A new scaling limit for matrix models

J. Ambjørn[1, 2]

[1]Niels Bohr Institute, Copenhagen, Denmark

[2]University of Utrecht, The Netherlands

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Hermitian Matrix Models

Recall the Hermitian matrix model, defined as a formal power series in $\tilde{g}$

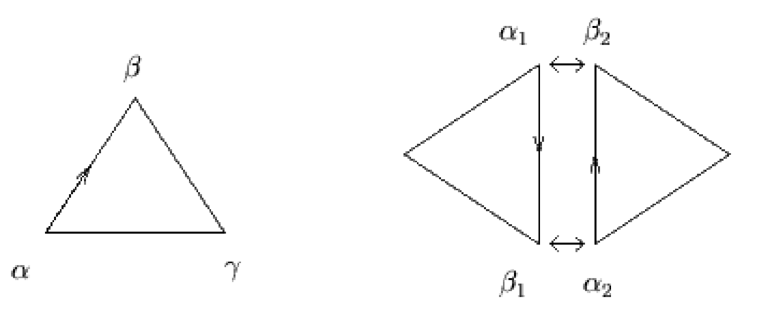
$$
\begin{aligned}
Z(\tilde{g}) &= \int d\phi \, \mathrm{e}^{-N\mathrm{tr}\left(\frac{1}{2}\phi^2 - \frac{\tilde{g}}{3}\phi^3\right)} \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} \int d\phi \, \mathrm{e}^{-\frac{1}{2}N\mathrm{tr}\left(\phi^2\right)} \left(\frac{N\tilde{g}}{3}\mathrm{tr}\,\phi^3\right)^k,
\end{aligned}
$$

$$
d\phi = \prod_{\alpha \le \beta} d\,\mathrm{Re}\,\phi_{\alpha\beta} \prod_{\alpha < \beta} d\,\mathrm{Im}\,\phi_{\alpha\beta}.
$$

The integral can be evaluated in the standard way by doing all possible Wick contractions of $(\mathrm{tr}\,\phi^3)^k$ and using

$$\langle \phi_{\alpha\beta}\phi_{\alpha'\beta'} \rangle = C \int d\phi \; e^{-\frac{1}{2}\sum_{\alpha\beta}|\phi_{\alpha\beta}|^2} \phi_{\alpha\beta}\phi_{\alpha'\beta'} = \delta_{\alpha\beta'}\delta_{\beta\alpha'},$$

Geometric interpretation:



$$d\phi \; e^{-\frac{N}{g_s}\mathrm{tr}\, V(\phi)} \propto dU(N) \prod_{i=1}^{N} d\ell_i \; e^{-\frac{N}{g_s} V(\ell_i)} \prod_{i<j} |\ell_i - \ell_j|^2$$

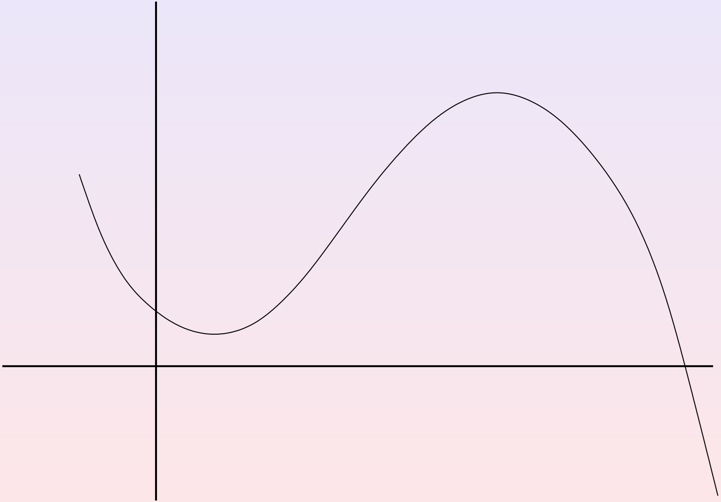
The "classical" limit is obtained for $g_s \to 0$, where all eigenvalues are lumped together at ℓ_0, where

$$V'(\ell_0) = 0.$$

However, for $g_s > 0$ the integration over the non-diagonal matrix elements produces the Vandermonde determinant, which acts as a "quantum" correction, a repulsion between different eigenvalues. Result: eigenvalues are smeared out over an interval around ℓ_0, even in the large N limit.

