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Universality in Non-Hermitian Random Matrix Theory

Gernot Akemann

28/09/2010 @ 23rd Marian Smoluchowski Symposium, Kraków

+ **M. Bender, M. Kieburg, M. J. Phillips, and H.-J. Sommers**

Literature

Review on Non-Hermitian RMT: Khoruzhenko, Sommers [arXiv:0911.5645] in

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The Oxford Handbook of Random Matrix Theory

Edited by Gernot Akemann, Jinho Baik and Philippe Di Francesco

(c) Oxford University Press, 2011
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Outline of Talk

- **Motivation: applications**
- **Why do we expect universality?**
- **Which RMTs:** Ginibre-Girko & non-Hermitian chiral/Wishart/Laguerre
- **Results for general weights at finite-N:**
  (skew) orthogonal polynomials on $\mathbb{C}$
- **Results for Gauß at finite-N:**
  Hermite, Laguerre, and kernel-relation $\beta = 2 \leftrightarrow 1, 4$ on $\mathbb{C}$
- **Large-N limit & examples for universality:**
  Sine-, Bessel- and Airy-kernel on $\mathbb{C}$
- **Conclusions & open questions**

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$$\boxed{\mathcal{D}_{Dirac} = \begin{pmatrix} 0 & iP + \mu Q \\ iP^\dagger + \mu Q^\dagger & 0 \end{pmatrix}}$$ complex for $\mu \neq 0$

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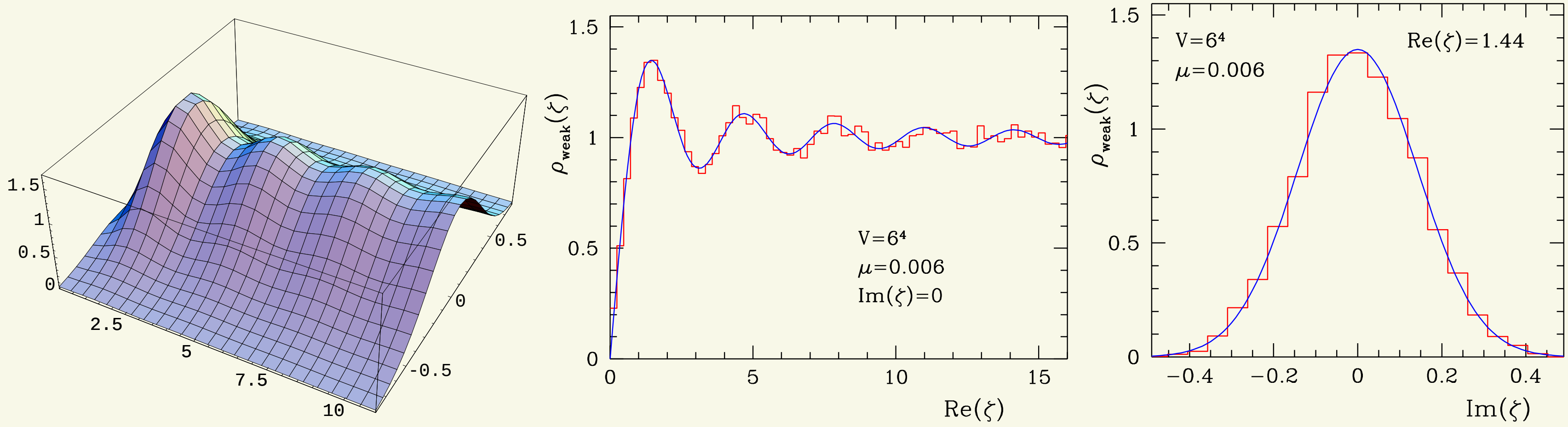
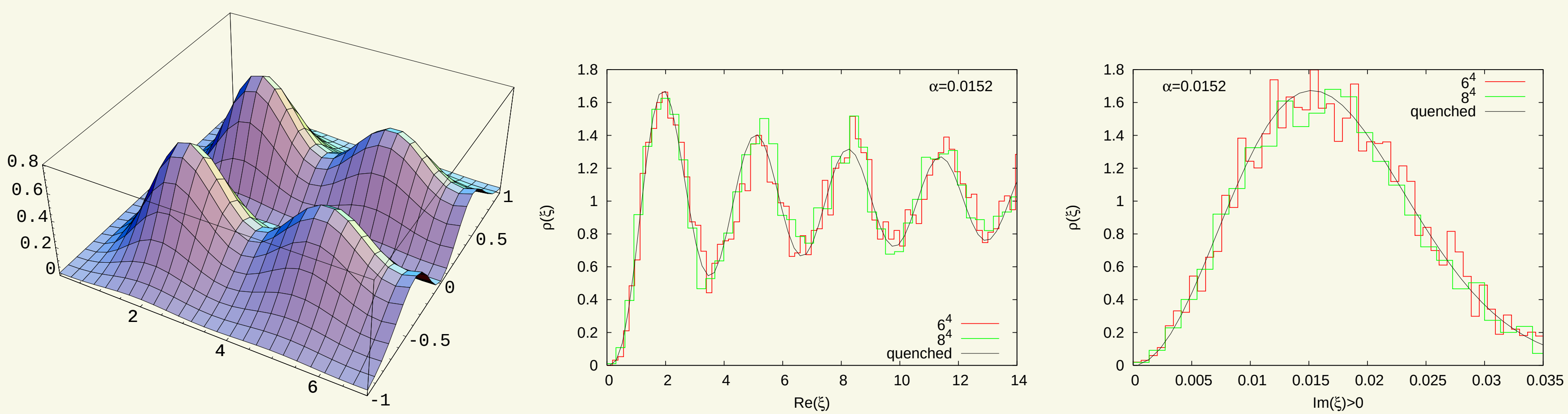
- III & IV deal with **rectangular matrices**

