Time Series Analysis:

12. Stochastic Differential Equations

Stochastic resonance

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Stochastic Differential Equation

Let $x, x_0, f, g \in \mathbb{R}$ (can be easily generalized to higher dimensions)

A "physical" notation:

$$\frac{dx}{dt} = f(t,x) + g(t,x)\xi(t)\,, \tag{1}$$

where $\xi(t)$ is GWN satisfying $\langle\xi(t)\rangle = 0$, $\langle\xi(t')\xi(t'')\rangle = \delta(t'-t'')$.

The same in a "mathematical" notation

$$dx = f(t,x)\,dt + g(t,x)\,dW(t) \tag{2}$$

$W(t)$ is called the Wiener process.

The stochastic integral

$$\int_a^b f(\xi(t))dt = ?$$

According to Riemann, the integral cannot depend on the choice of the intermediate points. However, if ξ is a stochastic process, the integral *most probably* depends on this choice. What can we do?

Ito interpretation

Define

$$\underbrace{\int\limits_a^b f'(W(s))\,dW(s)}_{\text{mathematical notation}} = f(W(b)) - f(W(a)) - \frac{1}{2}\int\limits_a^b f''(W(s))\,ds \qquad (3)$$

$W(s)$ is the *Wiener process*. The second term in (3) is called the Ito term.

Therefore

$$\int_a^b g(s)\,dW(s) \equiv \underbrace{\int_a^b g(s)\,\xi(t)\,dt}_{\text{physical notation}}$$

$$= \lim_{N\to\infty} \sum_{k=0}^{N} g(t_k)(W(t_{k+1}) - W(t_k)) \tag{4}$$

where $a \equiv t_0 < t_1 < \cdots < t_{N+1} \equiv b$. The function is always calculated in the left end of the interval! This is Ito interpretation.

There is another interpretation, named after Stratonovich:
we take $g\left((t_k + t_{k+1})/2\right)$ in (4). (The additional term in (3) is missing in this case.)

Two kinds of convergence

l — any polynomial. $\exists\ \bar{h}\ \forall\ l\ \forall\ n\ \exists\ K > 0, \tilde{K} > 0\ \forall\ h < \bar{h}$:

$$\left| l\left(\langle x(t_n)\rangle\right) - l\left(\langle x_n\rangle\right)\right| \leqslant K h^{\beta} \qquad \text{weak convergence of order } \beta$$

$$\left|\langle l(x(t_n)) - l(x_n)\rangle\right| \leqslant \tilde{K} h^{\gamma} \qquad \text{strong convergence of order } \gamma$$

$x(t_n)$ is the *exact* solution, x_n is the *numerical approximation*.

Numerical methods for ODE's, naively applied to SDE's, underestimate the noise!

Example — explicit Euler method

$$\text{ODE:} \quad \frac{dx}{dt} = f(t,x) \quad \Longrightarrow \quad x_{n+1} = x_n + hf(t_n, x_n) \tag{5}$$

$$\text{SDE:} \quad \frac{dx}{dt} = f(t,x) + g(t,x)\xi(t) \quad \Longrightarrow \quad x_{n+1} = x_n + hf(t_n, x_n) + hg(t_n, x_n)\xi_n \tag{6}$$

Bad idea!

Euler-Maruyama method

$$
\begin{aligned}
x(t_{n+1}) &= x_n + \int\limits_{t_n}^{t_n+h} f(t', x(t'))\, dt' + \int\limits_{t_n}^{t_n+h} g(t', x(t'))\xi(t')\, dt' && (7) \\
&\simeq x_n + f(t_n, x(t_n)) \int\limits_{t_n}^{t_n+h} dt' + g(t_n, x(t_n)) \int\limits_{t_n}^{t_n+h} \xi(t')\, dt' && (8)
\end{aligned}
$$

