Marian Smoluchowski Institute of Physics Mark Kac Complex System Research Center

# Complex systems

Matrices — Large multivariate ensembles of parameters

Random — Complexity is embedded in randomness

Asymmetric — Information flow typically involves delays, long-range interactions, and is asymmetric

# Asymmetric random matrices

## What do we need them for?

- long-range time-delayed correlations
- asymmetric correlation matrices
- examples: human brain and financial markets

Stanisław Drożdż[1,2] Jarosław Kwapień[1]

[1] Polish Academy of Science, Institute of Nuclear Physics, Kraków

[2] University of Rzeszów, Faculty of Mathematics and Natural Sciences

# Empirical/practitical aspects/applications/needs

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## Temporal correlations versus noise in the correlation matrix formalism: An example of the brain auditory response

J. Kwapień,[1] S. Drożdż,[1,2] and A. A. Ioannides[3]

[1]*Institute of Nuclear Physics, PL–31-342 Kraków, Poland*

[2]*Institut für Kernphysik, Forschungszentrum Jülich, D-52425 Jülich, Germany*

[3]*Laboratory for Human Brain Dynamics, Brain Science Institute, RIKEN, Wako-shi, 351-0198, Japan*

We adopt the concept of the correlation matrix to study correlations among sequences of time-extended events occurring repeatedly at consecutive time intervals. As an application we analyze the magnetoencephalography recordings obtained from the human auditory cortex in the epoch mode during the delivery of sound stimuli to the left or right ear. We look into statistical properties and the eigenvalue spectrum of the correlation matrix **C** calculated for signals corresponding to different trials and originating from the same or opposite hemispheres. The spectrum of **C** largely agrees with the universal properties of the Gaussian orthogonal ensemble of random matrices, with deviations characterized by eigenvectors with high eigenvalues. The properties of these eigenvectors and eigenvalues provide an elegant and powerful way of quantifying the degree of the underlying collectivity during well-defined latency intervals with respect to stimulus onset. We also extend this analysis to study the time-lagged interhemispheric correlations, as a computationally less demanding alternative to other methods such as mutual information.

# Two (sub)systems considered in separation

$A$ $B$ $N$ degrees each

represented by the time series $\{x_\alpha(t_i)\}$ $\{y_\beta(t_i)\}$ $(i = 1, ..., T)$

$$X_{\alpha,i} = \frac{1}{\sigma_\alpha}(x_\alpha(t_i) - \bar{x}_\alpha)$$

$$Y_{\beta,i}(\tau) = \frac{1}{\sigma_\beta}(y_\beta(t_i + \tau) - \bar{y}_\beta)$$

## Form generalized correlation matrix

$$\mathbf{C}(\tau) = \frac{1}{T}\mathbf{X}[\mathbf{Y}(\tau)]^{\mathrm{T}}$$

Typically asymmetric and non-Hermitean

$$\mathrm{Tr}\mathbf{C} \leqslant N$$

$$\mathbf{C}(\tau)\mathbf{v}^{(k)}(\tau) = \lambda_k(\tau)\mathbf{v}^{(k)}(\tau)$$

Typically complex eigenvalues

J. Ginibre, J. Math. Phys. 6, 440 (1965)

GinOE $\mathbf{G}$ $(N \times N)$ $\longrightarrow$ $p(\mathbf{G}) = (2\pi)^{-N^2/2} \exp[-\mathrm{Tr}(\mathbf{G}\mathbf{G}^{\mathrm{T}})]$

H.-J. Sommers, A. Crisanti, H. Somplinski, Y. Stein, PRL 60, 1895 (1988)

A. Edelman, J. Multivariate Anal. 60, 203 (1997)

E. Kanzieper, G. Akemann, Phys. Rev. Lett. 95, 230201 (2005)

H.-J. Sommers, W. Wieczorek, J. Phys. A 41, 405003 (2008)

$L$ real, $N - L$ complex } eigenvalues

$$\lim_{N\to\infty} E(L) = \sqrt{\frac{2N}{\pi}}.$$

$$E(L) = 1/2 + \sqrt{\frac{2N}{\pi}}\left(1 - \frac{3}{8N} - \frac{3}{128N^2} + \mathcal{O}(N^{-3})\right)$$

$\lambda = \lambda_x + i\lambda_y$

$$\rho_G(\lambda) = \rho_G^c(\lambda) + \delta(\lambda_y)\ \rho_G^r(\lambda)$$

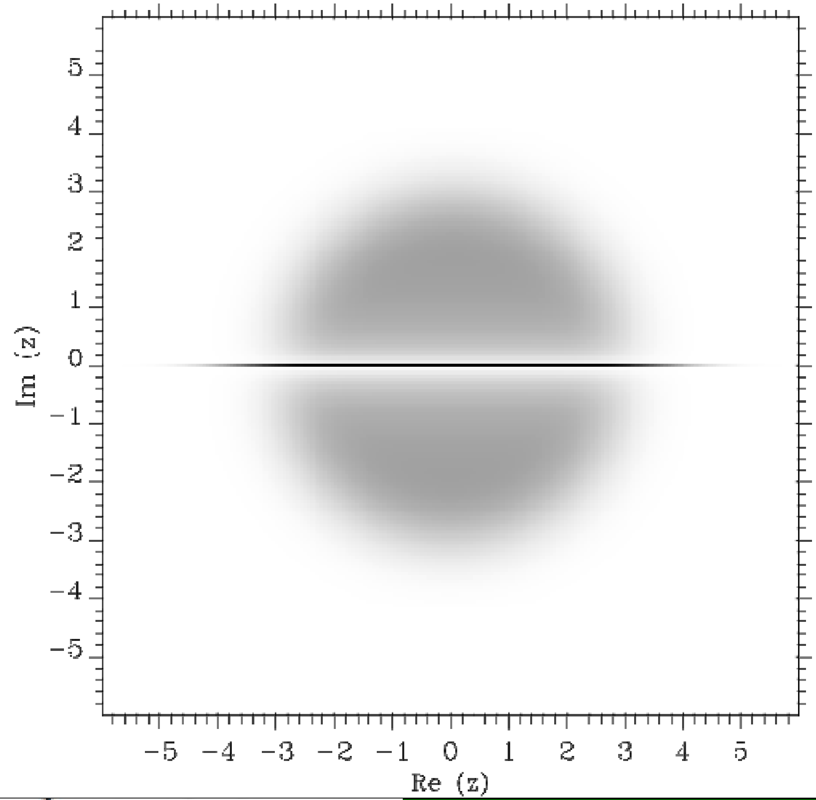
$$\rho_G^c(\lambda) = \frac{2|\lambda_y|}{\sqrt{2\pi}}\left(1 - \mathrm{erf}(\sqrt{2}|\lambda_y|)\right)\ e^{2\lambda_y^2} \int_{|\lambda_x|^2}^{\infty} du\ e^{-u} \frac{u^{N-2}}{\Gamma(N-1)}$$

$$\rho_G^r(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{|\lambda_x|^2}^{\infty} du\ e^{-u} \frac{u^{N-2}}{\Gamma(N-1)} + \frac{1}{\sqrt{2\pi}} |\lambda_x|^{N-1}\ e^{-\lambda_x^2/2} \int_0^{\lambda_x} du\ e^{-u^2/2} \frac{u^{N-2}}{\Gamma(N-1)}$$

Distribution of eigenvalues

$\lambda \equiv z$

for GinOE $N = 10$




For $N \to \infty$

$$\rho_G^c(\lambda) = \frac{1}{\pi}\Theta(\sqrt{N} - |\lambda|)$$ complex

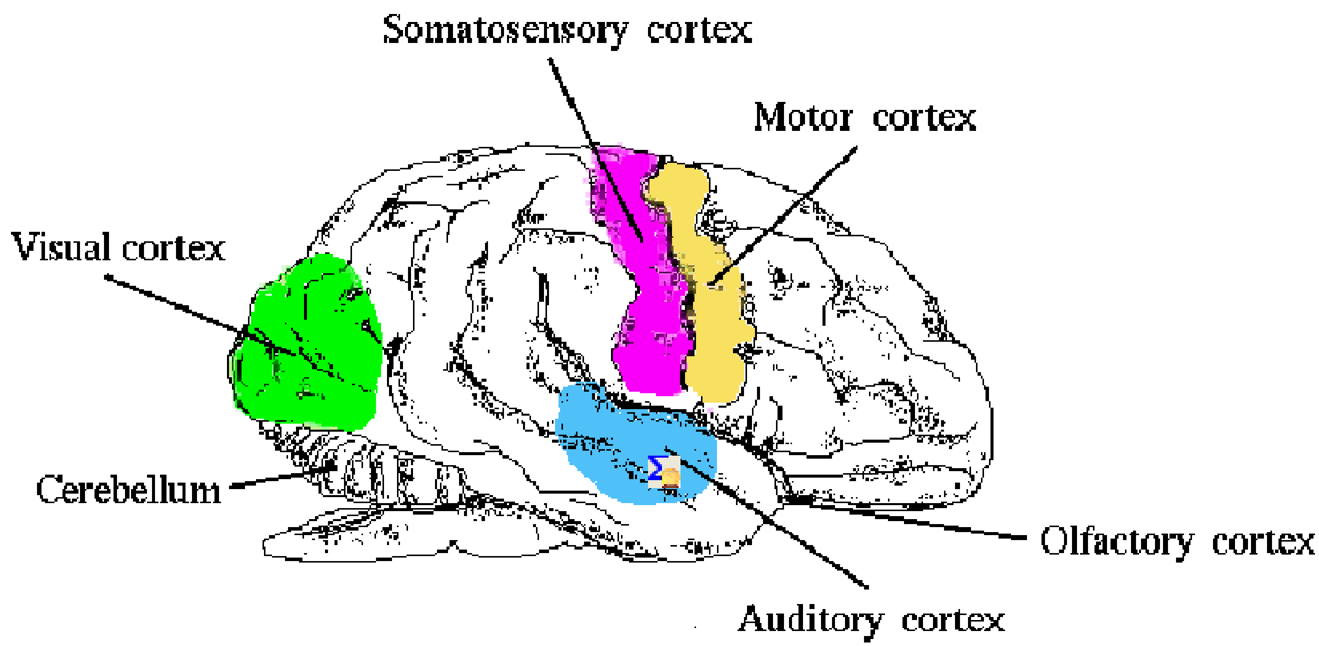
$$\rho_G^r(\lambda) = \frac{1}{\sqrt{2\pi}}\Theta(\sqrt{N} - |\lambda_x|)$$ real

## Generalized Wishart Ensemble ?

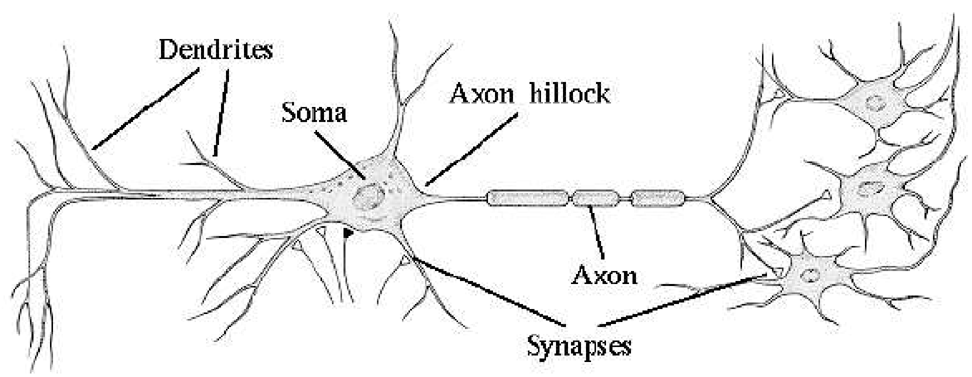
1. C. Biely, S. Thurner, Quantitative Finance 8, 705 (2008)
2. Eugene Kanzieper and Navinder Singh
   Non-Hermitean Wishart random matrices (I), 2010, preprint?

