Linear Stochastic Resonance

P. F. Góra

M. Smoluchowski Institute of Physics
Jagellonian University, Kraków, Poland

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Abstract

A linear transmitter with correlated multiplicative and additive Gaussian white noises may, under certain conditions, display a stochastic resonance. We show how this problem is related to that of a noisy logistic equation.

I. Linear kinetics and stochastic resonance

- Linear kinetics is easy to solve analytically.
  Linear kinetics is a "skeletal" model of a few realistic systems.
  Linear kinetics is a final stage of *many* realistic, dissipative processes.
- Gaussian white noise (GWN) corresponds to equilibrium fluctuations.

  Therefore…
- It will be nice to show that Stochastic Resonance (SR) is present in linear systems driven by GWN.

  However…
- SR is a phenomenon that is essentially nonlinear.

Can there be a way around?

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There have been many attempts to show LSR. These

- required colored noises;
- required a special preparation of the system (or the signal);
- LSR was present only for transient times;
- LSR did not survive averaging over the phase;
- the power spectra were divergent, etc.

A *robust* linear stochastic resonance is still missing.

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II. A noisy logistic equation

The logistic equation

$$\dot{x} = x(a - bx) \tag{1}$$

$a > 0$, is one of the best known population models. Its noisy generalization

$$\dot{x} = (1 + \sigma\xi(t))x(a - bx) \tag{2}$$

has long since been discussed.

Lets admit that both the paramaters a, b fluctuate

$$\dot{x} = (a + p\xi_{\mathrm{m}}(t))x - (b + q\xi_{\mathrm{a}}(t))x^2 \,. \tag{3}$$

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If we make the substitution

$$y = \frac{1}{x} \tag{4}$$

Eq. (3) is formally converted to a *linear* equation

$$\dot{y} = -(a + p\xi_{\mathrm{m}}(t))y + b + q\xi_{\mathrm{a}}(t)\,. \tag{5}$$

But can we do this substitution?

It turns out we can! In [1] Mao *et al.* have shown that, in Ito interpretation, if $p = 0$ (only one noise is present), solutions to (3) remain bounded and positive almost surely. This proof was later generalized in [2] (PFG) to the case of $p \neq 0$.

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The linear transmitter

The lucky thing is, we can solve the linear equation

$$\dot{y} = -(a + p\xi_{\mathrm{m}}(t))y + b + q\xi_{\mathrm{a}}(t) \tag{6}$$

exactly, provided we know what to do with the correlations between the noises.

A parametric coupling between the dynamical variable and the noise indicates a hidden nonlinearity.

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Decomposing the noises

The GWNs ξ_{a}, ξ_{m} may be correlated:

$$\left\langle \xi_{\mathrm{a}}(t)\xi_{\mathrm{a}}(t')\right\rangle = \left\langle \xi_{\mathrm{m}}(t)\xi_{\mathrm{m}}(t')\right\rangle = \delta(t-t')\,, \tag{7a}$$

$$\left\langle \xi_{\mathrm{a}}(t)\xi_{\mathrm{m}}(t')\right\rangle = c\,\delta(t-t')\,, \quad c \in [-1,1]\,. \tag{7b}$$

Following the approach originated in [3], we represent the noises as linear combinations of independent GWNs:

$$\xi_m(t) = \xi(t)\,, \tag{8a}$$

$$\xi_a(t) = c\,\xi(t) + \sqrt{1-c^2}\,\eta(t)\,, \tag{8b}$$

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A couple of integrals

To calculate statistical properties of solutions to Eq. (5), we need to know expectation values of three integrals:
The first is well-known [4] (Kubo)

$$\left\langle \exp\left[\int_0^T \varphi(t')\xi(t')\,dt'\right]\right\rangle = \exp\left[\frac{1}{2}\int_0^T [\varphi(t')]^2\,dt'\right], \quad (9)$$

ξ is a GWN and φ is a regular function.
The other two were calculated in [5] (PFG)

$$\left\langle \xi(t_1)\exp\left[\int_0^T \varphi(t')\xi(t')\,dt'\right]\right\rangle = \varphi(t_1)\exp\left[\frac{1}{2}\int_0^T [\varphi(t')]^2\,dt'\right], \quad (10)$$

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Excursion: A signal coupled multiplicatively [5] (PFG)

$$\dot{y} = -(a + p\xi(t) + A\cos(\Omega t + \varphi))y + qc\,\xi(t) + q\sqrt{1-c^2}\,\eta(t) \tag{12}$$