Example:

$$\frac{1}{g_s} V(\phi) = \frac{1}{g_s}\Big(-g\phi + \frac{1}{2}\phi^2 - \frac{g}{3}\phi^3 \Big)$$



Large N saddelpoint equation:

$$w(z) := \left\langle \frac{1}{N} \mathrm{tr} \frac{1}{z-\phi} \right\rangle = \frac{1}{Z} \int \mathrm{d}\phi \, \frac{1}{N} \frac{1}{z-\phi} \, \mathrm{e}^{-\frac{N}{g_s} \mathrm{tr}\, V(\phi)}$$

For $V(\phi) = -g\phi + \frac{1}{2}\phi - \frac{g}{3}\phi^3$ one has

$$w(z) = \frac{1}{2g_s} \Big(V'(z) + g(z-b)\sqrt{(z-c)(z-d)} \Big)$$

The constants b, c and d are determined by the requirement that $w(z) \to 1/z$ for $z \to \infty$.

In the large N expansion one has to any order the same square root structure.

The conventional scaling limit

The usual scaling limit of the matrix model is obtained (for fixed g_s) by adjusting g such that $b(g) = c(g)$. At this point the analytic structure of $w(z)$ changes from $(z - c(g))^{1/2} \to (z - c(g))^{3/2}$, and this change can only be accommodated by invoking arbitrary high k in the sum

$$\sum_{k=0}^{\infty} \frac{1}{k!} \int d\phi \; e^{-\frac{N}{2g_s} \mathrm{tr}\,(\phi^2)} \left(\frac{Ng}{3g_s} \mathrm{tr}\, \phi^3 \right)^k ,$$

This is why one geometrically can imagine a "continuum" limit where the size of each triangle shrinks to zero while the continuum size of the surface stays constant.

$$g = g_c(1 - \Lambda a^2), \qquad z = c(g_c) + aZ, \qquad a \to 0$$

$$w(z) = \frac{1}{2}\Big(V'(z) + g\sqrt{c(g_c) - d(g_c)}\; a^{3/2} W_E(Z,\Lambda)\Big)$$

$$W_E(Z,\Lambda) = (Z - \sqrt{2\Lambda/3})\sqrt{Z + 2\sqrt{2\Lambda/3}}$$

a has the interpretation as the length of the side of the triangles (polygons) which appear in $V(\phi)$.

Notice the non-scaling part $V'(z)/2$ which dominates when $a \to 0$ and renders the average number of polygons present in the ensemble with partition function $w(z)$ finite, even at the critical point. This somewhat embarrassing fact can be circumvented by differentiating $w(z)$ a sufficient number of times with respect to g and z, after which these "non-universal" contributions vanish.

The new scaling limit

Two limits:
"classical limit": $g_s = 0$
and
conventional scaling limit: $g_s > 0$ and $g \to g_c(g_s)$.

Is it possible to find a new, non-trivial scaling limit, closer to the classical limit when $g_s \to 0$?

Close to criticallity $b(g_c) = c(g_c)$:

$$g_c(g_s) = \frac{1}{2}(1 - \frac{3}{2}g_s^{2/3} + O(g_s^{4/3})),$$

$$z_c(g_s) = c(g_c, g_s) = 1 + g_s^{1/3} + O(g_s^{2/3}),$$

$$c(g_c) - d(g_c) = 4g_s^{1/3} + 0(g_s^{2/3})$$

A non-trivial scaling is obtained for $g_s = G_s a^3$.

Again the scaling parameter a can be given the geometric interpretation as the link lengths of the polygons in $V(\phi)$. Note that the length of the cut goes to zero as $a \to 0$, thus we are closer to the “classical” limit. However, it will survive in the continuum limit:

$$g = g_c(g_s)(1 - a^2\Lambda) = \bar{g}(1 - a^2\Lambda_{\rm cdt} + O(a^4))$$

$$z = z_c + aZ = \bar{z} + aZ_{\rm cdt} + O(a^2)$$

$$\Lambda_{\rm cdt} \equiv \Lambda + \frac{3}{2}G_s^{2/3}, \quad \bar{g} = \frac{1}{2}, \quad Z_{\rm cdt} \equiv Z + G_s^{1/3}, \quad \bar{z} = 1.$$