# Real complexity: Brain

- spacially and temporally correlated

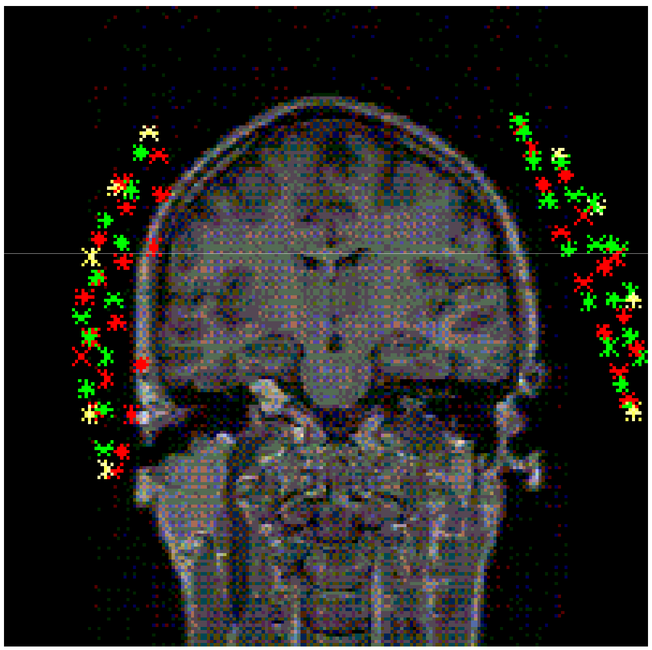



Human brain contains
10-100 billion neurons

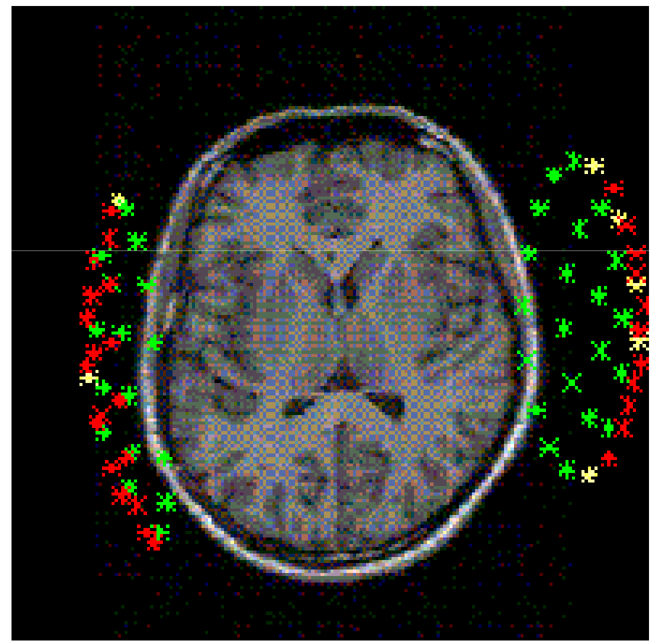


# Magnetoecephalography

## MEG



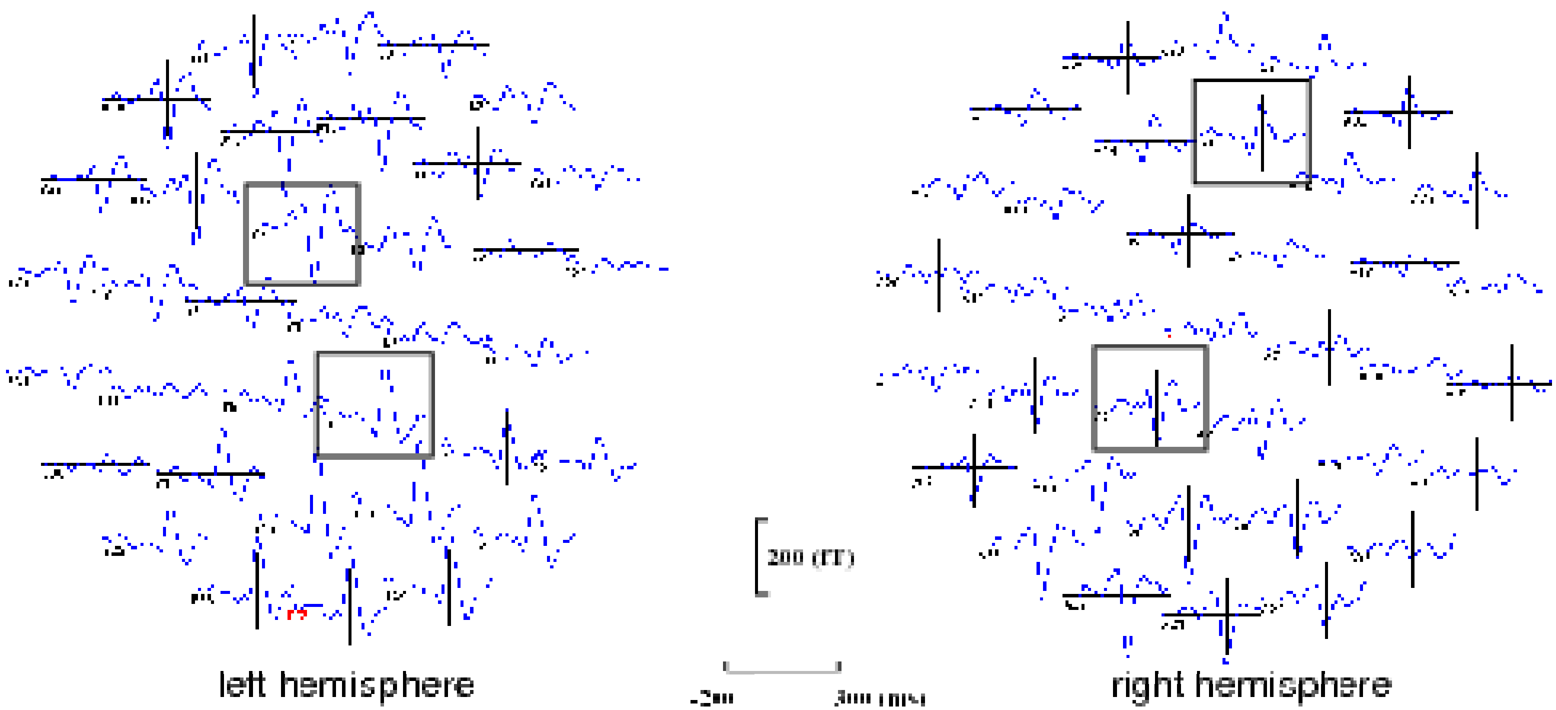
L R



L R

SQUID detectors

# Average MEG signals for sound presentation to the left ear



For each channel located at point $\mathbf{r}_i$, a weighted sum is created

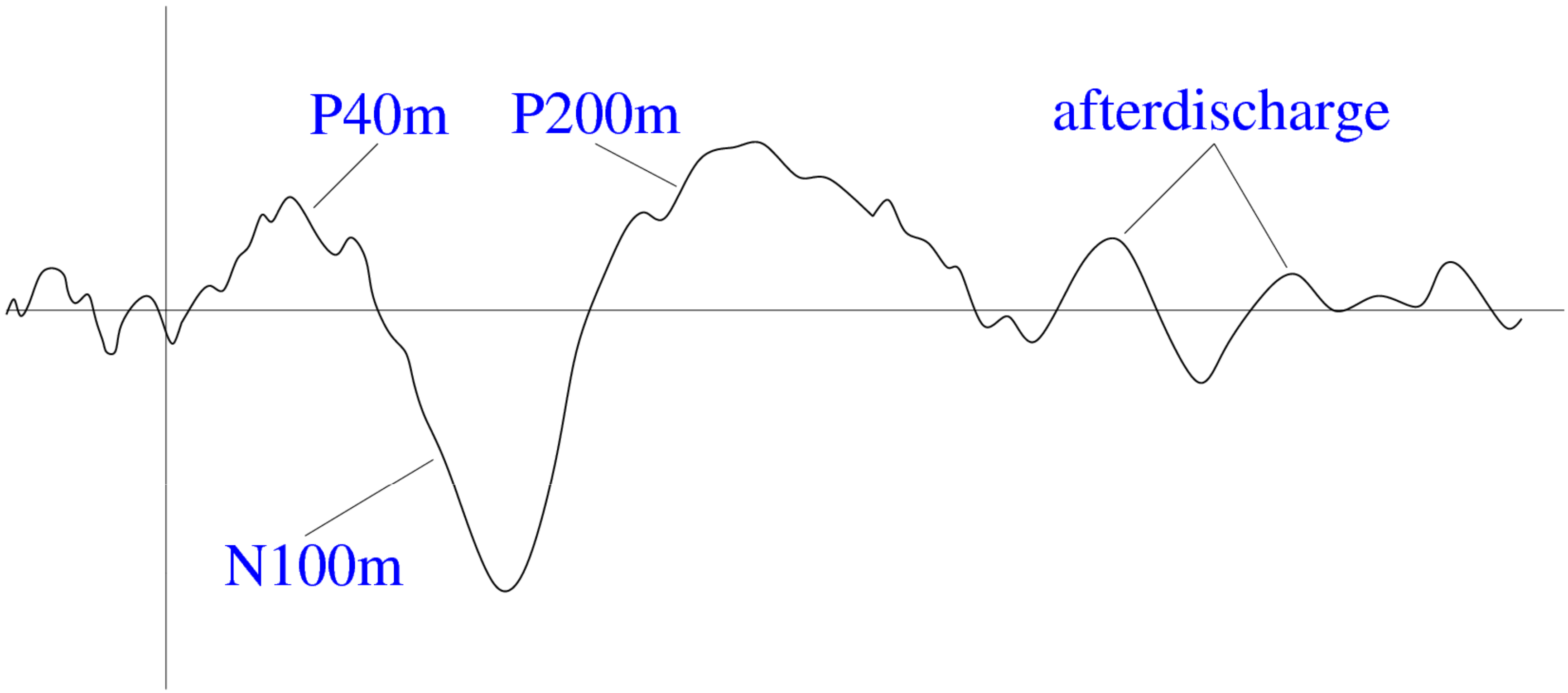
$$v_i(t) = \sum_{j=1}^{K} e^{-\left(\frac{|\mathbf{r}_j - \mathbf{r}_i|}{\lambda}\right)^2} S_j(t)$$