$\longrightarrow$ non-Hermitian chiral/Wishart/Laguerre

Motivation II - RMT vs QCD



- $\beta = 2$ Bessel - law on $\mathbb{C}$ vs. Lattice **QCD** [Wettig, GA '04]



- $\beta = 4$ Bessel - law on $\mathbb{C}$ vs. Lattice **QCD** [Bittner, GA '06]

Why should we expect universality?

"Universality = non-Gaussian RMT gives same as Gaussian"

- universality true on $\mathbb{R}$ & **weak non-Hermiticity is deformation**:

Sine-kernel $\boxed{K^{\mathbb{C}}_{Sine}(z,u^*) \sim \int_0^1 dt\, \mathrm{e}^{-\sigma^2 t^2} \cos(t(z-u^*))} \stackrel{\sigma\to 0}{\longrightarrow}$

$$K^{\mathbb{R}}_{Sine}(x,y) = \int_0^1 dt \cos(t(x-y)) = \frac{\sin(x-y)}{x-y} = \frac{\sin(x)\cos(y)-\cos(x)\sin(y)}{x-y}$$

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  - which polygons not important

  - difference to Hermitian 1-RMT: 2 colors = **checkered surfaces**

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- **same result** as Gaussian RMT **from field theory**:

  e.g. QCD- spectral density from chiral perturbation theory

Which Non-Hermitian RMT: A) elliptic Ginibre-Girko ensembles

$$\boxed{\mathcal{P}(J) \sim \exp\left[-\frac{N}{1-\tau}\mathrm{Tr}(JJ^\dagger + \frac{\tau}{2}(J^2 + J^{\dagger\,2}))\right]}$$ Ginibre-Girko

- $J_{ij}^\dagger \neq J_{ij} \in \mathbb{R}, \mathbb{C}, \mathbb{H}$ $N \times N$ matrix for $\beta = 1, 2, 4$
- we consider complex eigenvalues of matrix J
- **deformation parameter** $\tau \in [0, 1)$:

  - limit $\tau \to 1$ gives back GOE, GUE and GSE

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  $\mathcal{P}(H, A) \sim \exp\left[-\frac{1}{1+\tau}\text{Tr}H^2 - \frac{1}{1-\tau}\text{Tr}A^2\right]$ **= two-matrix model** (2MM)

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- **other weights**: - non-invariant: J_{ij} i.i.d. & moment conditions

  - invariant: Gauss weight $\to \exp[-N\mathrm{Tr}V(J;\tau)]$,

  e.g. V harmonic/Elbau-Felder ($\Rightarrow$ J normal not needed!)