Using these definitions one computes in the limit $a \to 0$ that

$$w(z) = \frac{1}{a}\,\frac{\Lambda_{\rm cdt} - \frac{1}{2}Z_{\rm cdt}^2 + \frac{1}{2}(Z_{\rm cdt} - H)\sqrt{(Z_{\rm cdt} + H)^2 - \frac{4G_s}{H}}}{2G_s}.$$

$$h^3 - h + \frac{2G_s}{(2\Lambda_{\rm cdt})^{3/2}} = 0, \quad h = H/\sqrt{2\Lambda_{\rm cdt}}$$

$$w(z) = \frac{1}{a}W_{\rm cdt}(Z_{\rm cdt}, \Lambda_{\rm cdt}, G_s)$$

$$G_s \to 0: \qquad W_{\rm cdt}(Z_{\rm cdt}, \Lambda_{\rm cdt}, G_s) \to \frac{1}{Z_{\rm cdt} + \sqrt{2\Lambda_{\rm cdt}}}$$

$$G_s \to \infty: \qquad \frac{(Z_{\rm cdt} - H)\sqrt{(Z_{\rm cdt} + H)^2 - \frac{4G_s}{H}}}{2G_s} \to$$

$$G_s^{-5/6} \left(Z - \sqrt{2\Lambda/3} \right) \sqrt{Z + 2\sqrt{2\Lambda/3}}$$

In the first case the cut disappear and one can say that this result is "classical" in a way that will be made precise shortly.

In the second case one recover the standard scaling $W_E(Z, \Lambda)$. However, the part not related to the square root will not scale, in accordance with the previous discussion of standard scaling.

The matrix model

The new scaling can be obtained by a change of variables:

$$\phi \to \bar{z}\,\hat{I} + a\Phi + O(a^2)$$

Up to a ϕ independent term we then have:

$$V(\phi) = \bar{V}(\Phi), \qquad \boxed{\bar{V}(\Phi) \equiv \frac{\Lambda_{\mathrm{cdt}}\Phi - \frac{1}{6}\Phi^3}{2G_s}}$$

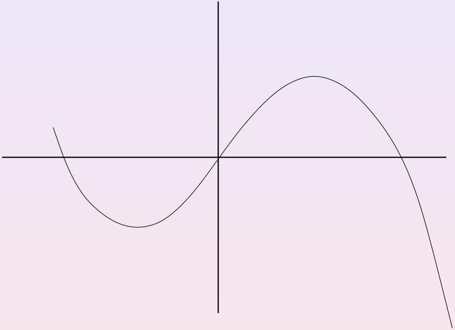
$$Z(g, g_s) = a^{N^2} Z(\Lambda_{\mathrm{cdt}}, G_s), \quad Z(\Lambda_{\mathrm{cdt}}, G_s) = \int d\Phi\, \mathrm{e}^{-N \mathrm{tr}\, \bar{V}(\Phi)}$$

$$\frac{1}{z-\phi} = \frac{1}{a}\,\frac{1}{Z_{\mathrm{cdt}} - \Phi} \quad \Rightarrow \quad w(z) = \frac{1}{a} W_{\mathrm{cdt}}(Z_{\mathrm{cdt}}, \Lambda_{\mathrm{cdt}}, G_s)$$

Relation correct to all order in N. The new scaling limit is itself a matrix model defined by $\bar{V}(\Phi)$..

$$\bar{V}(\Phi) \propto 2\Lambda_{\rm cdt}\Phi - \frac{1}{3}\Phi^3,$$

$$\bar{V}'(\ell_0) = 0 \quad \Rightarrow \quad \ell_0 = -\sqrt{2\Lambda_{\rm cdt}}.$$



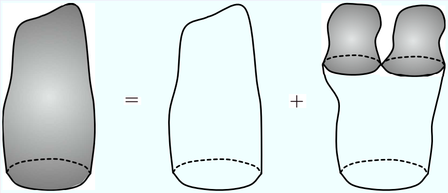
Thus the “classical” limit of the matrix integral with potential $\bar{V}(\Phi)$, when only the minimum plays a role, leads to the following expectation value:

$$\frac{1}{N}\left\langle \mathrm{tr}\, \frac{1}{Z_{\rm cdt} - \Phi} \right\rangle = \frac{1}{Z_{\rm cdt} + \sqrt{2\Lambda_{\rm cdt}}} = \lim_{G_s \to 0} W_{\rm cdt}(Z_{\rm cdt}, \Lambda_{\rm cdt}, G_s).$$

Geometric interpretation

Objects: $W_{\lambda,g_s}(\ell)$ and $W_{\lambda,g_s}(x)$ where

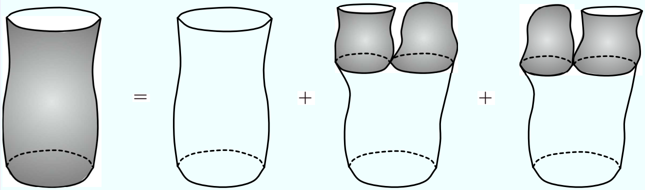
$$W_{\lambda,g_s}(x) = \int_0^\infty \mathrm{d}\ell\, \mathrm{e}^{-x\ell}\, W_{\lambda,g_s}(\ell)$$