where $S_j$ denotes the signal recorded by the $j^{\text{th}}$ detector

select two channels $k_1$ and $k_2$, associated with the two opposite extrema

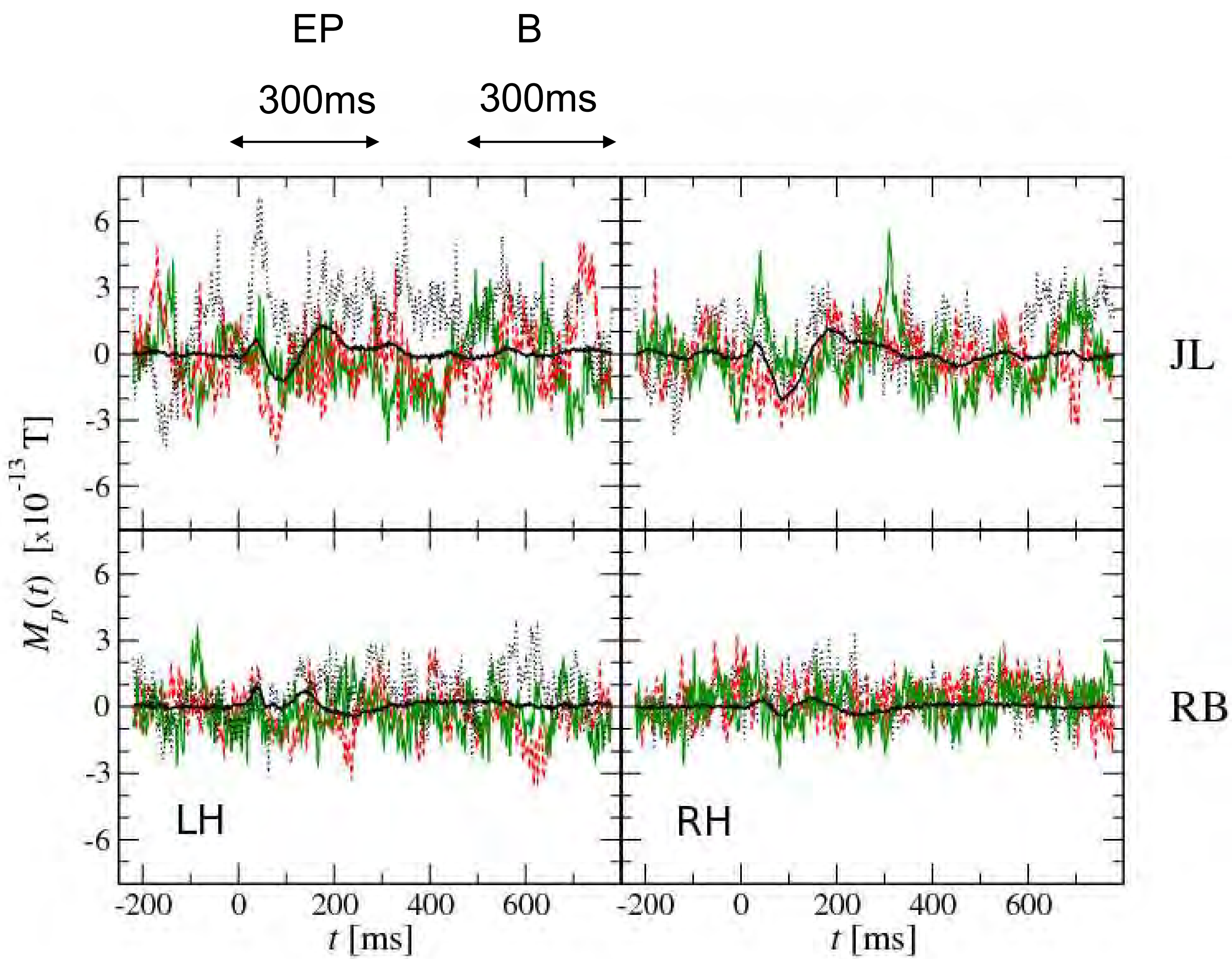
The global activity of the region is then defined by the difference

$$\mathcal{V}(t) = v_{k_1}(t) - v_{k_2}(t) = \sum_{j=1}^{K} \left[ e^{-\left(\frac{|\mathbf{r}_j - \mathbf{r}_{k_1}|}{\lambda}\right)^2} - e^{-\left(\frac{|\mathbf{r}_j - \mathbf{r}_{k_2}|}{\lambda}\right)^2} \right] S_j(t)$$




Schematic plot of magnetic response to a short sound stimulus

# Exemplary VS signals for two human subjects



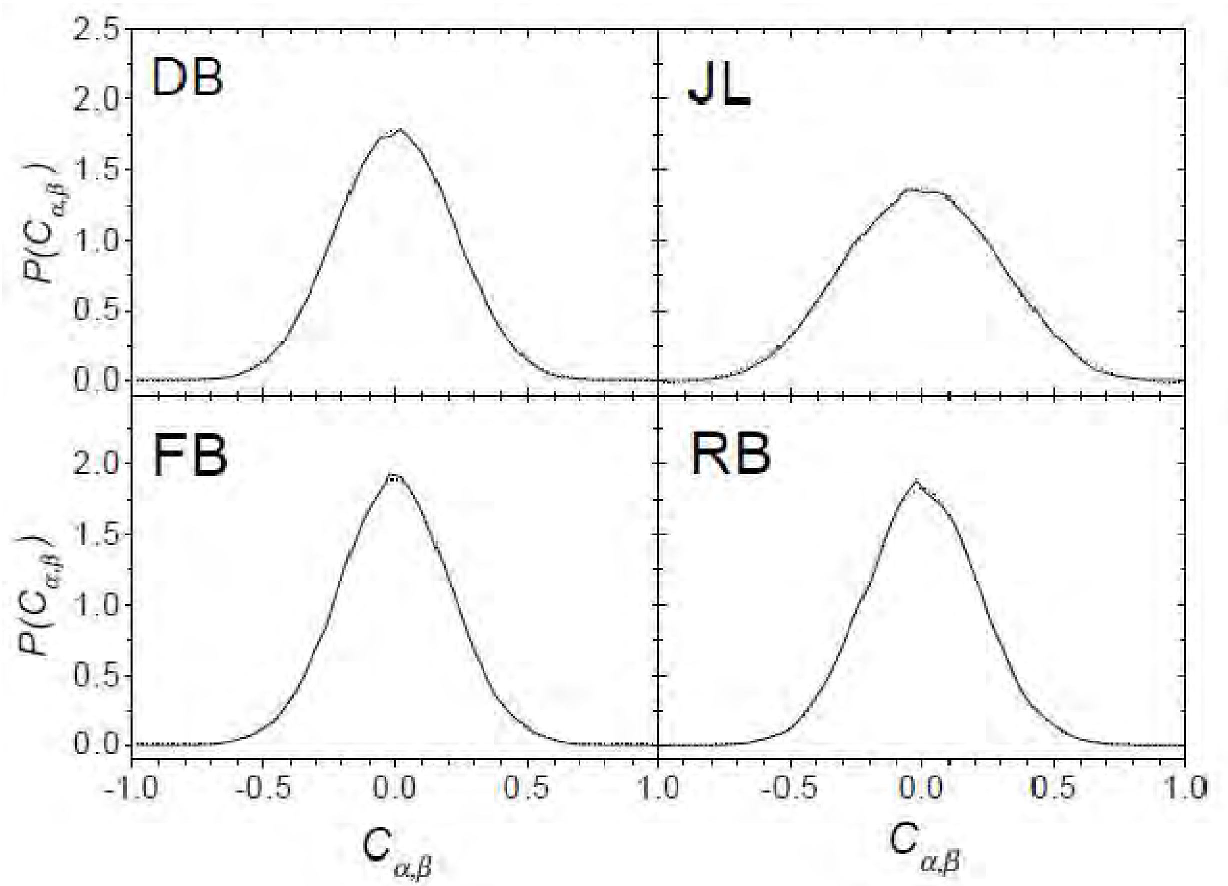
Study correlations between MEG time-series representing consecutive trials (sound delivery to the left or right ear)

$N = 240$ - the number of trials $(\alpha = 1, \ldots, N)$

$T = 300$ - the length of the series (in ms) $(i = 1, \ldots, T)$

both in the evoked potential (EP) region
as well as in the background (B) region

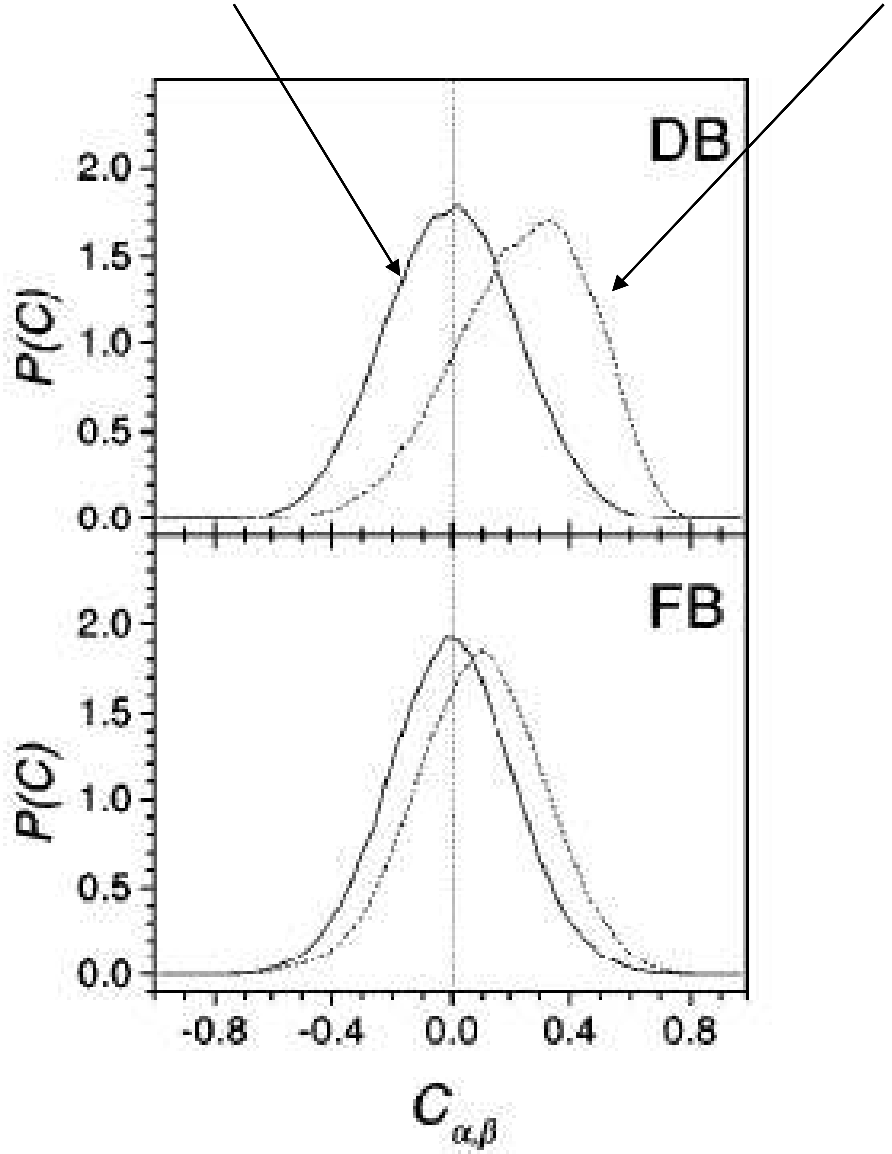
Distribution of off-diagonal elements of one-hemisphere correlation matrix in the background activity region (sold line) and a Gaussian fit (dotted line)