$$\left\langle \int\limits_{t_n}^{t_n+h} \xi(t')\, dt' \right\rangle = 0$$

$$\left\langle \left(\int\limits_{t_n}^{t_n+h} \xi(t')\, dt' \right)^2 \right\rangle = \int\limits_{t_n}^{t_n+h} dt' \int\limits_{t_n}^{t_n+h} dt'' \left\langle \xi(t')\xi(t'') \right\rangle = h$$

$$x_{n+1} = x_n + hf(t_n, x_n) + \sqrt{h}\, g(t_n, x_n)\eta_n \tag{9}$$

η_n is a Gaussian random variable such that $\langle \eta_n \rangle = 0$, $\langle \eta_n \eta_m \rangle = \delta_{nm}$.

- weak convergence of order $\beta = 1$
- strong convergence of order $\gamma = \frac{1}{2}$
- corresponds to the deterministic Euler method
- corresponds to Ito interpretation

Remark

Suppose we have a continuous noisy signal

$$x(t) = f(t) + \sigma\, g(t)\, \xi(t) \tag{10}$$

and we want to *sample* it with a timestep h. Then $1/(2h)$ is the Nyquist frequency and

$$x_n = f(t_n) + \widehat{\sigma}\, g(t_n)\, \eta_n \tag{11}$$

where $\widehat{\sigma}$ represents an "average" power of the noise within the sampling step. If $\xi(t)$ is GWN, η_n is a discrete GWN and $\widehat{\sigma} = \sqrt{h}\,\sigma$.

Heun method

$$x(t_{n+1}) = x_n + \int\limits_{t_n}^{t_n+h} f(t', x(t'))\, dt' + \int\limits_{t_n}^{t_n+h} g(t', x(t'))\xi(t')\, dt' \tag{12}$$

Use

$$
\begin{aligned}
x_{(1)} &= x_n + hf(t_n, x_n) + \sqrt{h}g(t_n, x_n)\eta_n && (13)\\
x_{(2)} &= x_n + hf(t_n, x_{(1)}) + \sqrt{h}g(t_n, x_{(1)})\eta_n && (14)\\
x_{n+1} &= \frac{x_{(1)} + x_{(2)}}{2} && (15)
\end{aligned}
$$

- weak convergence of order $\beta = 2$

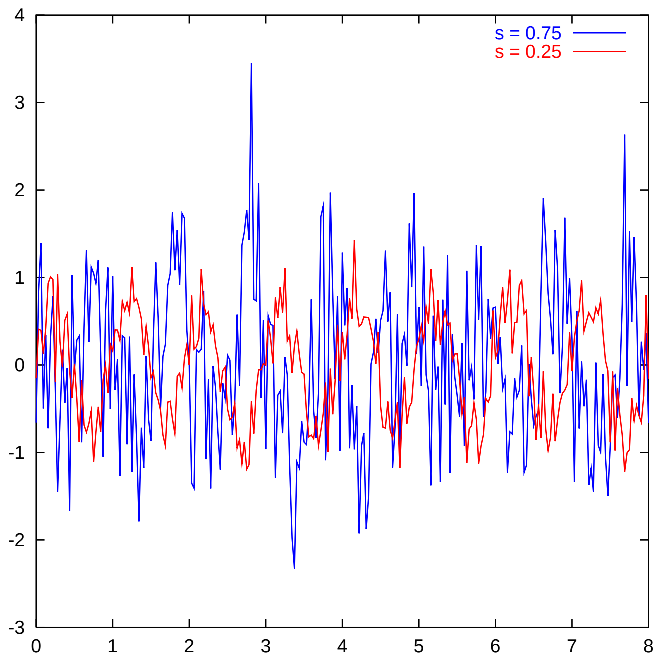
- strong convergence of order $\gamma = 1$
- corresponds to the deterministic trapezoidal rule (Runge-Kutta of order 2)
- corresponds to Statonovich interpretation