Analytical results:

$$\langle\langle y(t)\rangle\rangle = \frac{1}{2\pi}\int_0^{2\pi}\langle y(t)\rangle\, d\varphi = -\frac{1}{2}cpq\int_0^t e^{-(a-\frac{1}{2}p^2)t'} I_0\left(\frac{2A}{\Omega}\sin\frac{1}{2}\Omega t'\right)\, dt'\,, \tag{13}$$

convergent for $p < \sqrt{2a}$.

$$\langle\langle y(t)y(t-\tau)\rangle\rangle = \text{(a complicated expression)}\,, \tag{14}$$

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The constructive role of noise

The equation

$$\dot{y} = -(a + p\xi(t) + A\cos(\Omega t + \varphi))y + qc\,\xi(t) + q\sqrt{1 - c^2}\,\eta(t)$$

is, after a substitution

$$z = y - \frac{cq}{p}\,, \tag{15}$$

converted into

$$\begin{aligned} \dot{z} = &-(a + p\xi(t) + A\cos(\Omega t + \varphi))z - \tfrac{cq}{p}a \\ &- \tfrac{cq}{p}A\cos(\Omega t + \varphi) + q\sqrt{1 - c^2}\,\eta(t)\,. \end{aligned} \tag{16}$$

Correlations between the noises effectively introduce an additive signal!

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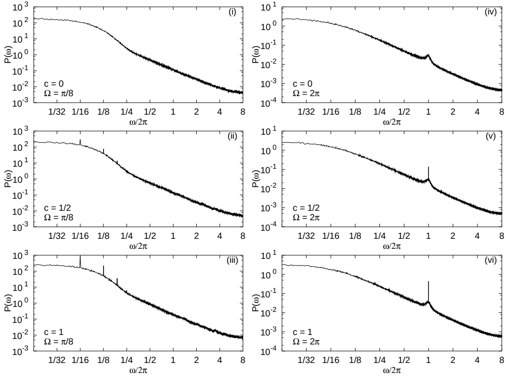
Numerical results



Numerical power spectra of the process (12) for various input signal frequencies and correlations between the noises. The input signal frequency equals $\Omega = \pi/8$ (panels (i)–(iii)) and $\Omega = 2\pi$ (panels (iv)–(vi)). The multiplicative and additive noises are uncorrelated ($c = 0$) on panels (i), (iv), partially correlated ($c = 1/2$) on panels (ii), (v), and fully correlated ($c = 1$) on panels (iii), (vi). Other parameters, common for all panels, are $a = 1/2$, $p = \sqrt{a}/2$, $q = 1/4$, $A = 1$.

The higher harmonics present on panels (i)-(iii) indicate a nonlinear nature of the coupling.

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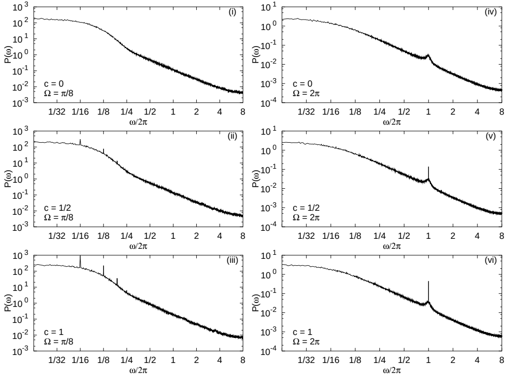

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Back to the linear transmitter [6] (PFG)

The formal solution to the equation (6) reads

$$y(t) = \int_0^t e^{-a(t-t')} \exp\left[-p\int_{t'}^t \xi(t'')dt''\right] \times \left(b + qc\,\xi(t') + q\sqrt{1-c^2}\,\eta(t')\right) dt' \,. \tag{17}$$

The expectation value

$$\langle y(t)\rangle = \frac{b-\frac{1}{2}cpq}{a-\frac{1}{2}p^2}\left(1 - e^{-(a-\frac{1}{2}p^2)t}\right) \xrightarrow[t\to\infty]{} y_\infty = \frac{b-\frac{1}{2}cpq}{a-\frac{1}{2}p^2} \,. \tag{18}$$

The expectation value exists if $p < \sqrt{2a}$.