→ Consistent with conventional Wishart

# Distribution of matrix elements:

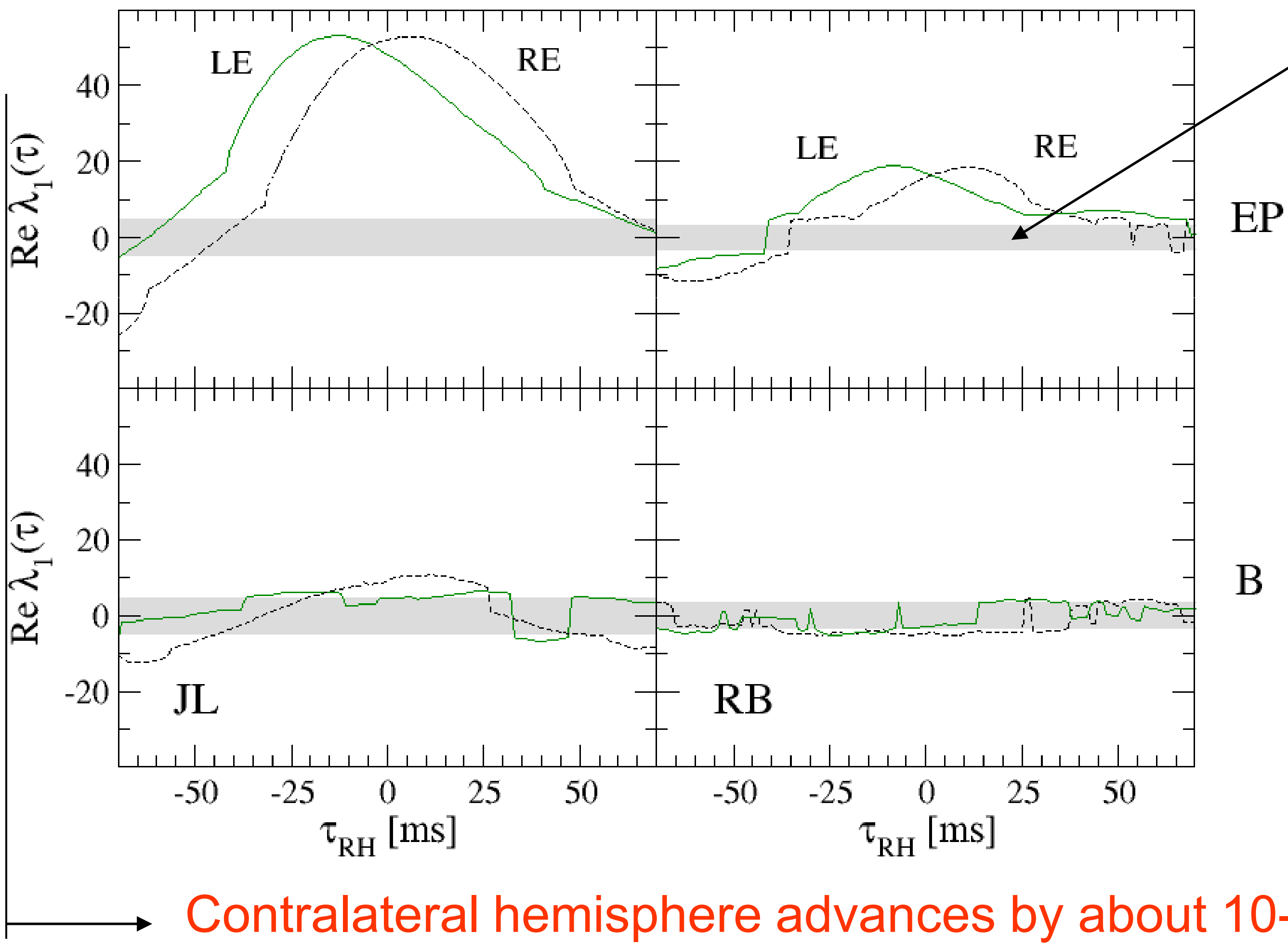
Background (B) vs Evoked Potential (EP)



# Cross-hemisphere correlations

$\tau$ - time-lag between signals from the opposite hemispheres

GinOE strip

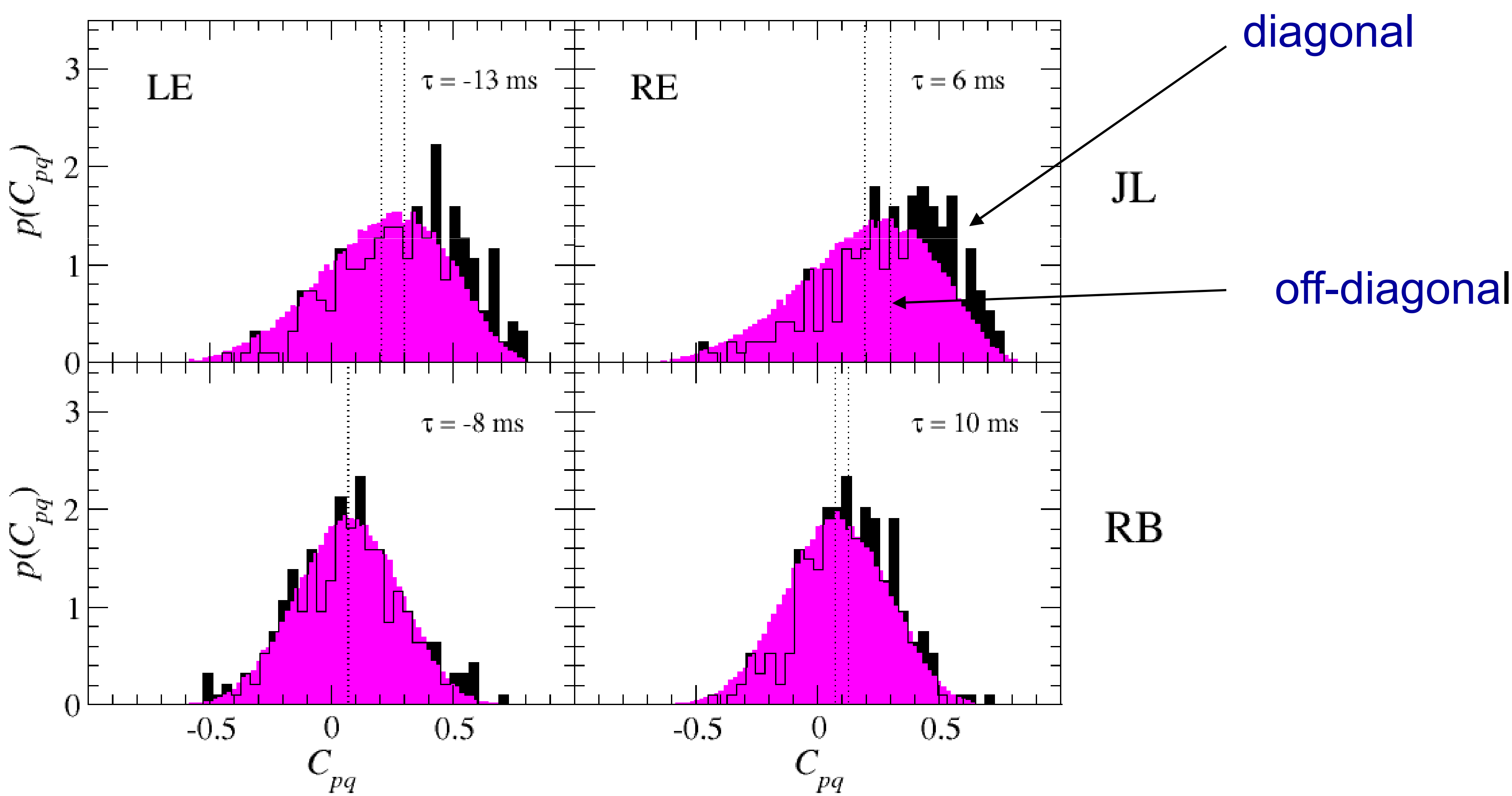



Evoked Potential

Backround

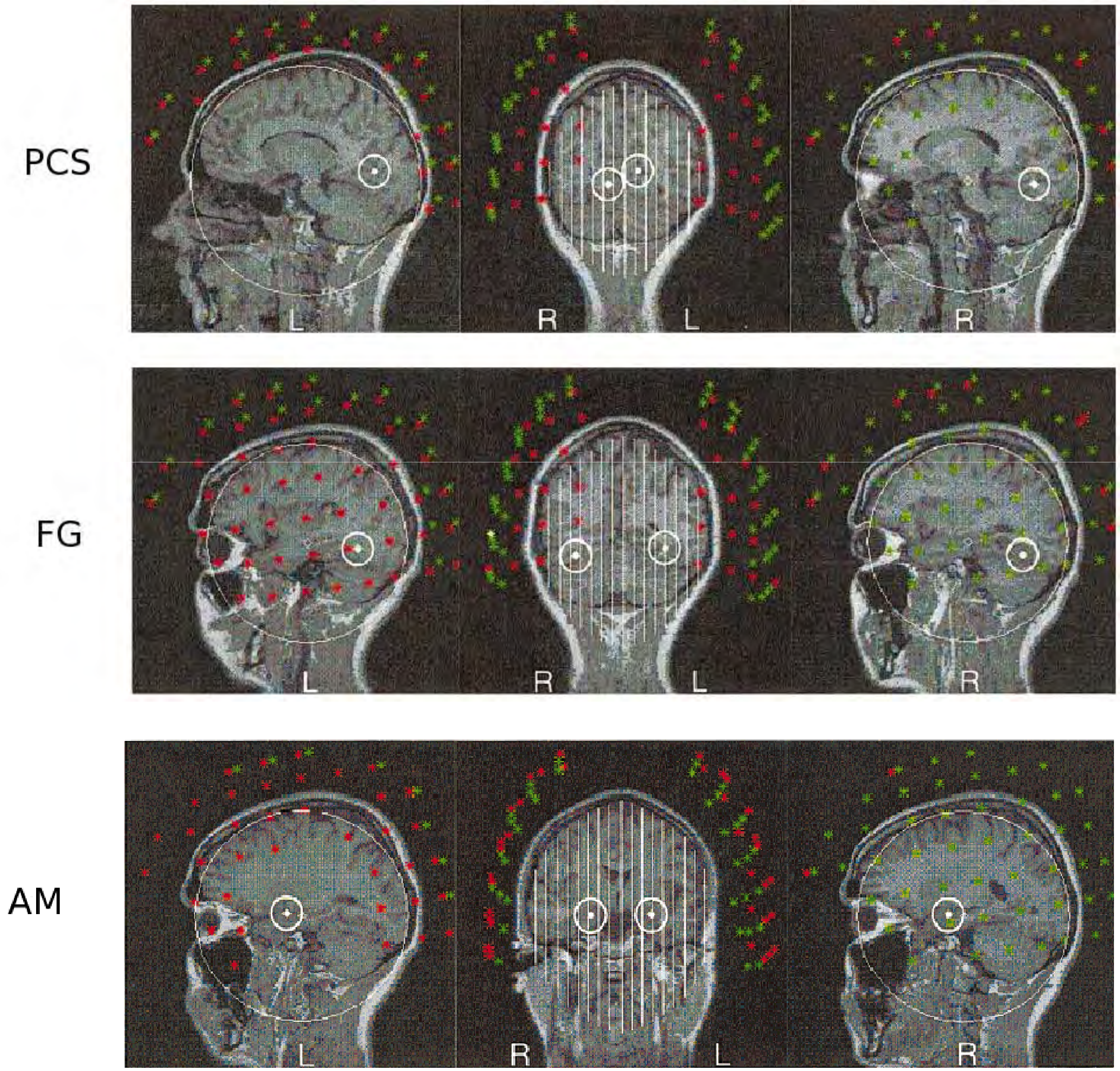
Contralateral hemisphere advances by about 10-15 ms

