)}\left(y,t,s\right) = \left[\delta_{il}\frac{1}{n}\sum_{p=3n}^{4n} r_{pp}^{(n)}\left(y,t,s\right)\sigma_{pl}^{(n),1}\right]_{i,l=1,\ldots,n},$$

$$G_n^{(2)}\left(y,t,s\right) = \left[\delta_{il}\frac{1}{n}\sum_{p=2n}^{3n} r_{pp}^{(n)}\left(y,t,s\right)\sigma_{pl}^{(n),2}\right]_{i,l=1,\ldots,n},$$

$$G_n^{(3)}\left(y,t,s\right) = \left[\delta_{il}\frac{1}{n}\sum_{p=n}^{2n} r_{pp}^{(n)}\left(y,t,s\right)\sigma_{pl}^{(n),1}\right]_{i,l=1,\ldots,n},$$

$$G_n^{(4)}\left(y,t,s\right) = \left[\delta_{il}\frac{1}{n}\sum_{p=1}^{n} r_{pp}^{(n)}\left(y,t,s\right)\sigma_{pl}^{(n),2}\right]_{i,l=1,\dots,n}.$$

*There exists a unique solution of this equation in the class of analytical functions in $t$ and $s$.*

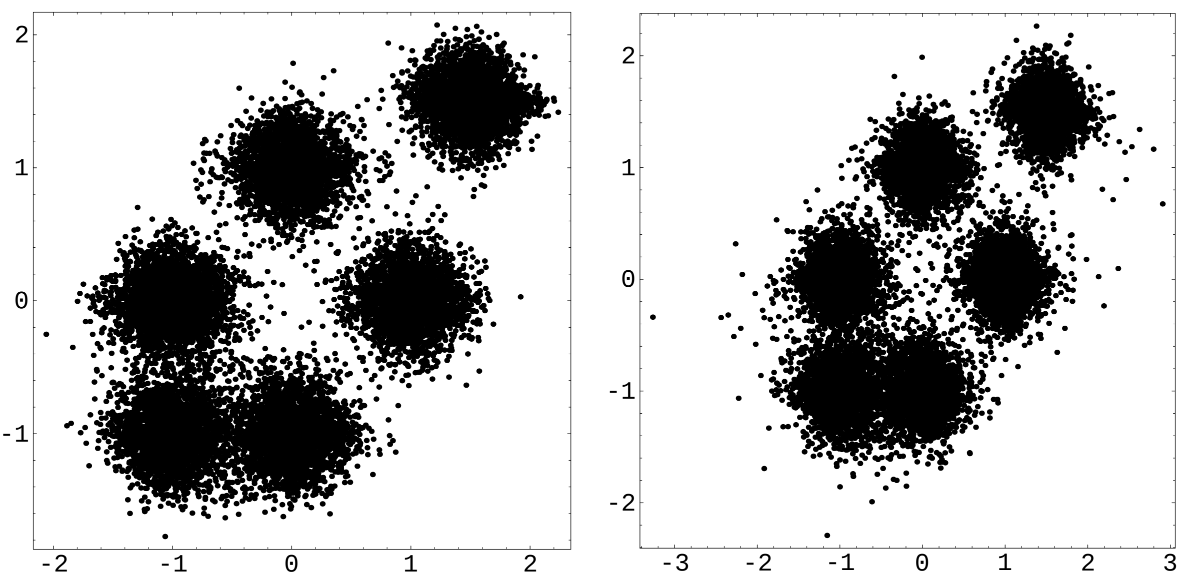
After obtaining the results for n.s.f. of a single symmetric, non symmetric and unitary random matrices we now move in this paper toward the main goal, namely to the most general solution of the problems of the limit theorems of the theory of random matrices: to find limit distributions of n. s. f. of random matrices $f[(\Xi_n^{(j)})^k, (\Xi_n^{(j)^*})^p, j,k,p = 1,2,...]$, where $f(x_1,x_2,...)$ is an analytical function and $\Xi_n^{(j)}, j = 1,2,...$ are independent *ACE(Asymptotically Constant Entries)*-random matrices (in particular, unitary random matrices). Particularly, using the canonical equation $K_{91}$ we derive so called L.I.F.E.-Law: under a certain conditions $\prod_{j=1}^m \Xi_n^{(j)} \sim L.\widetilde{I.F}.E. \sim \{\Xi_n^{(1)}\}^m$.