B) non-Hermitian chiral/Wishart/Laguere ensembles

$$\mathcal{P}(\phi,\psi) \sim \exp\left[-\frac{N}{1-\tau}\mathrm{Tr}(\phi\phi^\dagger + \psi^\dagger\psi + \tau(\phi\psi + \psi^\dagger\phi^\dagger)\right]$$

[Osborn ’04; GA ’05; +Phillips, Sommers ’09]

- non-Hermitian $\phi_{ij}, \psi_{ij} \in \mathbb{R}, \mathbb{C}, \mathbb{H}$ $N \times (N+\nu)$ matrix for $\beta = 1, 2, 4$
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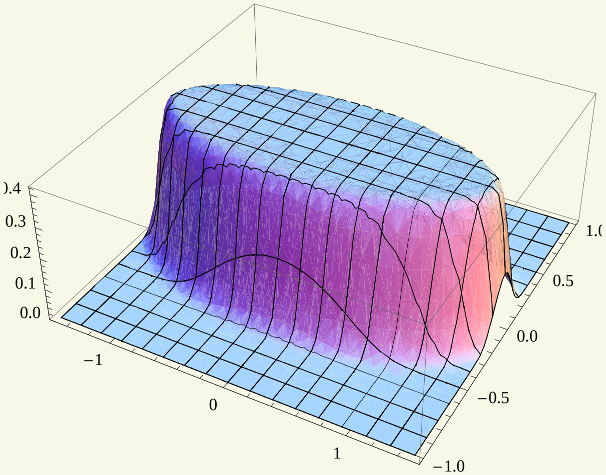
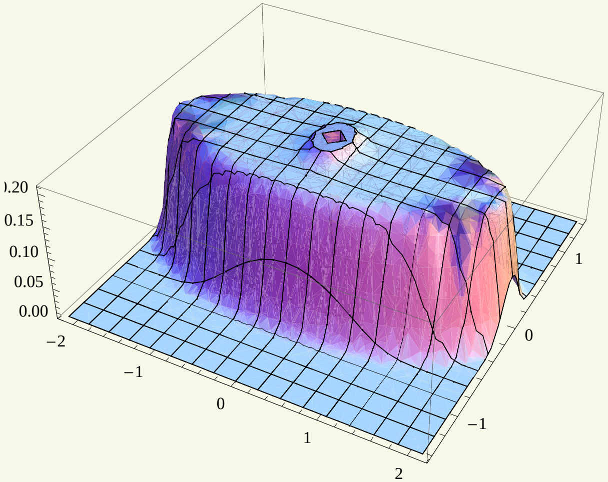
- **other weights**: add higher powers $(\phi\psi)^k$ + h.c.: "harmonic"
- for **QCD**: $\mathcal{P} \times \det^{N_F}\begin{pmatrix} 0 & iP + \mu(\tau)Q \\ iP^\dagger + \mu(\tau)Q^\dagger & 0 \end{pmatrix}$

Ensemble characteristics for A) & B) and $\beta = 1, 2, 4$

- $\beta = 2$: **A) Ginibre-Girko** & **B) chiral/W/L** squared eigenvalue, $\nu = O(1)$