Distribution of matrix elements for time-lags for which the largest eigenvalue reaches maximum



# More complex: stimulated visual cortex

Involves several functionally distinct regions




Posterior calcarine sulcus

(Tylna bruzda ostrogowa)

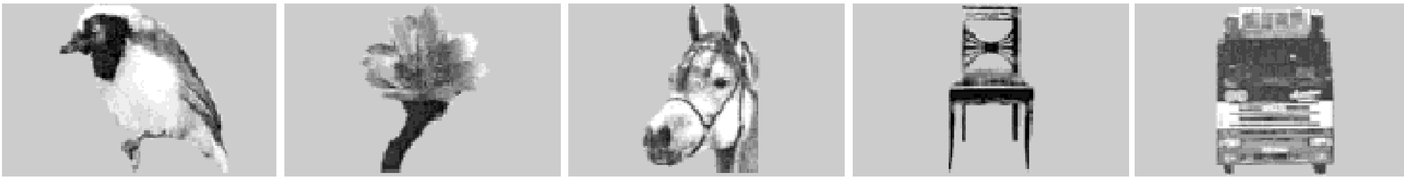
Fusiform gyrus

(Zakręt wrzecinowaty)

Amygdala

# Visual stimulation: object recognition

Examples of stimuli used:



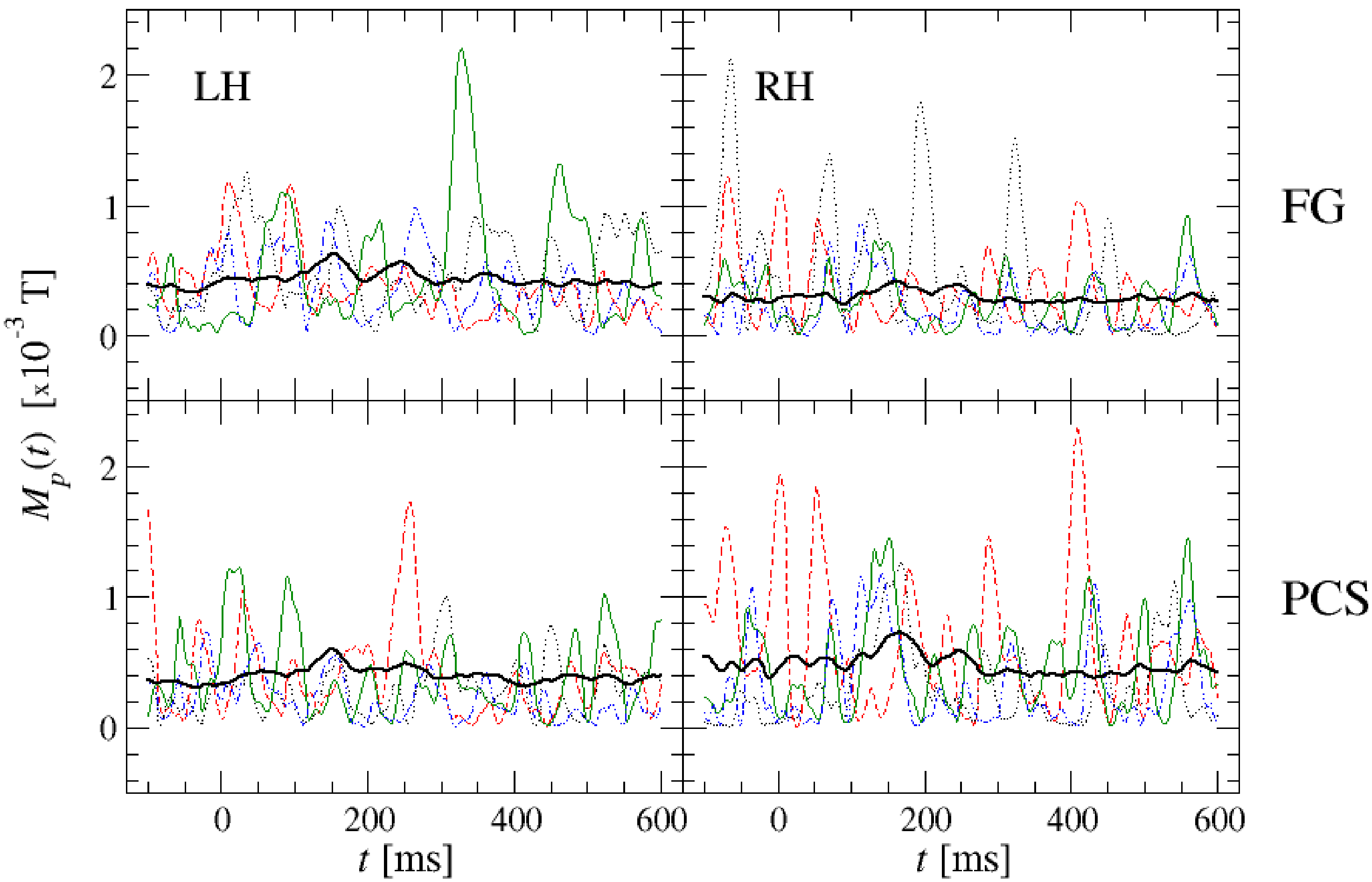