Roughly speaking L.I.F.E. means that n.s.f. of the sum of nonrandom matrix $A_n$ and the power of a non Hermitian matrix $H_n^k$ with independent ACE-entries(Asymptotically Constant Entries) is approximately equal to n.s.f. of the sum of nonrandom matrix $A_n$ and the product of $k$ independent random matrices $H_n^{(1)}H_n^{(2)}\cdots H_n^{(k)}$ having the same structure as the initial random matrix $H_n$ but their entries may have any distributions from class $G_3$. The similar assertion we can prove for other certain function $f[(\Xi_n^{(j)})^k, (\Xi_n^{(j)^*})^p, j,k,p = 1,2,...]$. This assertion is a simple Corollary from Equation $K_{91}$.

By tradition of choosing the names of Laws in probability theory(Arcsine law, Law of iterated logarithm, etc.) we call this unusual behavior of the n.s.f. of the power of random matrix $\Xi_n^k$ as the **Halloween Law** keeping in mind that the appearance instead in n.s.f. of $k$ copies of the same random matrix $\Xi_n$ its $k$ independent copies $\Xi_n^{(j)}, j = 1, ..., k$ looks like *phantom or illusion.* More important that the histogram and the density of this law look like a hat that people wear during Halloween days(see these pictures 7 and 8 below).

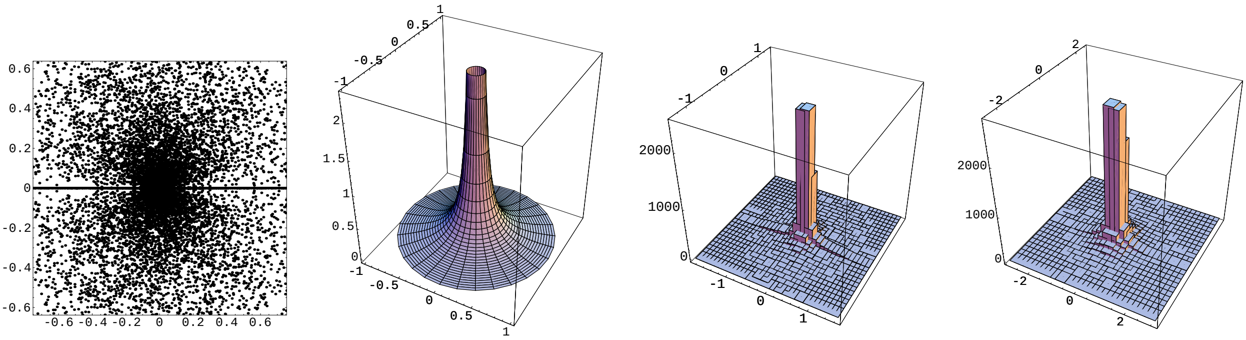
## 12. The $\sim \widetilde{L.I.F.E.} \sim$-phenomenon

For the first time the powers of matrices $\Xi_n$ from class $\mathbf{G_1}$ were investigated by Wegmann. In our case when the matrix $\Xi_n$ is non Hermitian and belongs to the class $\mathbf{G_2}$ or $\mathbf{G_3}$ the Wegmann's method is not valid. Nevertheless, we can find some relatively simple relation for the spectra of functions of random matrices using the *L.I.F.E.* phenomenon(the main statement): in the *L.I.F.E.* sense the spectra of random matrix $\Xi_n^k, \Xi_n \in \mathbf{G_1} \div \mathbf{G_3}$, where $k > 1$ and the matrix $\Xi_n$ is not Hermitian, approximately is equal to the spectra of the product of $k$ independent random matrices $\prod_{j=1}^k \Xi_n^{(j)}$, where $\Xi_n^{(j)} \approx \Xi_n$, the symbol $\approx$ staying between two matrices $\Xi_n$ and $H_n$ means coincidence of distributions of these matrices. (This assertion is a simple Corollary from Equation $K_{91}$.