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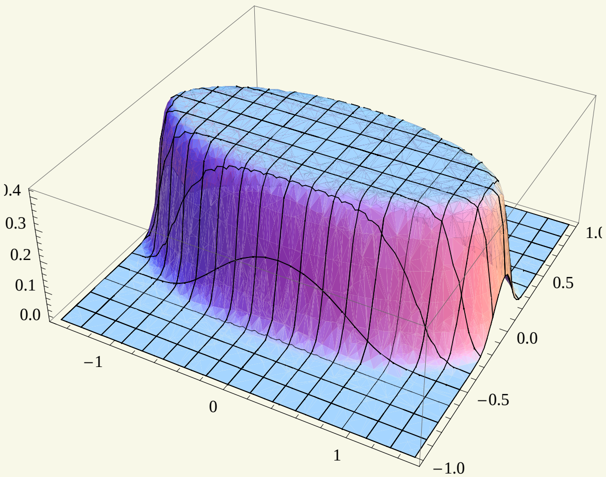
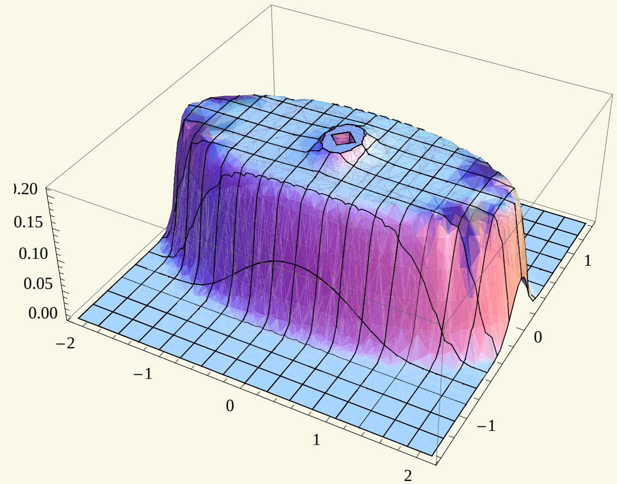
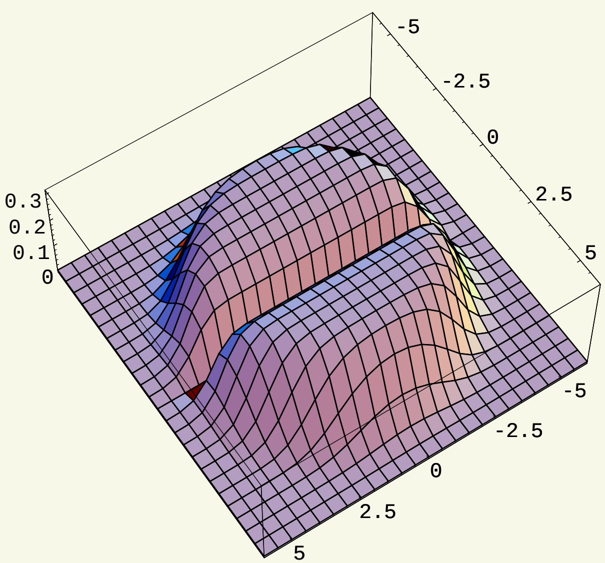
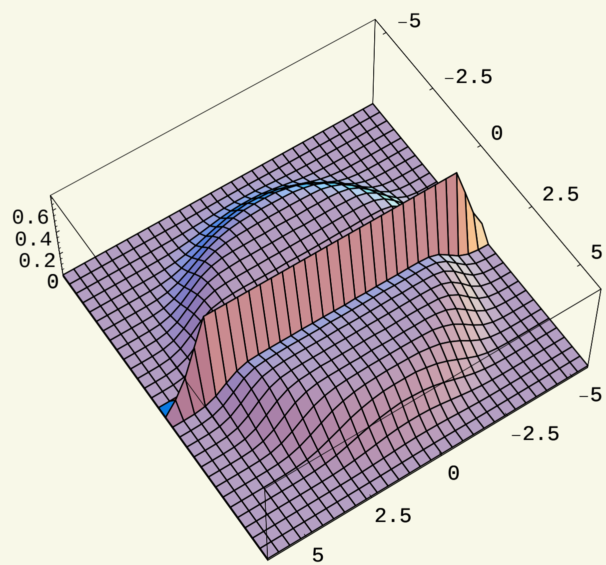