Projection time of stimuli: 0 -500 ms

MEG frequency 510 Hz

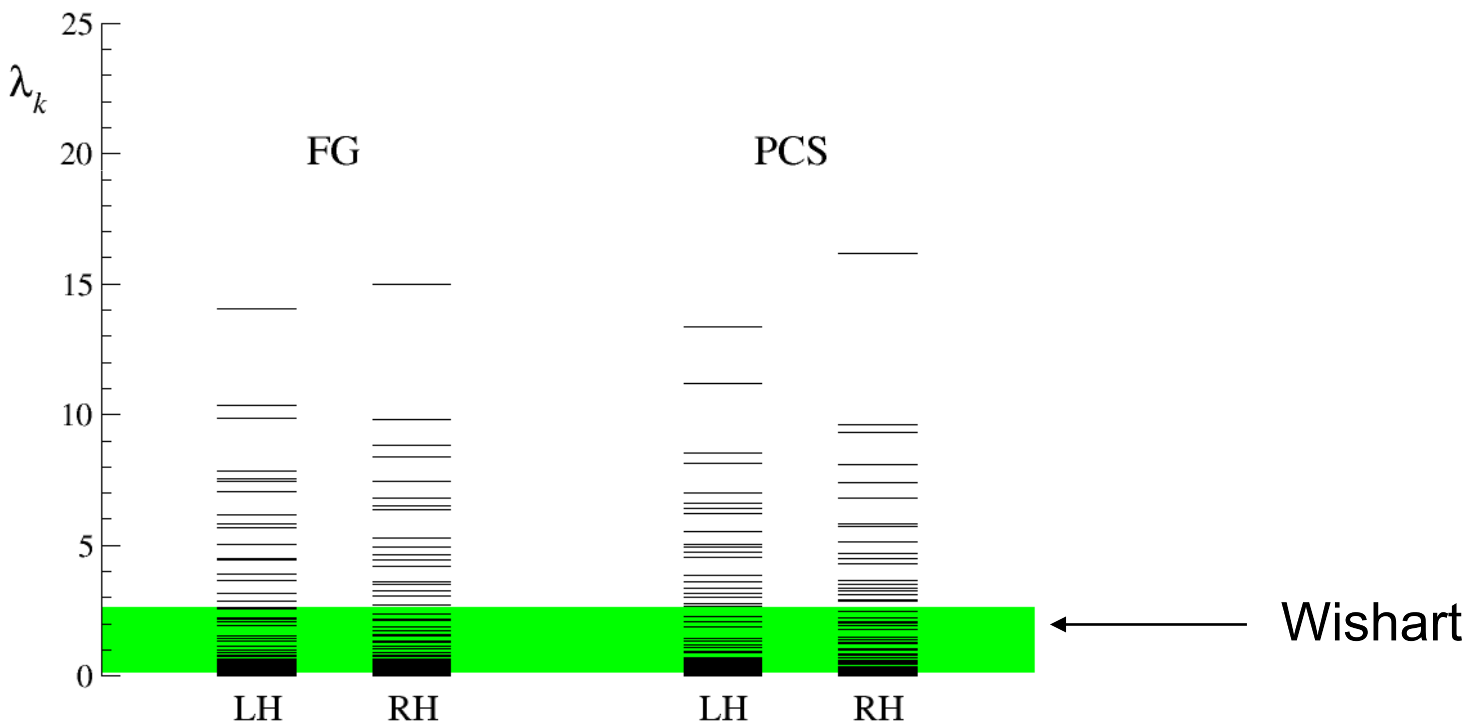
$$N = 280$$

$$T = 357$$

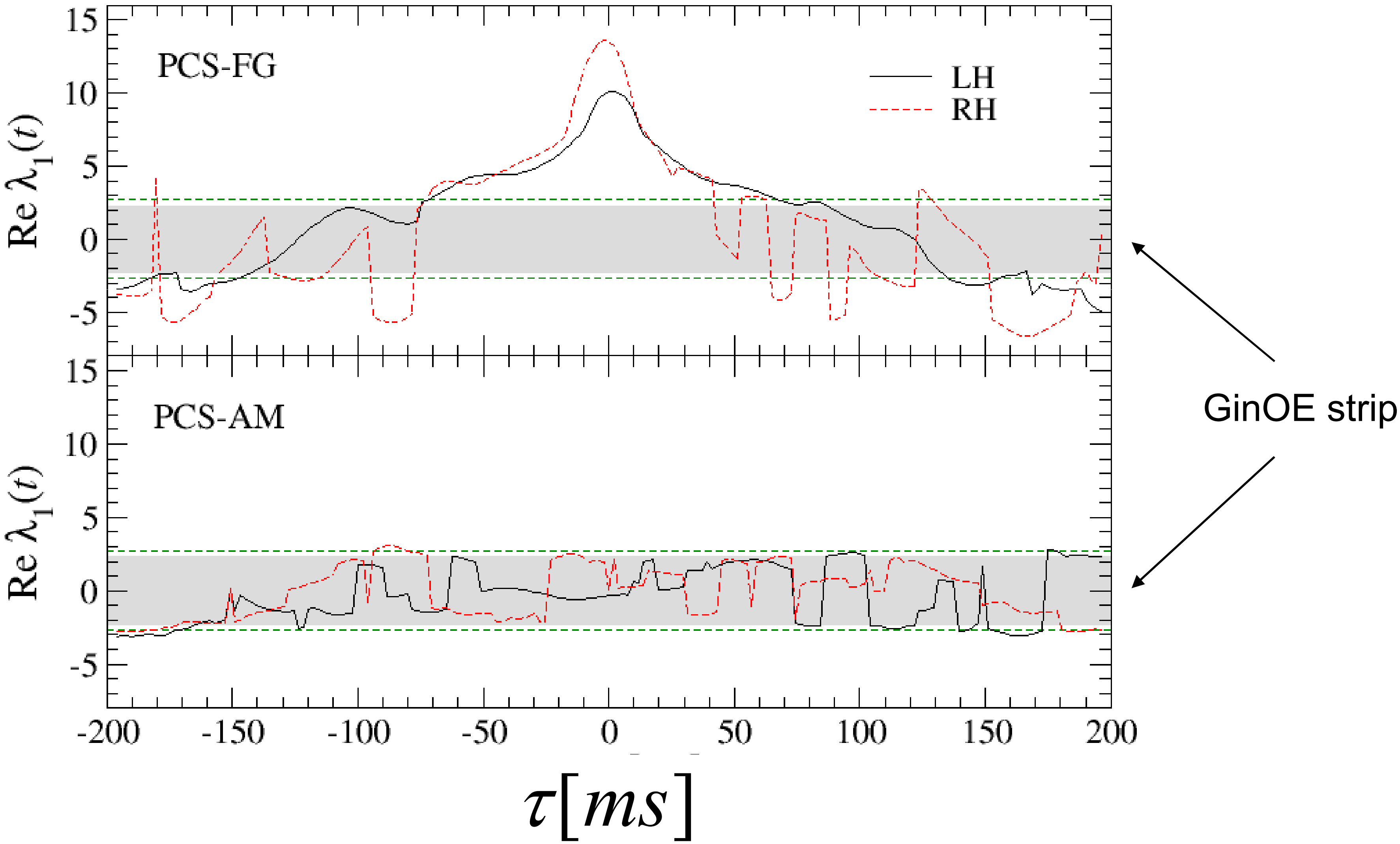
# Exemplary MEG visual signals

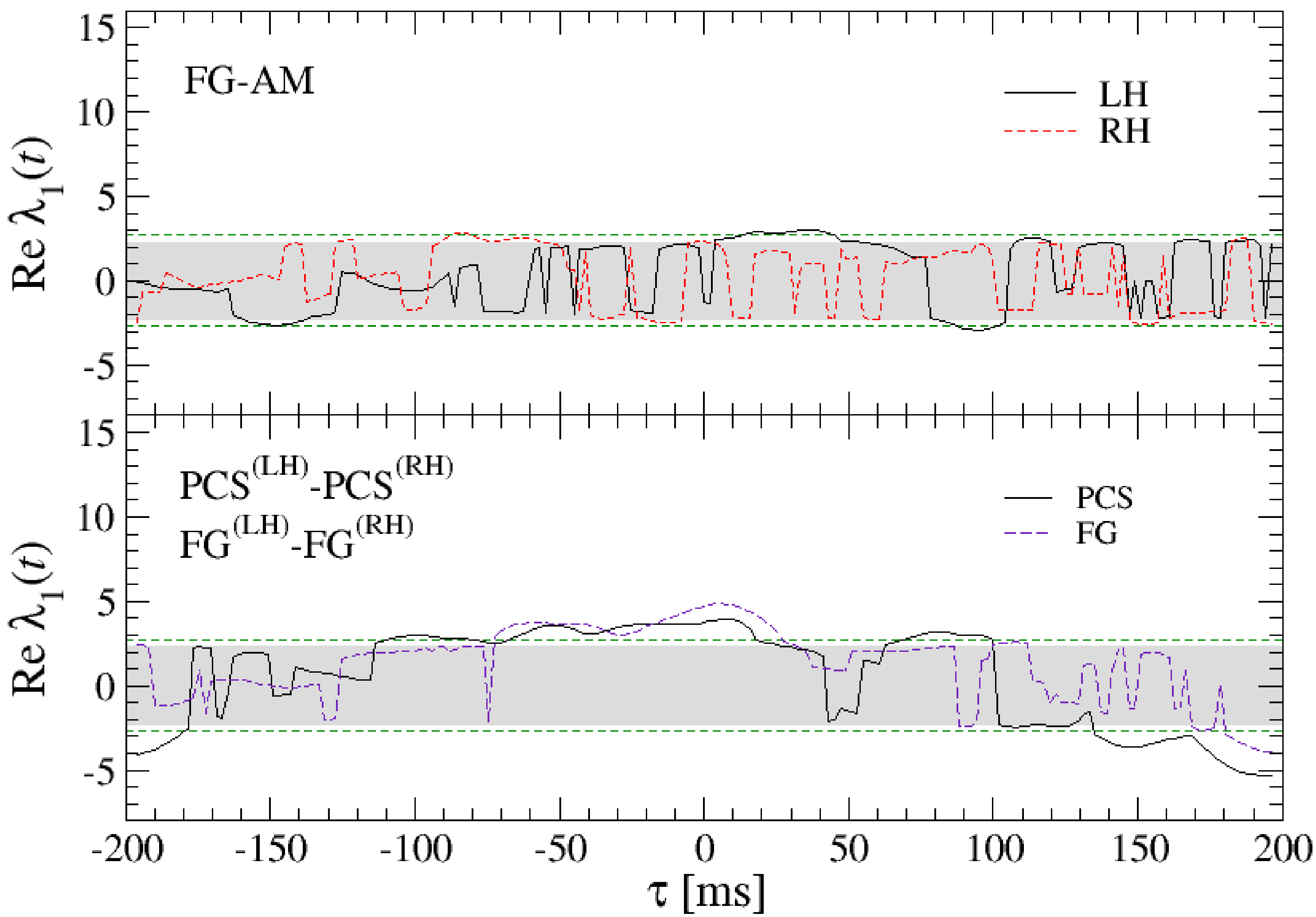


Temporal correlation spectra in response to visual stimulations the four cortical regions – symmetric correlation matrices:



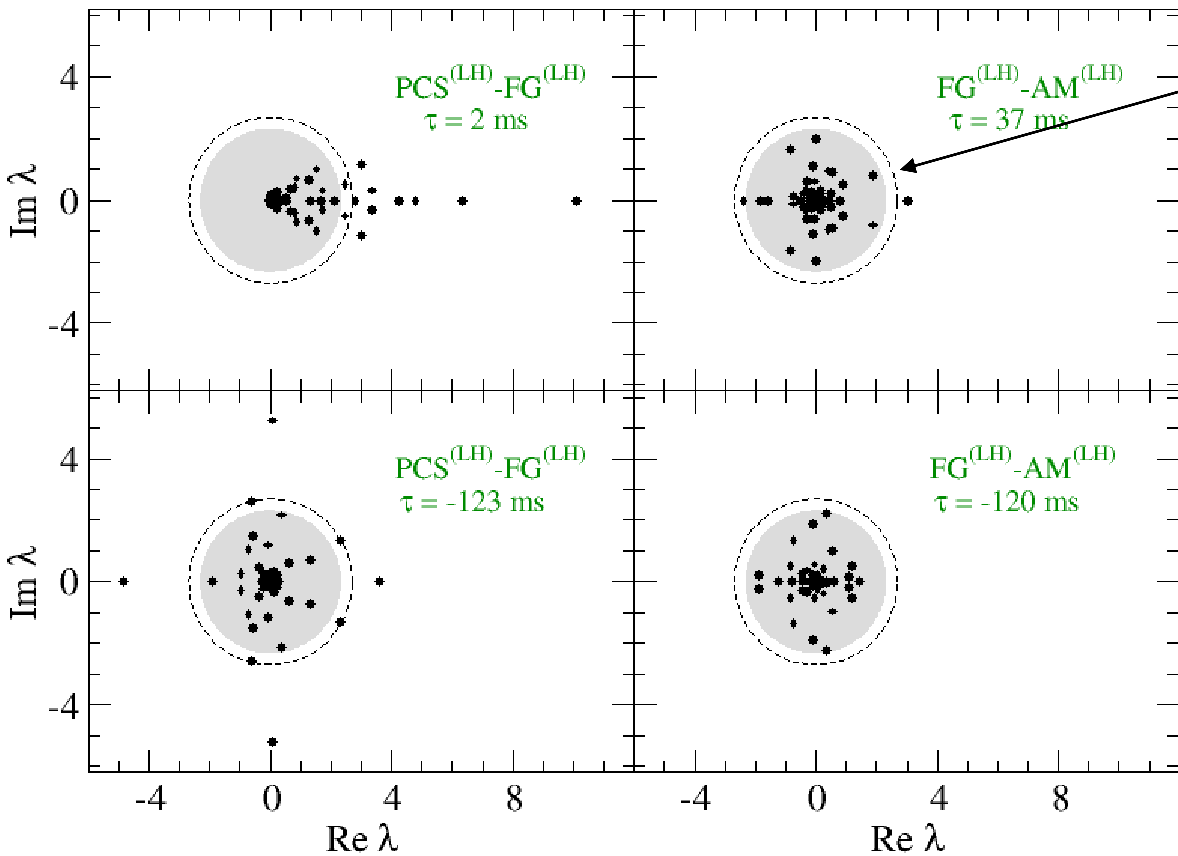
# Cross-area (asymmetric matrices) visual correlations