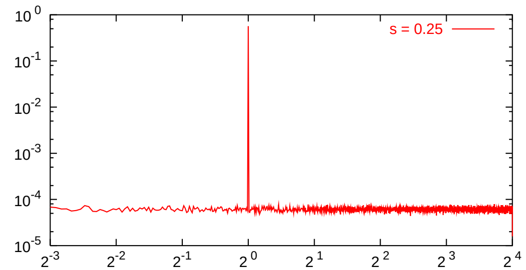
Explicit Runge-Kutta methods

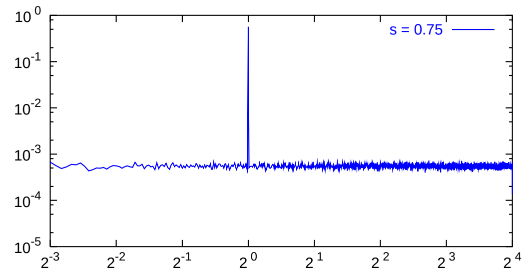
$\dfrac{dx}{dt} = f(t,x)$	$\dfrac{dx}{dt} = f(t,x) + g(t,x)\xi(t)$
s-stage RK method for ODE	s-stage RK method for SDE
$x_{n+1} = x_n + h\sum\limits_{i=1}^{s} w_i k_i$ $k_i = f\left(t_n + \alpha_i h, x_n + h\sum\limits_{j<i}\beta_{ij}k_j\right)$	$x_{n+1} = x_n + h\sum\limits_{i=1}^{s} w_i k_i + \sqrt{h}\sum\limits_{i=1}^{s} w_i r_i$ $k_i = f\left(t_n + \alpha_i h, x_n + h\sum\limits_{j<i}\beta_{ij}k_j + \sqrt{h}\sum\limits_{j<i}\beta_{ij}r_j\right)$ $r_i = g\left(t_n + \alpha_i h, x_n + h\sum\limits_{j<i}\beta_{ij}k_j + \sqrt{h}\sum\limits_{j<i}\beta_{ij}r_j\right)$

It can be shown that the strong convergence for Runge-Kutta cannot exceed $\gamma = \frac{3}{2}$. They do not correspond to Ito. They do not necessarily correspond to Stratonovich, either.