- **A)** $\beta = 4$ and $\beta = 1$ have **extra repulsion/accumulation**





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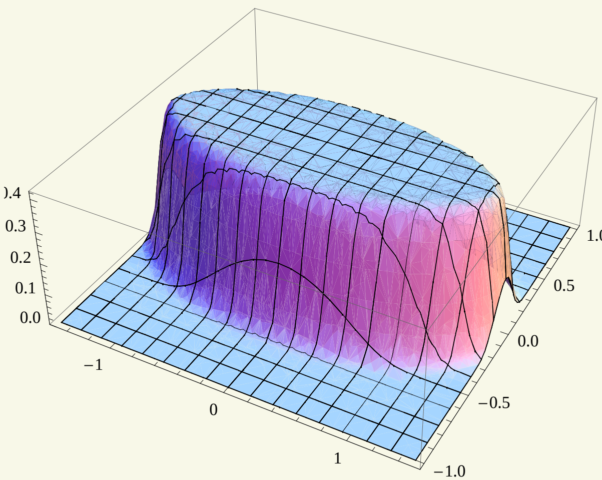

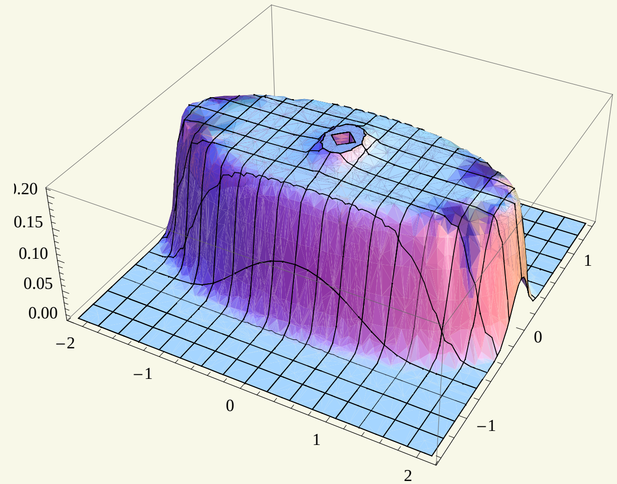

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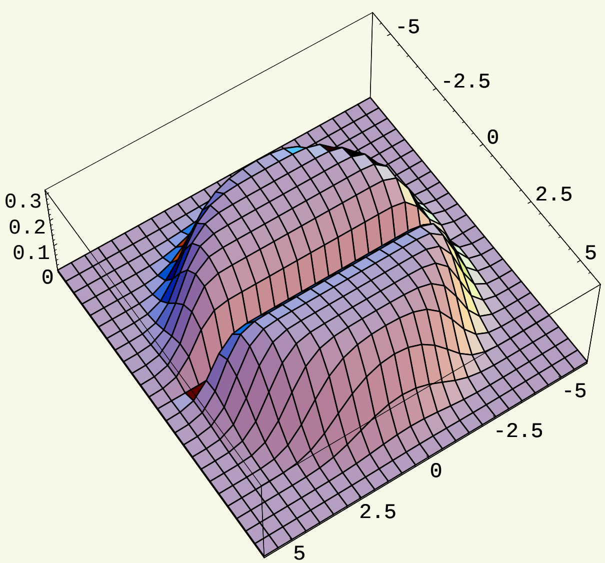