Figures 3 and 4.

The picture 3 shows the 300 Monte Carlo simulation of the support of the accompanying spectral density of the sum of diagonal matrix $A_{60}$ with six different diagonal entries $a = 1, b = -1, c = -1 - \mathrm{i}, d = \mathrm{i}, e = -\mathrm{i}, f = 1.5 + 1.5\mathrm{i}$ chosen in equal parts and the product of five independent random matrices with independent entries $\Xi_{60}^{(p)} = (\xi_{ij}^{(p)}), p = 1, 2, 3, 4, 5; \mathbf{E}\,\xi_{ij} = 0, \mathbf{E}\,[\xi_{ij}]^2 = 1/60, i, j = 1, ..., 60$. The pictures 4 shows the 300 Monte Carlo simulation of the support of spectral densities of the sum of the same matrix $A_{60}$ and the power of matrix: $[\Xi_{60}^{(1)}]^5$ and these pictures give the conformation of the *L.I.F.E*-law: approximately the support of spectral densities of two matrices $A_{60} + [\Xi_{60}^{(1)}]^5$ and $A_{60} + \prod_{p=1}^{5} \Xi_{60}^{(p)}$ are approximately the same.



Figures 7, 8, 9 and 10

The picture 7 shows the 300 Monte Carlo simulation of the support of the accompanying limit spectral density $p(x, y) = \frac{1}{5\pi}(x^2 + y^2)^{\frac{1}{5}-1}\chi\{x^2 + y^2 \leq 1\}$ of the product of five independent random matrices with independent entries $\Xi_{60}^{(p)} = (\xi_{ij}^{(p)}), p = 1, ..., 5; \mathbf{E}\,\xi_{ij} = 0, \mathbf{E}\,[\xi_{ij}]^2 = 1/60, i, j = 1, ..., 60$. The picture 8 shows the Halloween density for $k = 5$. This picture gives the conformation of the *L.I.F.E*-law: approximately the supports of spectral densities of the product of five matrices $\prod_{j=1}^{5} H_{60}^{(j)}$, and $\{H_{60}^{(1)}\}^5$ are the same. The pictures 9 and 10 shows the histograms(Halloween Law) and support of the accompanying limit spectral density.

These pictures give the conformation of the *L.I.F.E*-law. We see that figure 8 looks like a hat that some people wear during Halloween days.

## 13. Monte Carlo simulations for Sombrero Law $A_n\Xi_n^m$

L.I.F.E. phenomenon is working also for a matrices $A_n\Xi_n^k$, where $A_n$ is a diagonal non random and $\Xi_n$ is a random matrices. We do not present here corresponding calculations, because all proofs are almost the same as for matrices $A_n + \Xi_n^k$.

## 14. The border $G_{93}(t,s)$ of the support of the accompanying spectral density $p_{93}(t,s)$ for random matrices whose entries have equal variances and nonzero expectations for diagonal entries

It is difficult to find the accompanying probability spectral density of the product of two matrices, but surprisingly more easily to find the border of the support $G$ of the accompanying spectral density for random matrices.

Then from Theorem we obtain that the border of the support $G$ of the accompanying spectral density for random matrices with equal variances. Now we can consider many interesting cases of distribution of eigenvalues of the product of two matrices. For example, if $\mathbf{E}\,\Xi_n(1) = \mathbf{E}\,\Xi_n(2) = A_n, \mathbf{E}\,|\xi_{pl}^{(j)} - a_{pl}^{(j)}|^2 = n^{-1}, \varepsilon(j) = 1, j = 1,2$, then we have equation for the border of the support of accompanying spectral density

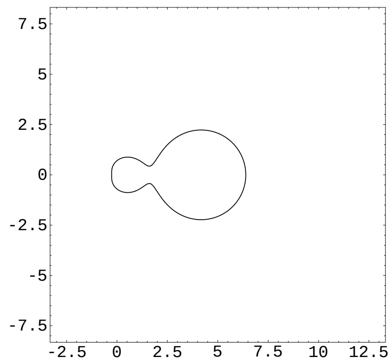
$$1 = \frac{1}{n}\mathrm{Tr}\left\{|\tau| + (I_n + A_n)(I_n + A_n)^* - [\sqrt{\tau}(I_n + A_n)^* + \sqrt{\overline{\tau}}(I_n + A_n)]\right.$$