The stronger the noise, the weaker the signal



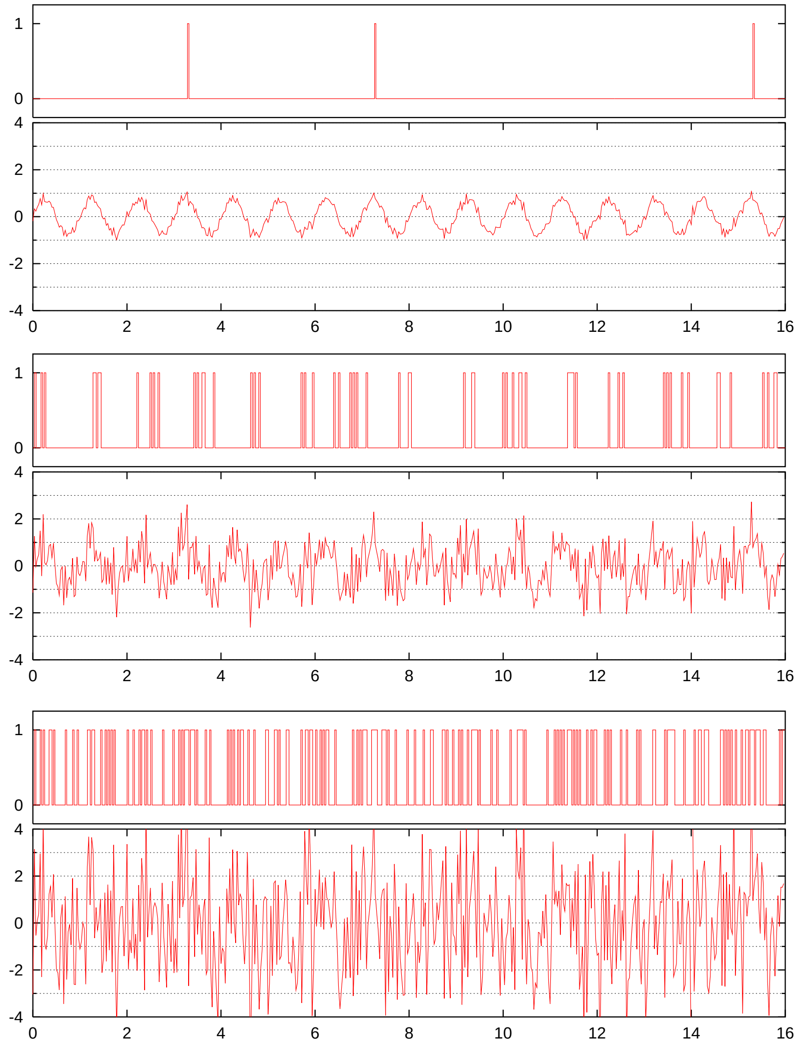




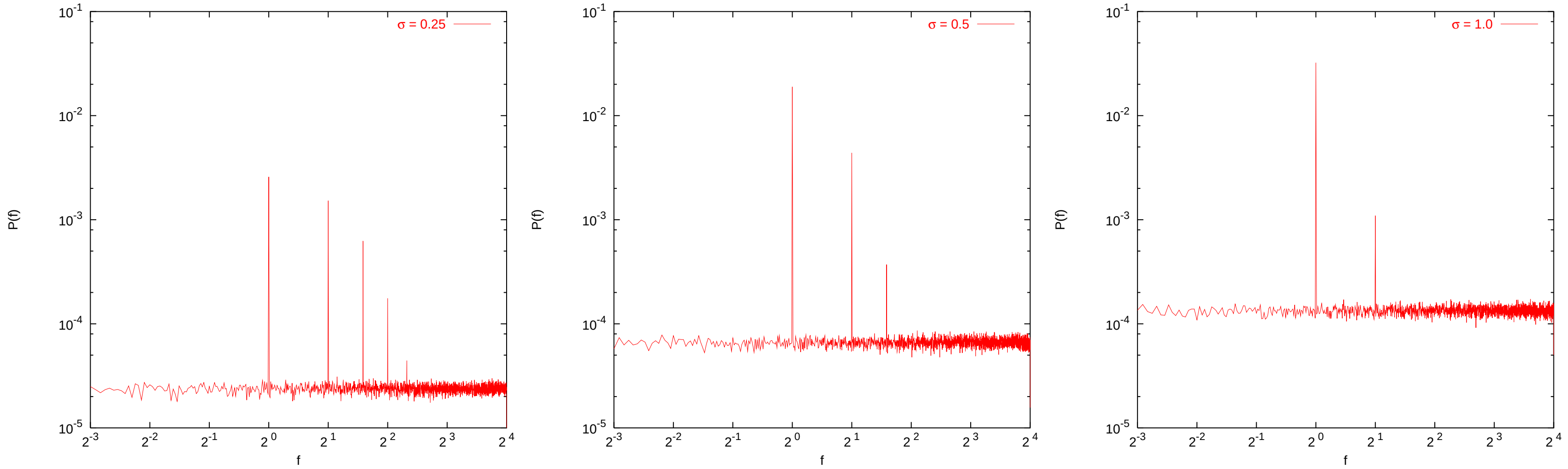
Is that *always* the case?