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The variance

The variance:

$$D = \langle y^2(t) \rangle - \langle y(t) \rangle^2 \xrightarrow[t\to\infty]{} \frac{4b^2p^2 - 8abcpq + \left(4a^2 - 4a(1-c^2)p^2 + (1-c^2)p^4\right)q^2}{2(a-p^2)(p^2-2a)^2}\,. \qquad (19)$$

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The limiting cases:

$D = \frac{b^2p^2 + (a-\frac{1}{2}p^2)^2q^2}{2(a-p^2)(a-\frac{1}{2}p^2)^2}$	$D = \frac{(bp \mp aq)^2}{2(a-p^2)(a-\frac{1}{2}p^2)^2}$
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The variance *vanishes* if $c = \pm 1$ and $bp \mp aq = 0$.

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Linear Stochastic Resonance

Now add a signal:

$$\begin{aligned}\dot{y} &= -(a+p\xi(t))y + qc\,\xi(t) + b + q\sqrt{1-c^2}\,\eta(t) \\ &+ A\cos(\Omega t+\varphi)\end{aligned} \tag{20}$$

We can analytically calculate the correlation function:

$$\langle\langle y(t)y(t+\tau)\rangle\rangle - \langle\langle y(t)\rangle\rangle^2 \underset{t\to\infty}{\longrightarrow}$$

$$\frac{A^2\cos\Omega\tau}{2\left[(a-\frac{1}{2}p^2)^2+\Omega^2\right]} + \left[\frac{A^2p^2}{4(a-p^2)\left[(a-\frac{1}{2}p^2)^2+\Omega^2\right]} + D\right] e^{-(a-\frac{1}{2}p^2)\tau}\,, \tag{21}$$

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Signal-To-Noise-Ratio

We can now calculate the power spectrum and the Signal-To-Noise Ratio. For $c = 1$,

$$\mathrm{SNR} = 10\log_{10}\frac{2A^2(a-p^2)(a-\frac{1}{2}p^2)\left[(a-\frac{1}{2}p^2)^2+\Omega^2\right]}{A^2p^2(a-\frac{1}{2}p^2)^2+2\left[(a-\frac{1}{2}p^2)^2+\Omega^2\right](bp-aq)^2}\,. \quad (22)$$

If $c = 1$, the SNR, as a function of q, the additive noise strenght, has a maximum for $bp - aq = 0$.

A (weaker) maximum is also present for all $c > 0$.

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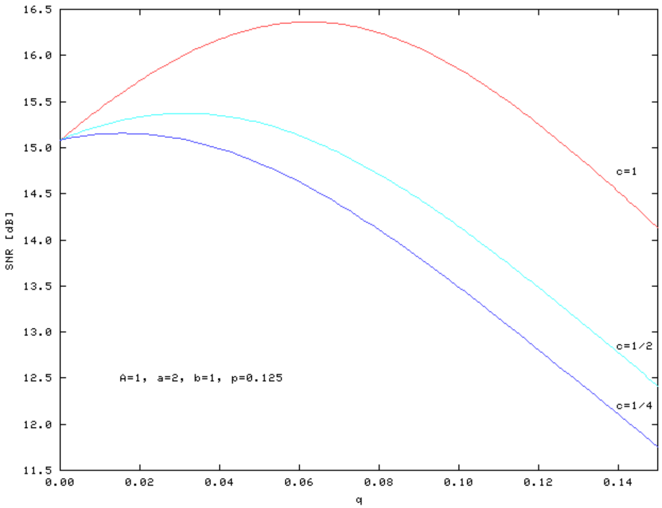
16.5
16.0
15.5
15.0
14.5
14.0
13.5
13.0
12.5
12.0
11.5
SNR [dB]
0.00
0.02
0.04
0.06
0.08
0.10
0.12
0.14
q
c=1
c=1/2
c=1/4
A=1, a=2, b=1, p=0.125

LSR — conclusions

- The LSR reported here is robust:
  - needs GWN only;
  - does not require any special preparations;
  - characterized by a clear maximum of the SNR;
  - persists to asymptotic times;
  - survives phase averaging;
  - parametric coupling means a hidden nonlinearity.
- Two factors are needed for the LSR:
  1. The additive and multiplicative noises must be correlated.
  2. A constant forcing, apart form the external signal, must be present.

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The noisy logistic equation

We turn back to the noisy logistic equation:

$$\dot{x} = (a + p\,\xi_{\mathrm{m}}(t))x - (b + q\,\xi_{\mathrm{a}}(t))x^2 \,. \tag{23}$$

The corresponding linear system can display a vanishing variance and SR — can traces of those be seen in the noisy logistic system?