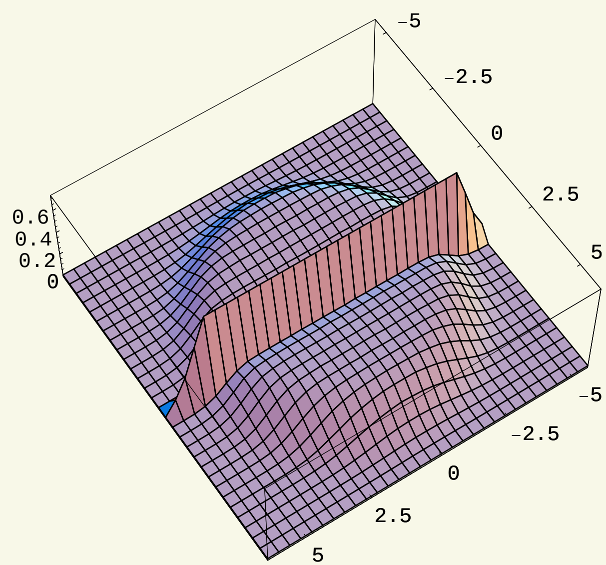

- **chiral/W/L**: + **also** repulsion/attraction **from imaginary axes**

Results for general weights: structure

- **Determinental and Pfaffian point processes:** A) and B)

$$\mathcal{Z} = \prod_{i=1}^{N} \int_{\mathbb{C}} d^2 z_i P_{jpdf}$$

$$R_k(z_1, \ldots, z_k) \equiv \frac{N!}{\mathcal{Z}(N-k)!} \prod_{i=k+1}^{N} \int_{\mathbb{C}} d^2 z_i P_{jpdf}$$ k-point correlations

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$$\underline{\beta = 1}: \quad \sim \mathrm{Pf}_{2k\times 2k} \begin{bmatrix} k_N^{(1)} & \int w\, k_N^{(1)} \\ -\int w\, k_N^{(1)} & w - \int w \int w\, k_N^{(1)} \end{bmatrix}$$

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- $\exists$ **many more unsolved non-Hermitian classes: + 27** [Bernard & LeClair'02, Magnea'07] + above 6 + 10 Hermitian [Zirnbauer'96,+ Altland'97]

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- **kernels & (skew) orthogonal polynomials:**

  characteristic polynomials (e.g. from Susy)

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- similar expressions hold for the **Cauchy transforms** of (S)OP
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- **How to do asymptotic** for generic SOP and their norms?

- **all OP and SOP know explicitly** in terms of Hermite H_k and Laguerre polynomials L_k^ν on $\mathbb{C}$ for all $\beta = 1, 2, 4$ [DiFrancesco, Gaudin, Itzykson, Lesage’94; Kanzieper’02; Forrester, Nagao’07; Osborn’04; GA’03+’05, + Kieburg, Phillips’10]

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$\beta = 4$	$(z - z^*)\times$ ditto
$\beta = 1$	**real eigenvalues**: Gauss **complex conjugate pairs**: Gauss $\times$ Erfc$(\lvert z - z^*\rvert)$

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B) $\beta = 2$	**mod. Bessel:**$\lvert z\rvert^\nu K_\nu(a\lvert z\rvert^2)\times$ Gauss
$\beta = 4$	$(z^2 - z^{2\,*})\times$ ditto $(\nu \to 2\nu)$
$\beta = 1$	real: $(\nu \to \nu/2)$ complex: Gauss $\times$ Erfc $\times \int \frac{dt}{t}$ e$^{a(z^2+z^{2\,*})t-1/4t} K_{\frac{\nu}{2}}(a\lvert z\rvert^2 t)$