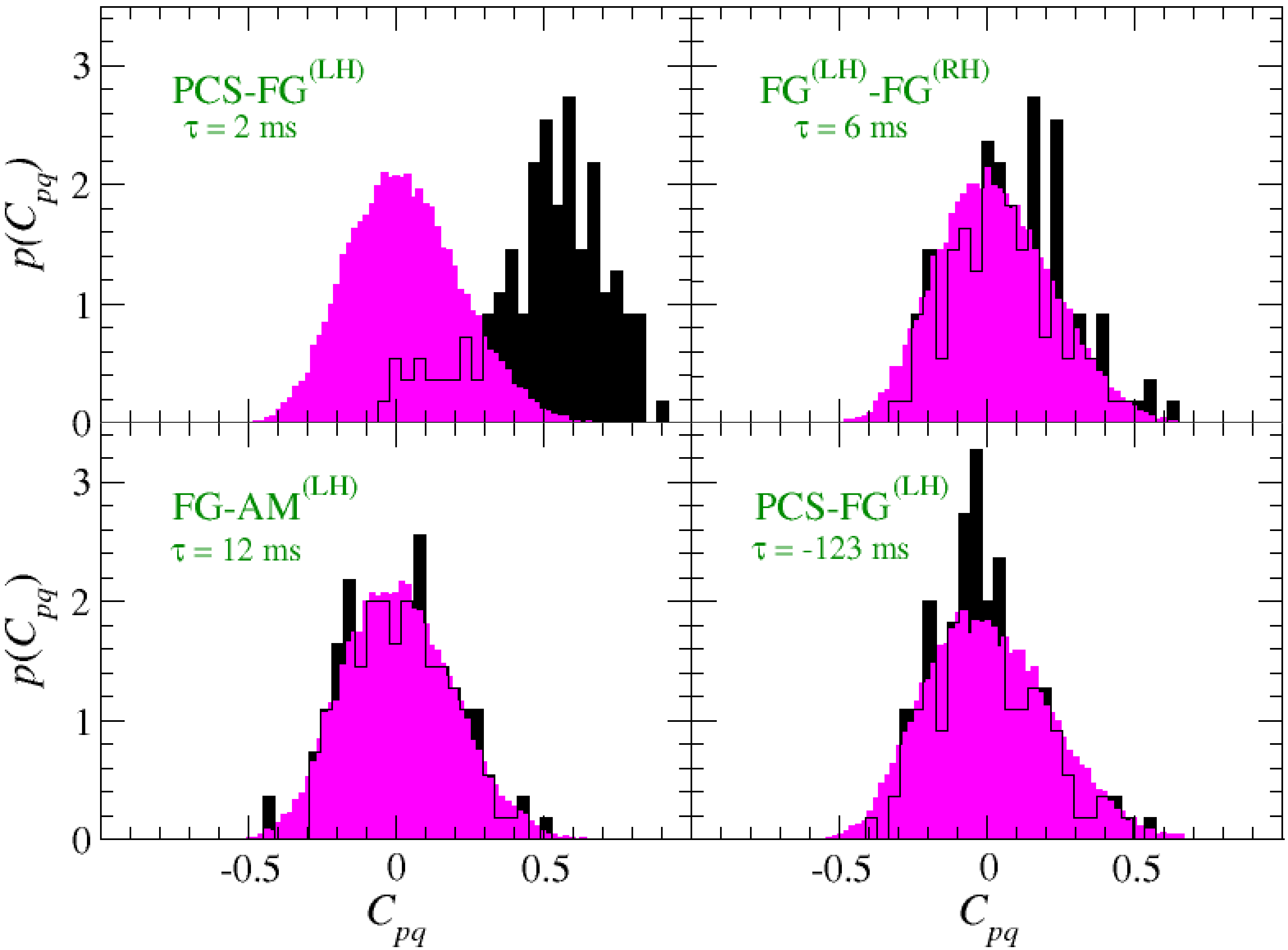
FG-AM
LH
RH
PCS$^{(LH)}$-PCS$^{(RH)}$
FG$^{(LH)}$-FG$^{(RH)}$
PCS
FG
Re $\lambda_1(t)$
$\tau$ [ms]
15
10
5
0
-5
-200
-150
-100
-50
0
50
100
150
200

Examples of eigenvalue distribution of $C(\tau)$ for selected areas and time lags




$$\rho(\lambda) = 0.01\rho(0)$$

GinOE


PCS-FG$^{(LH)}$
τ = 2 ms
FG$^{(LH)}$-FG$^{(RH)}$
τ = 6 ms
FG-AM$^{(LH)}$
τ = 12 ms
PCS-FG$^{(LH)}$
τ = -123 ms
$p(C_{pq})$
$C_{pq}$
0
1
2
3
-0.5
0
0.5

Another - likely productive - application of asymmetric matrices

→ **Financial markets**

First attempts:

J. Kwapień, S. Drożdż, A.Z. Górski, P. Oświęcimka,
*Asymmetric matrices in an analysis of financial correlations*,
Acta Phys. Pol. B **37**, 3039-3048 (2006)

C. Biely, S. Thurner,
*Random matrix ensembles of time-lagged correlation matrices: derivation of eigenvalue spectra and analysis of financial time-series*,
Quant. Finance **8**, 705-722 (2008)

## Two stock markets correlated

Dow Jones Industrial Average DJIA

Deutscher Aktienindex DAX

30 stocks each

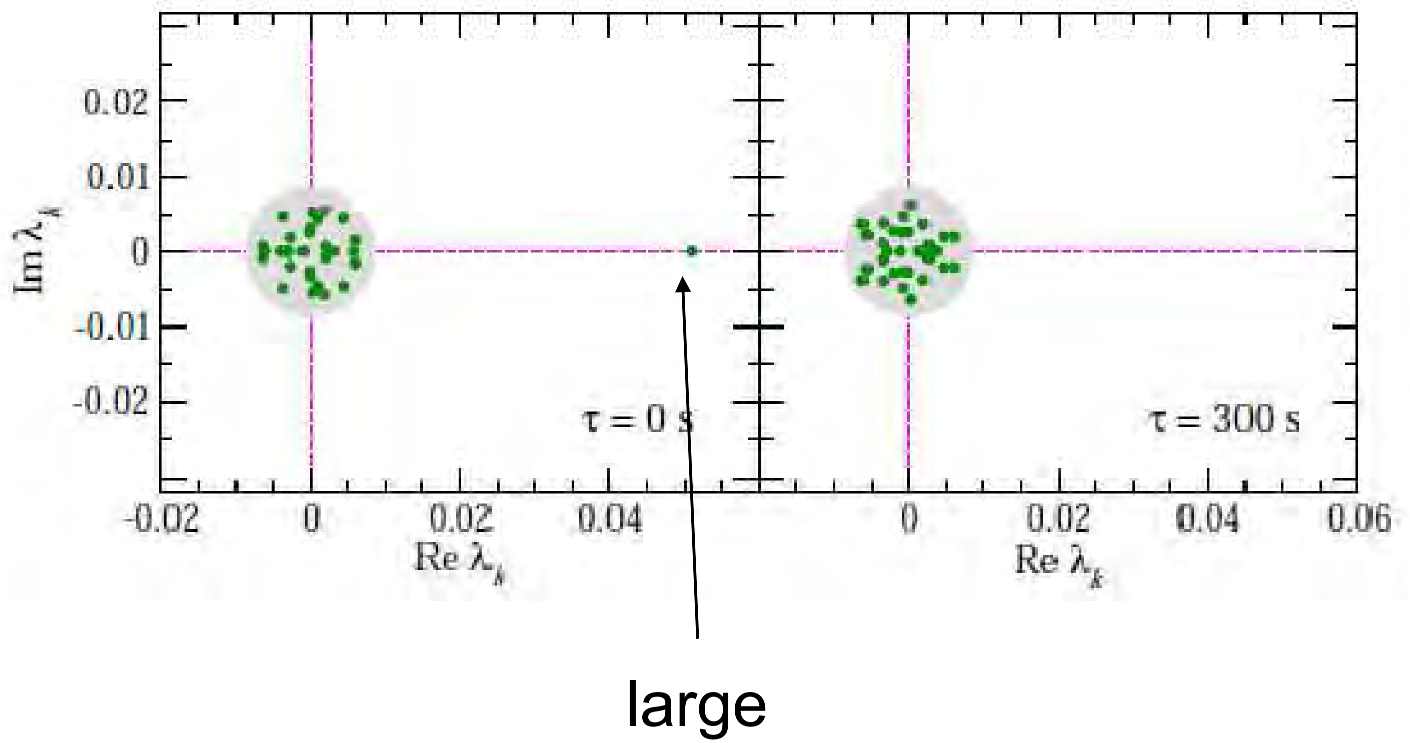
trading overlap:
- 9:30 – 11:30 in New York
- 15:30 – 17:30 in Frankfurt