$$x(t) = A \sin(\omega t + \phi) + \sigma\xi(t)\,, |A| < 1$$

$$y(t) = \begin{cases} 1 & x(t) \geqslant 1 \\ 0 & x(t) < 1 \end{cases}$$

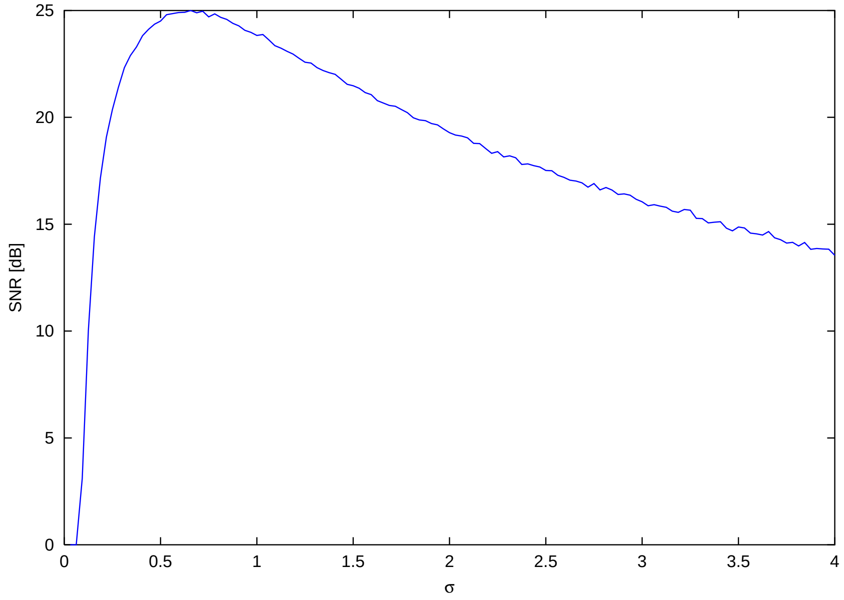


Power spectra for different levels of noise



Signal To Noise Ratio, SNR

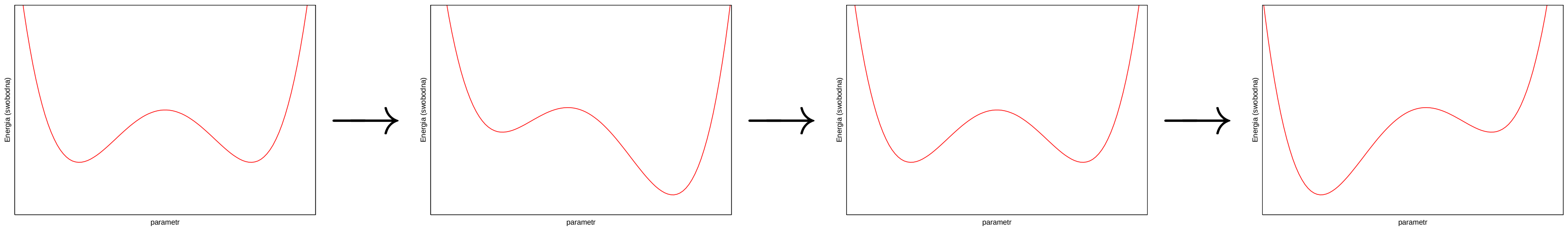
$$\text{SNR} = 10\,\log_{10}\frac{P(f_0)}{P_{\text{back}}(f=f_0)}$$



Overdamped motion in a time dependent double well potential

Langevin equation:

$$\dot{x} = -\frac{dU(x)}{dx} + A\sin(2\pi f_0 t + \phi) + \sigma\xi(t)$$



Autocorrelation and the power spectrum

$$C(\tau) = \lim_{t\to\infty} \frac{1}{2\pi} \int\limits_0^{2\pi} \langle x(t)x(t+\tau)\rangle \, d\phi$$

$\langle\cdots\rangle$ stands for averaging over the realizations of the noise. From $C(\tau)$ we get the power spectrum by means of Wiener-Khinchin theorem:

$$P(f) = P_0\,\delta(f_0) + P_{\text{back}}(f)$$

In practice, is it a *bad idea* to calculate the autocorrelation first and use it to calculate the power spectrum. Use the methods that you already know to estimate the periodogram. Sometimes it is beneficial to discard severeal terms from the beginning of the series, corresponding to transient behaviour (before the system equilibrates).

The noise background does not need to be flat!