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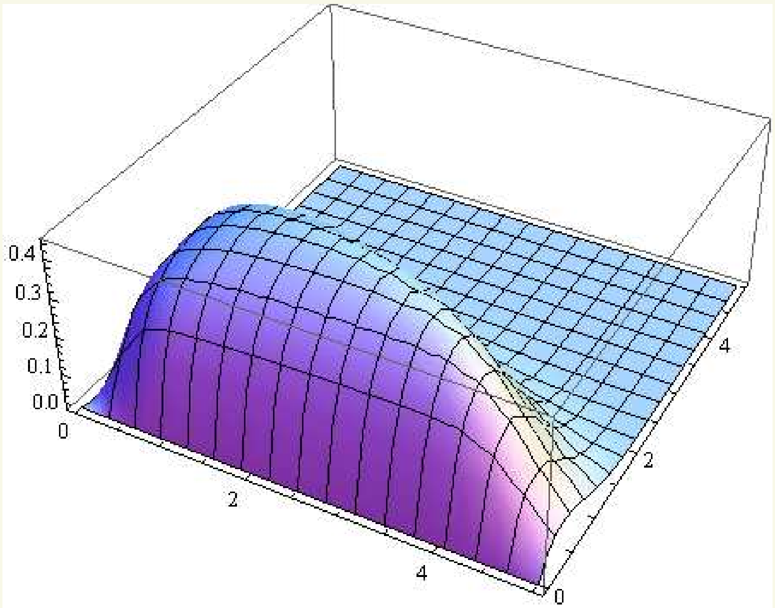
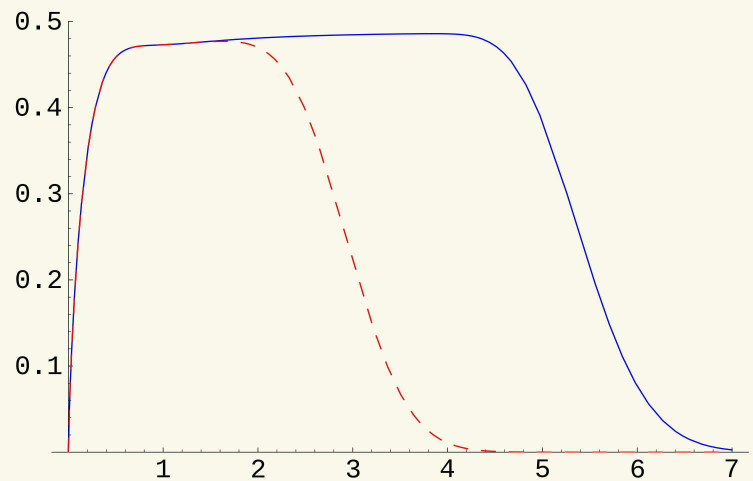
B) $\beta = 2$	**mod. Bessel:**$\lvert z\rvert^\nu K_\nu(a\lvert z\rvert^2)\times$ Gauss
$\beta = 4$	$(z^2 - z^{2\,*})\times$ ditto $(\nu \to 2\nu)$
$\beta = 1$	real: $(\nu \to \nu/2)$ complex: Gauss $\times$ Erfc $\times \int \frac{dt}{t}$ e$^{a(z^2+z^{2\,*})t-1/4t} K_{\frac{\nu}{2}}(a\lvert z\rvert^2 t)$

- proof ($\beta = 2$) Hermite & Laguerre on $\mathbb{C}$: generating functional**(!)** or $\oint$

Example densities for B) $\beta = 1$: 2 real asymetric matrices

- Density of complex, real and imaginary Dirac eigenvalues





[Phillips, Sommers, GA, J. Phys. **A43** (2010) 085211 [arXiv:0911.1276], see also J. Phys. **A42** (2009) 012001 [arXiv:0810.1458]

$$R_1^{\mathbb{R}}(x) = \frac{\eta_-}{8\pi(4\mu^2\eta_+)^{\nu+1}} \int_{-\infty}^{\infty} dx' sgn(x-x')|xx'|^{\nu/2} e^{\eta_-(x+x')} 2K_{\frac{\nu}{2}}(\eta_+|x|) 2K_{\frac{\nu}{2}}(\eta_+|x'|)$$
$$\times \sum_{j=0}^{N-2} \left(\frac{\eta_-}{\eta_+}\right)^{2j} \frac{(j+1)!}{(j+\nu)!} \left\{ L_{j+1}^{\nu}\left(\frac{x'}{4\mu^2\eta_-}\right) L_j^{\nu}\left(\frac{x}{4\mu^2\eta_-}\right) - (x' \leftrightarrow x) \right\}$$

with mapping $R_1^{\mathbb{R}}{}_{Dirac}(x) = 2|x| R_1^{\mathbb{R}}(x^2)$

- the $\beta = 1, 4$ **anti-symmetric kernels** can be expressed through the **symmetric** $\beta = 2$ **kernels** [Phillips, Sommers, GA '10]

A) $\beta = 1$ $\boxed{k_N^{(1)}(x,y) = \Big[\partial_y - \partial_x - 2(y-x)\Big] K_N^{(2)}(x,y)}$ finite-N

B) $\beta = 1$ $\boxed{k_N^{(1)}(x,y) = \Big[y\partial_y - x\partial_x - (y-x)\Big] K_N^{(2)}(x,y)}$

Results for Gauss (matrix) weight: kernel relations

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- both true for general weight functions?