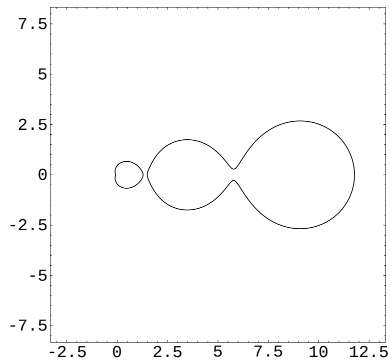
$$\left.\times\,[|\tau| + (I_n + A_n)(I_n + A_n)^*]^{-1}[\sqrt{\tau}(I_n + A_n)^* + \sqrt{\overline{\tau}}(I_n + A_n)]\right\}^{-1}.$$

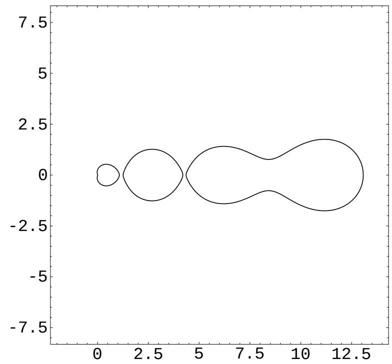
The second example, if $\mathbf{E}\,\Xi_n^{(1)} = \mathbf{E}\,\Xi_n^{(2)} = A_n$ and the matrix $I_n + A_n$ is a symmetric real matrix with eigenvalues $\lambda_k, k = 1, ..., n$. then the border $G_{91}(t, s)$ of the support of the limit spectral density of the product of two matrices $(I_n + \Xi_n^{(1)})(I_n + \Xi_n^{(2)})$ is equal to

$$\frac{1}{n}\sum_{k=1}^{n}\frac{1}{\sqrt{t^2+s^2}+\lambda_k^2 - |\sqrt{\bar{\tau}}+\sqrt{\tau}|^2\frac{\lambda_k^2}{\sqrt{t^2+s^2+\lambda_k^2}}} = 1.$$

We have chosen in the pictures below two points $\lambda_k = 0.55, k = 1, ..., [n/2], \lambda_j = 2, j = [n/2]+1, ..., n$ in two equal parts for all eigenvalues, and similarly we have chosen in the second picture eigenvalues 0.55; 1.8;3 in three equal parts and 0.55;1.6;2,4; 3,4 in four equal parts. Then we can see the structure of the border support for the limit spectral density of the product of two matrices $(I_n + \Xi_n^{(1)})(I_n + \Xi_n^{(2)})$




Of course, we can consider any matrix $A_n$ in our equation for the border support, for example, we can consider diagonal complex matrix, but behavior of border will be similar, i.e. if the distance between diagonal entries of diagonal matrix $A_n$ are large enough, then the border support looks like several closed almost circles.

LITERATURE

1. V. L. Girko, *An Introduction to Statistical Analysis of Random Arrays.* Publishing VSP, 1998.

2. V. L. Girko, *Theory of stochastic canonical equations.* Volume I and Volume II. Publishing Kluwer, 2001.

3. V. L. Girko, A. I. Vladimirova, Spectral analysis of stochastic recurrence system of growing dimension under $G$-condition. Canonical equation $K_{91}$ *Random operators and stochastic equations.* **17(3)**, (2009), 247–280.

4. E. Gudowska-Nowak, R. A. Janik, J. Jurkiewicz and M. A. Nowak, Infinite Products of Large Random Matrices and Matrix-Valued Diffusion. *Nucl. Phys. B* **670** (2003), 479–507.

5. E. Gudowska-Nowak, R. A. Janik, J. Jurkiewicz and M. A. Nowak, Diffusion process of large random matrices. *Acta Phys. Pol. B*, **34** (2003), 4835–4853.

6. R. Wegmann, The asymptotic eigenvalue-distribution for a certain class of random matrices. *J. Math. Anal. Appl.* **56** (1976), 113–132.