High frequency (1s) recordings studied

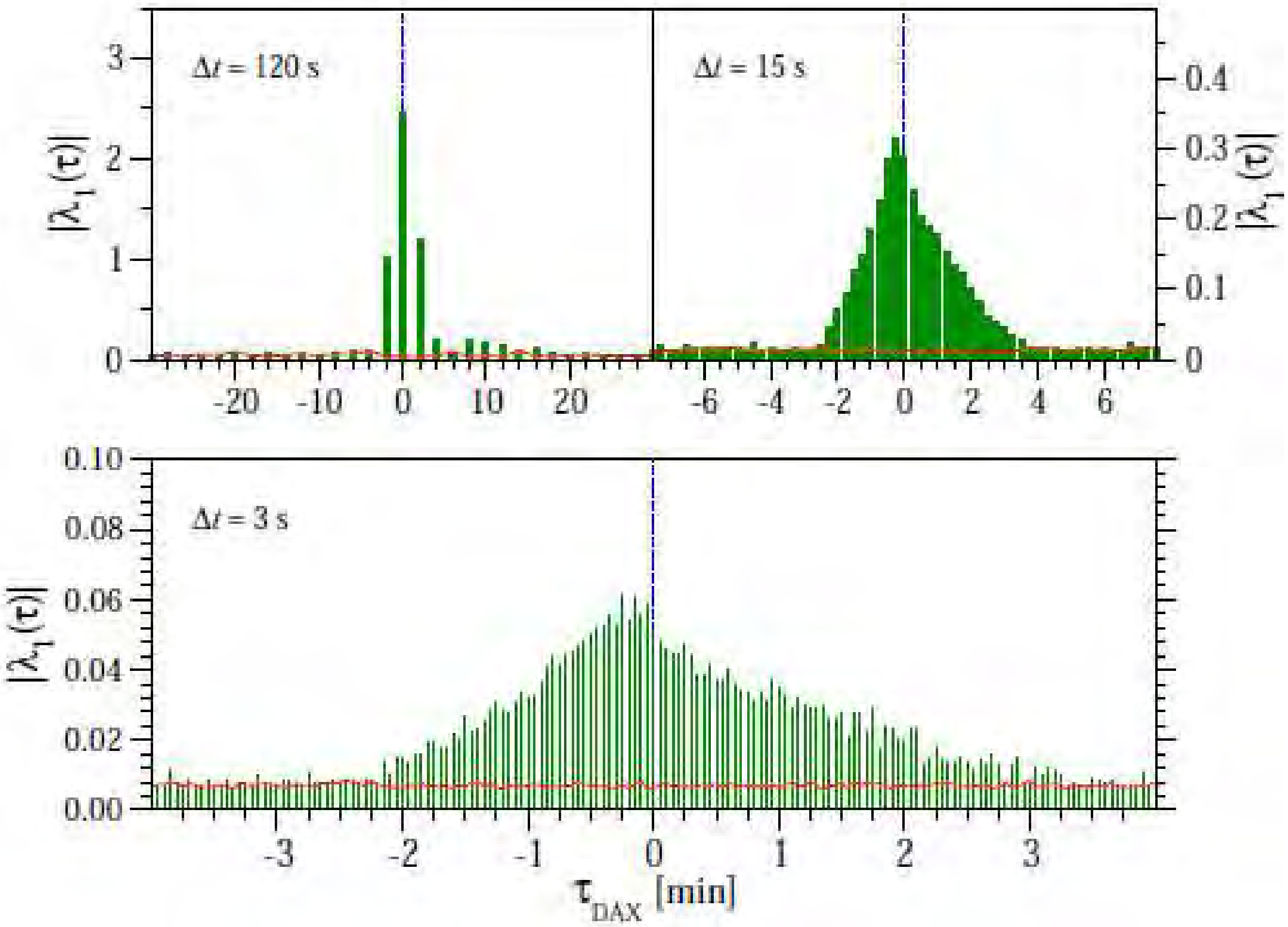
# Spectra of the correlation matrix

$$C(\tau) = X^{DJ}[Y^{DAX}(\tau)]^T$$

here for $\Delta t = 3s$ returns



# The largest eigenvalue as function of time-lag




Statistically relevant correlations exist within the range

$$-2\,\text{min} \leq \tau \leq\sim +3\,\text{min}$$

# Asymmetric (random) matrices

We need them because they constitute

- an unqestionable intellectual challenge

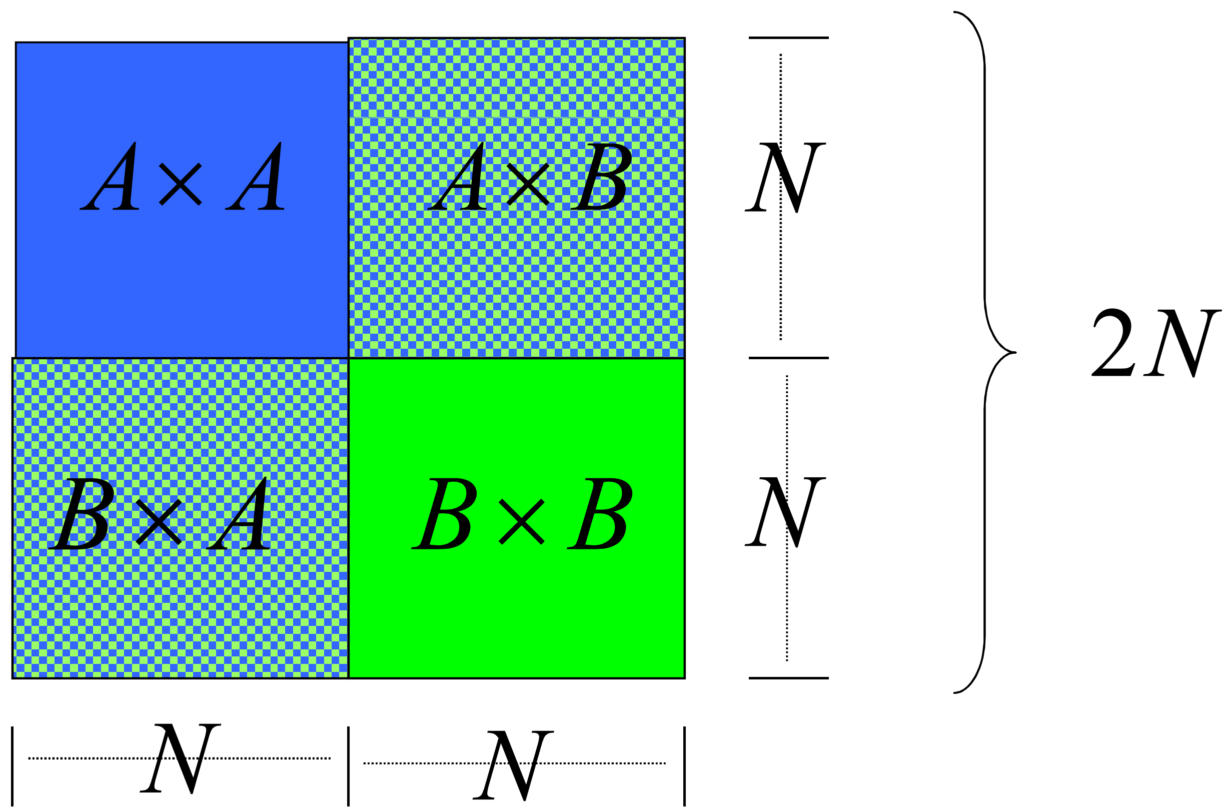
- attractive and efficient means to quantify various subtle characteristics of long-range (both in space and in time) correlations of extreme complexity

Appropriate variants of Ensembles of
Asymmetric Random Matrices,
(especially generalized Wishart)
are needed as reference
In order to extract the real information
from omnipresent noise

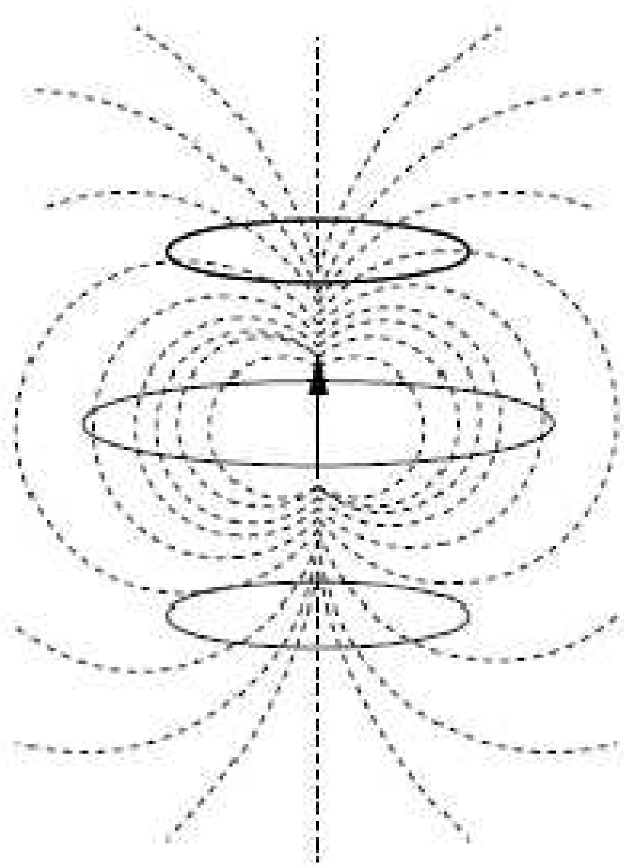
Treat $A$ and $B$ as one system $A+B$

Form a corresponding $2N \times 2N$ (symmetric) matrix

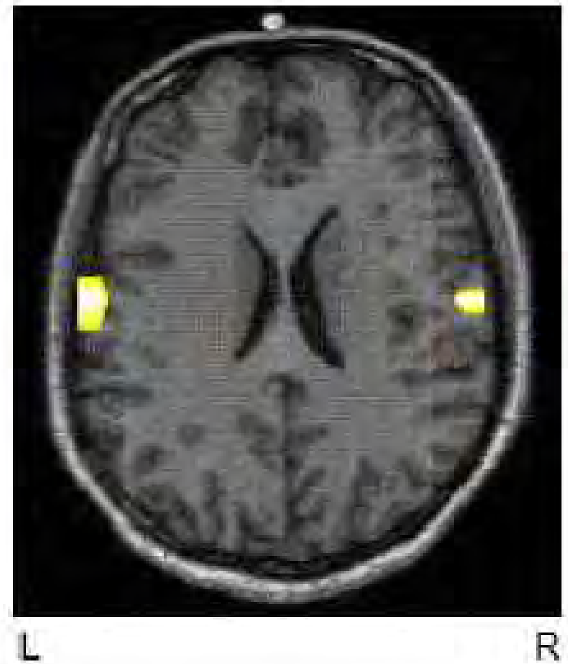
How does $A \times B$ relate to such a matrix?



Magnetic current of a current dipole (solid) and volume currents (dashed)



# Sensitivity profiles of visual sensors

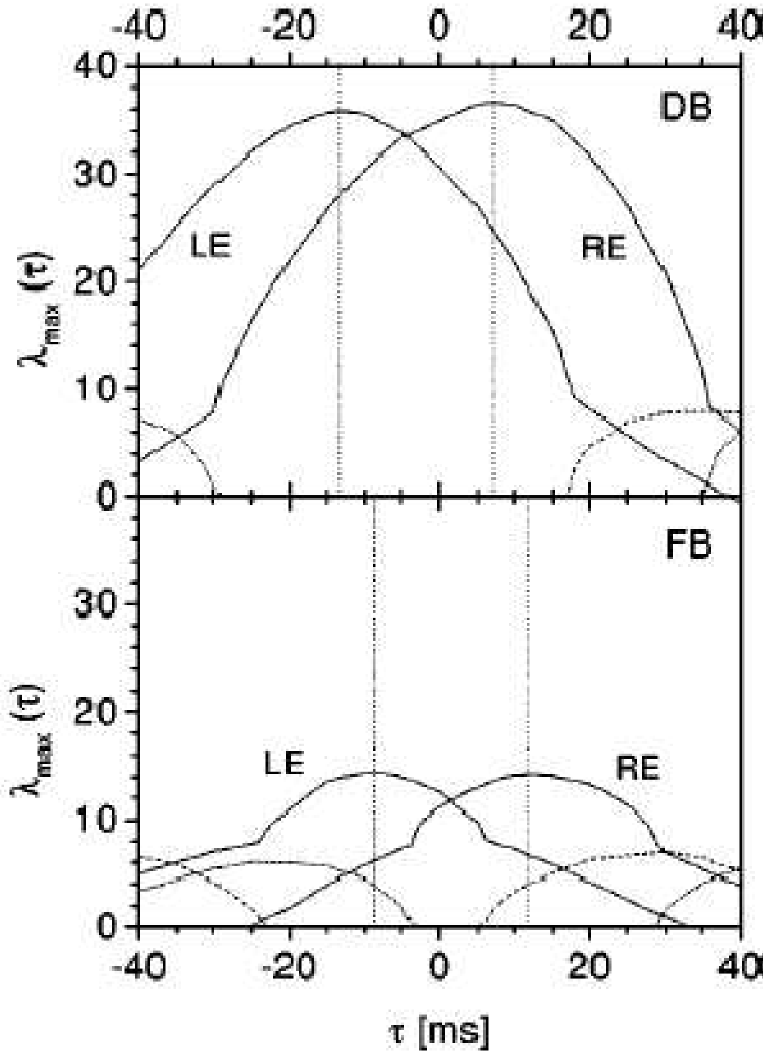



FIG. 7. $\lambda_{max}(\tau)$ calculated from the cross-hemisphere correlation matrix. The upper part corresponds to DB and the lower part to FB. Both panels illustrate two kinds of stimulation: left ear (LE) and right ear (RE). The solid lines denote the real part of $\lambda_{max}$, while the dashed and dotted ones are its imaginary part. The sign of $\tau$ denotes retardation of a signal from the right hemisphere ($\tau > 0$) or the left one ($\tau < 0$).