  $\longrightarrow$ **universality** for $\beta = 2$ **implies** $\beta = 1, 4$?

Large-N limits

i) **strong non-Hermiticity**: $\tau \in [0, 1)$ fixed: not considered here

a) bulk limit $\sqrt{N}(z - z_0) = \xi$ fixed

b) outer edge $(1 + x)\mathrm{e}^{i\varphi}$

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ii) **weak non-Hermiticity**: $(1-\tau)N^{\delta}$ fixed when $\tau \to 1$

a) origin $Nz = \xi$ fixed, $\boxed{(1-\tau)N = \sigma^2}$

b) bulk ditto

c) outer edge close to real axis

$\Re\mathrm{e}\,\xi = (x-(1+\tau))N^{2/3}$, $\Im\mathrm{m}\,\xi = yN^{2/3}$,

$\boxed{(1-\tau)N^{1/3} = \sigma^2}$ (bulk already in strong limit)

Weak non-Hermiticity results:

- standard asymptotic of Hermite & Laguerre gives:

A) $\beta = 2$ $\boxed{K^{\mathbb{C}}_{Sine}(z, u^*) \sim \int_0^1 dt\, \mathrm{e}^{-\sigma^2 t^2} \cos(t(z - u^*))}$

[Fyodorov, Khoruzhenko, Sommers '98]

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- **sending $\sigma \to 0$ gives back kernels on $\mathbb{R}$**

What is known about universality?

- universal **Sine kernel** on $\mathbb{C}$ in the bulk: heuristic proof

  - i.i.d. matrix elements [Fyodorov, Khoruzhenko, Sommers '98]

  - harmonic $V(z) = \frac{1}{1-\tau}|z|^2 + \sum_k g_k(z^{2k} + z^{*\,2})$ [GA '03]

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- **How to get more & rigorous results for non-Gaussian weights?**

An alternative non-Gaussian deformation

- **trace squared ensembles:** well studied on $\mathbb{R}$ [Khorchemsky'92, David'97]

$$\boxed{\mathcal{P}(H) \sim \exp\left[- \mathrm{Tr}V(H) + (g\mathrm{Tr}V(H))^2 \right]}$$

- below a critical coupling $g < g_c$: behavior unchanged = universal

An alternative non-Gaussian deformation

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- **exactly solvable quartic non-Hermitian RMT**

"Hubbard-Stratonovic"

$$\boxed{\exp[-\mathrm{Tr}V(H) + (g\mathrm{Tr}V(H))^2] = \int dx\, \mathrm{e}^{-x^2} \exp[-(1+2gx)\mathrm{Tr}V(H)]}$$

and choose $V(J) = JJ^\dagger + \tau/2(J^2 + J^{\dagger\,2})$ Gauss:

$\rightarrow$ integrate A) or B) with x-dependent variance

- asymptotic depending on g remains to be done

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- $\exists$ **more general deformations:** $\rightarrow$ new universality classes

  $\mathcal{P}_\gamma(H) = (1 + \mathrm{Tr}V(H))^{-\gamma} \sim \int dx\, w(x)\mathcal{P}_{Gauss}(H; x)$

  has power law tails and deformed correlations

Conclusion & open problems

- $\exists$ **strong arguments for universality of weak-non-Hermiticity**

  = **one parameter deformation of real kernels:**

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- **How to gain rigorous & more general universality results?**

  - amend Riemann-Hilbert methods: $\bar{\partial}$-problem **?**

  [Its, Takhtajan '08; Balogh, Harnad '08]

  - complex analysis tools from strong limit **?**

  [Berman et al. '08]