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The Fokker–Planck equation

Results of [1] and [2] ensure that if $x(0) > 0$, the solution remains positive and bounded almost surely. With that we can construct the Fokker-Planck equation — otherwise we would not know how to normalize the probability distribution:

$$\begin{aligned} \frac{\partial P(x,t)}{\partial t} &= -\frac{\partial}{\partial x}\left[(a-bx)xP(x,t)\right] \\ &+ \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[x^2(p^2-2cpqx+q^2x^2)P(x,t)\right] . \end{aligned} \tag{24}$$

We adopt the following sign convention: $\mathrm{sgn}(pq) = +1$.

Stationary solutions

If $|c| \neq 1$,

$$P_{\rm st}(x) = \frac{\mathcal{N} x^{2(a-p^2)/p^2}}{(p^2-2cpqx+q^2x^2)^{(a+p^2)/p^2}} \times \exp\left[-\frac{2(bp-acq)\arctan\left(\frac{qx-cp}{\sqrt{1-c^2}p}\right)}{\sqrt{1-c^2}\,p^2q}\right]. \quad (25)$$

- Normalizable if $p < \sqrt{2a}$ — the corresponding linear system has a convergent mean.
- If $p < \sqrt{a}$, $P_{\rm st}(x)$ goes to zero as $x \to 0^+$ — the corresponding linear system has a convergent variance. For $\sqrt{a} < p < \sqrt{2a}$, $P_{\rm st}(x)$ is mildly divergent at zero.
- If normalizable, $P_{\rm st}(x) \sim x^{-4}$ for $x \to \infty$ for all values of parameters.

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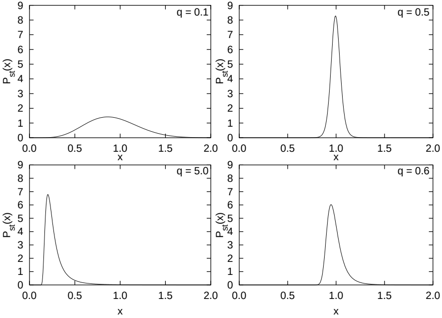
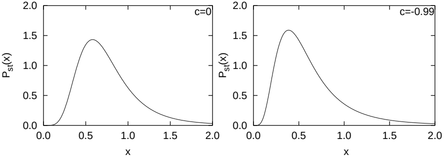


Stationary distributions (25) in a strongly correlated case, $c = 0.99$. Clockwise, from top-left $q = 0.1$, $q = 0.5$, $q = 0.6$, and $q = 5.0$. Other parameters, common for all panels, are $p = 0.5$, $a = b = 1$.



Stationary distributions for the uncorrelated ($c = 0$, left panel) and a strongly anticorrelated ($c = -0.99$, right panel) cases. Other parameters are $a = b = 1$, $p = q = 0.5$.

$c = -1$, $p < \sqrt{2a}$ — normalizable stationary distribution:

$$P_{st}(x) = \mathcal{N} \frac{x^{2(a-p^2)/p^2}}{(p+qx)^{2(a+p^2)/p^2}} \exp\left[\frac{2(bp+aq)}{pq(p+qx)}\right] . \qquad (26)$$

$c = +1$, *any* value of p such that $bp - aq = 0$:

$$P_{st}(x) = \delta\left(x - \frac{p}{q}\right) \qquad (27)$$

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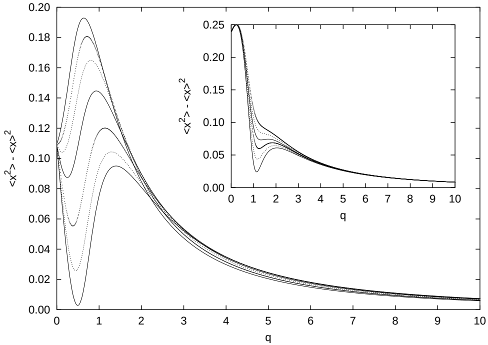

The corresponding linear system does not give any predictions in the $\sqrt{a} < p < \sqrt{2a}$ case, corresponding to the inset

The variance $\left\langle x^2 \right\rangle - \left\langle x \right\rangle^2$ determined form the distribution (25) as a function of the additive noise strength, q. Main panel: $p = 0.5$, the curves, from bottom to top, correspond to $c = 0.99$, $c = 0.90$, $c = 0.75$, $c = 0.50$, $c = 0.25$, $c = 0$, and $c = -0.25$, respectively. Inset: $p = 1.1$, the curves correspond, from bottom to top, to $c = 0.99$, $c = 0.98$, $c = 0.97$, $c = 0.96$, $c = 0.95$, and $c = 0.94$, respectively. Other parameters, common for all curves presented, are $a = b = 1$.