Shaded parts represent the full disc amplitude, unshaded parts the CDT disc amplitude and the CDT propagator. g_s associeted with branching.

$$W_{\lambda,g_s}(x) = W_\lambda^{(0)}(x) + g_s \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}\ell_1 \mathrm{d}\ell_2\, (\ell_1+\ell_2) G_\lambda^{(0)}(x,\ell_1+\ell_2;t) W_{\lambda,g_s}(\ell_1) W_{\lambda,g_s}(\ell_2)$$

"Propagation" of a boundary as a function of the geodesic distance from the boundary:



Shaded parts of graphs represent the full, g_s-dependent propagator and disc amplitude, and non-shaded parts the CDT propagator where $g_s = 0$. In all four graphs, the geodesic distance from the final to the initial loop is given by t.

Objects: $G_{\lambda,g_s}(\ell_1,\ell_2;t)$ and $G_{\lambda,g_s}(x,y;t)$ where:

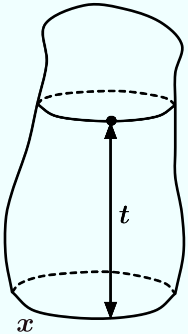
$$G_{\lambda,g_s}(x,y;t) = \int_0^\infty \mathrm{d}\ell_1 \int_0^\infty \mathrm{d}\ell_2\; \mathrm{e}^{-x\ell_1-y\ell_2}\, G_{\lambda,g_s}(\ell_1,\ell_2;t).$$

Differentiating the integral equation, corresponding to the figure with respect to t leads to

$$a^\varepsilon \frac{\partial}{\partial t} G_{\lambda,g_s}(x,y;t) = -\frac{\partial}{\partial x}\Big[\Big(a(x^2-\lambda)+2g_s\, a^{\eta-1} W_{\lambda,g_s}(x)\Big) G_{\lambda,g_s}(x,y;t)\Big]$$

The following geometric picture couples W and G and leads to consistency relations for the scaling of W:

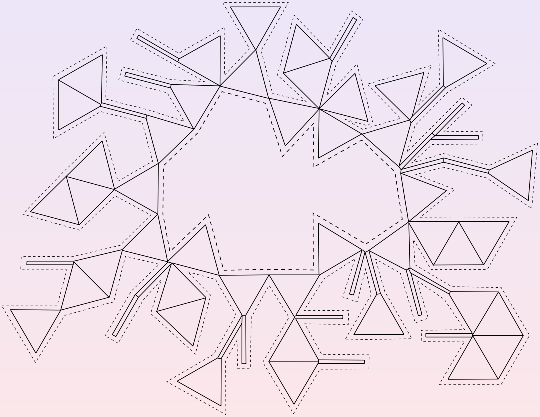
$$-\frac{\partial W_{\lambda,g_s}(x)}{\partial \lambda} = \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}\ell \, G_{\lambda,g_s}(x,\ell;t)\, \ell W_{\lambda,g_s}(\ell).$$



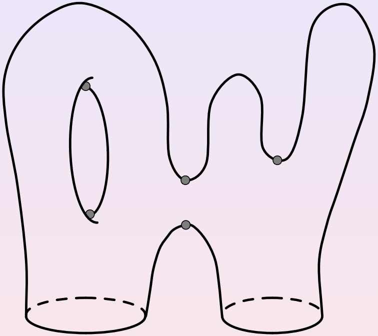
$$W_{reg} \xrightarrow[a\to 0]{} a^{\eta} W_{\lambda}(x), \qquad \eta < 0,$$
$$t_{reg} \xrightarrow[a\to 0]{} t/a^{\varepsilon}, \qquad \varepsilon = 1.$$

$$W_{reg} \xrightarrow[a\to 0]{} \text{const.} + a^{\eta} W_{\lambda}(x), \qquad \eta = 3/2$$
$$t_{reg} \xrightarrow[a\to 0]{} t/a^{\varepsilon}, \qquad \varepsilon = 1/2,$$

The branching of baby-universes in the ordinary scaling limit is the reason that "proper time" scales as $a^{1/2}$ and not as a. We have an infinite number of baby universes in the continuum limit.



In the new scaling limit the number of baby univeses is finite for a universe with finite volume (i.e. area in 2d).



Unfinished stuff

For the ordinary matrix models we have a description of conformal matter coupled to 2d quantum gravity by multicritical one-matrix models and by two-matrix models.

It would be interesting to define the new scaling limit in these cases provided it is simple. If the exponents are the same as in flat space, it would the provide a simple realization of these critical systems.