Add a deterministic signal. . .

. . . correspondng to, for example, seasonal changes in the maximal population level:

$$\dot{x} = (a + p\,\xi_{\mathrm{m}}(t))x - (b + A\sin(\Omega t + \varphi) + q\,\xi_{\mathrm{a}}(t))x^2\,. \quad (28)$$

The corresponding linear system displays a SR. Does the noisy logistic system display it?

If so, is it related to the LSR?

Is it related to the minima of the variance?

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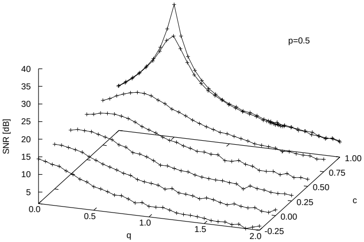
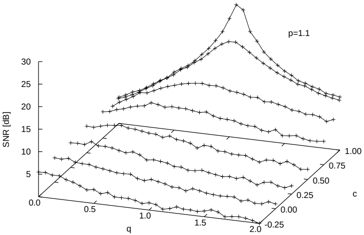


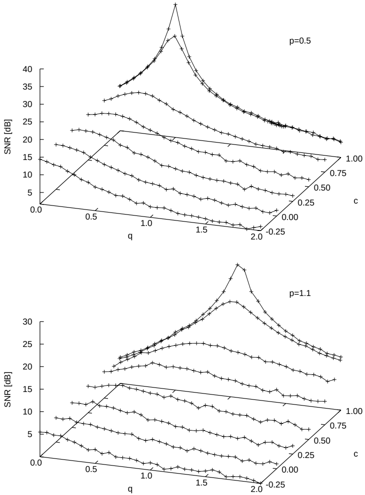


Stochastic resonance in the system (28). The upper panel — the condition $p < \sqrt{a}$ is satisfied, $p = 0.5$. The lower panel — the condition $p < \sqrt{a}$ is not satisfied, $p = 1.1$. Other parameters, common for the two panels, are $a = b = 1$, $A = 0.5$, $\Omega = 2\pi$. Curves presented correspond, back to front, to $c = 1.0$, 0.99, 0.9 (lower panel only), 0.75, 0.5, 0.25, 0.0, and -0.25, respectively.

One may expect that SR occurs where the variance has its minimum. This is the case when $p < \sqrt{a}$.

If $\sqrt{a} < p < \sqrt{2a}$, the system displays a (weak) SR even in the regime where the variance no longer displays a minimum.

The minimum of the variance and the SR are *different* phenomena.



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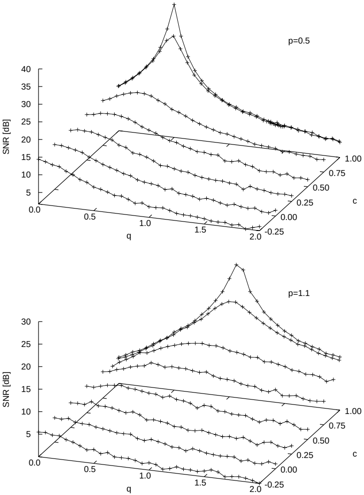

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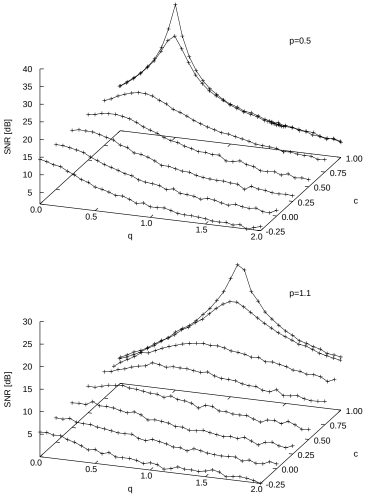

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I am grateful for very helpful discussions with Andrzej Fuliński, Peter Hänggi, Ryszard Zygadło and Bartłomiej Dybiec at various stages of this work.

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