Time Series Analysis:

8. Financial time series

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Volatility

In analysing financial time series, it is frequently more important not to predict future values of an asset, but to predict the future *variability* of the asset. Investors realize and accept that future values of an asset are random variables. However, if these random variables come from a narrow distribution, the associated risk is small. Contrariwise, if the random variables come from a wide distribution, the associated risk is large. This is particularly true with assets like options, future contracts and other derivatives.
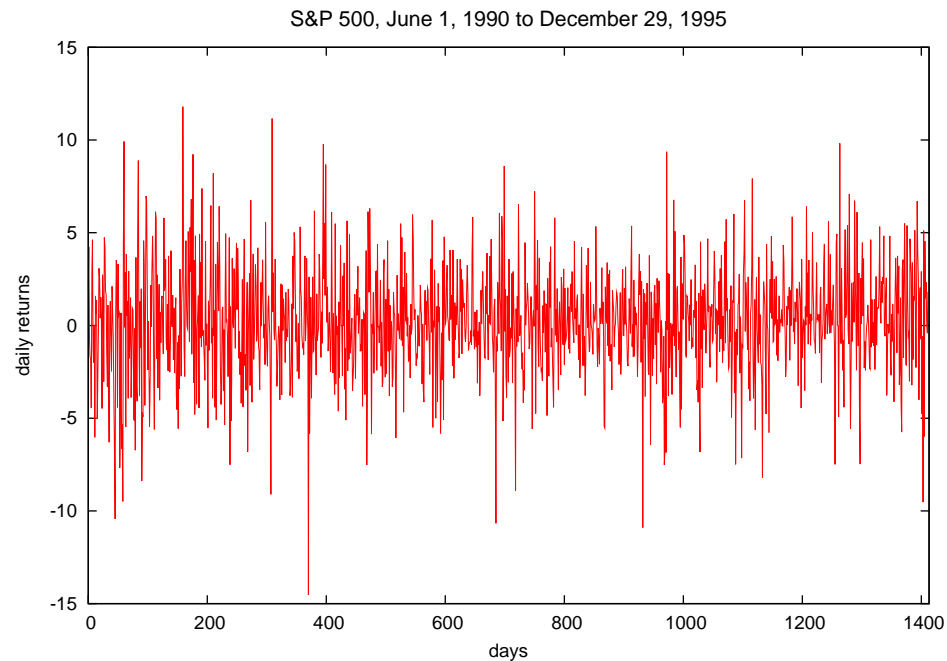
The point is, the width of the distribution can change over time.

Frequently, such series are modelled as
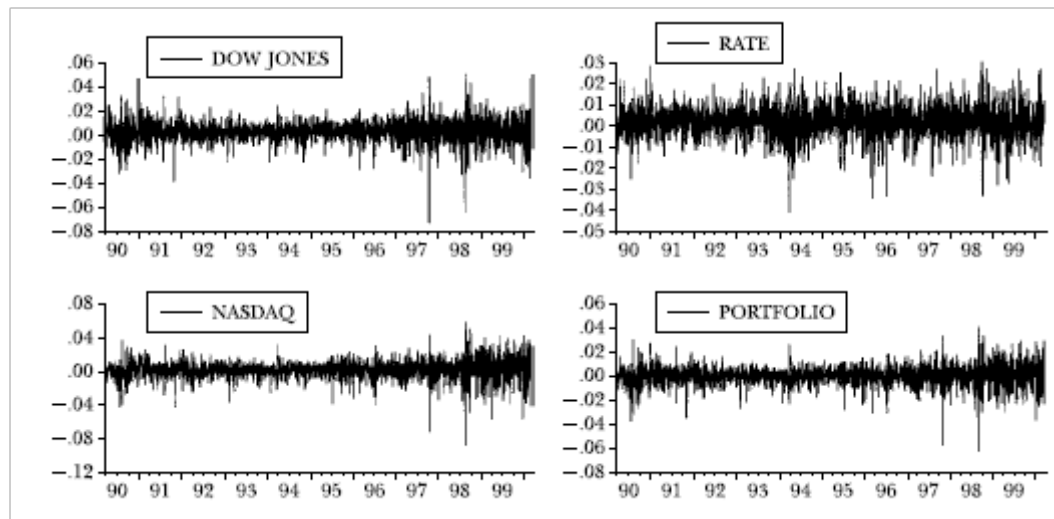
$$\epsilon_n = \sqrt{h_n}\,\eta_n \qquad (1)$$

$\{\eta_n\}$ is GWN. $h_n > 0$ is called *volatility*. The presence of an index, h_n, indicates that volatility changes in time

A typical time series in finances

S&P 500, June 1, 1990 to December 29, 1995

Regions of high and low volatility can easily be seen. High volatilities and low volatilities tend to cluster.
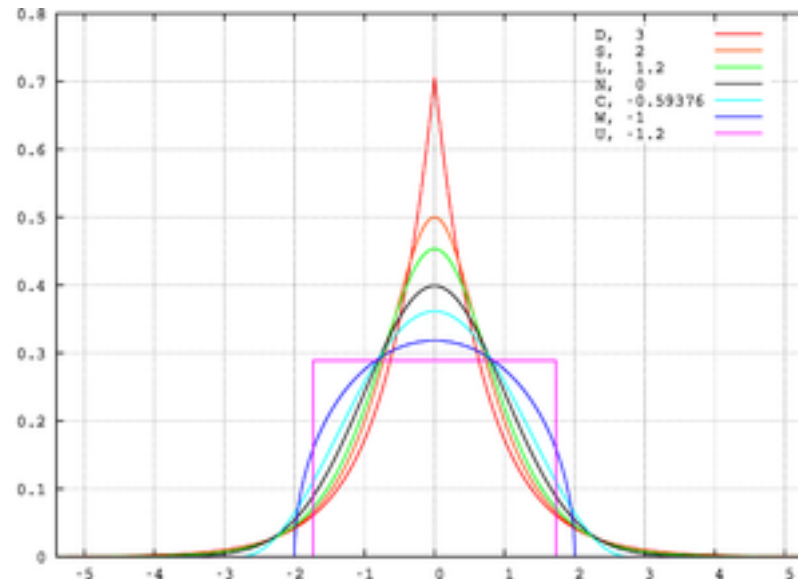
A classic example

From the seminal paper by Robert Engle,
Journal of Economic Perspectives **15**, 157-168 (2001).

Features of *real life* financial time series

- Time-varying variance

- Regions with large variances have a tendency to cluster

- Non-zero kurtosis

- Heavy (fat) tails

In finances, we call the variance *volatility*.

Predicting **volatility** is *essential* in assessing future financial risks.

Heteroscedasticity

In statistics, if there are subpopulations within a collection of random variables that have different variabilities than the others, *especially* if those subpopulations can be associated with a quantity that sets some sort of "order" or "sequence", we say that these random variables are **heteroskedastic** (or heteroscedastic). The "variability" can be quantified as the variance or any other measure of statistical dispersion.

If the variability is constant within the collection of random variables, we say that these variables are homoscedastic.

- **Classic example 1**: Babies are born with similar body weights: the variance of body weights in newborn babies is small. However, the variance of body weights in children just a few years old is considerably larger. Thus, the population of young children, from newborns to pre-school or introductory

school ages is heteroscedasticit with respect to body weight. The age of children acts as an "ordering quantity" here.

- **Classic example 2**: A classic example of heteroscedasticity is that of income versus expenditure on meals. As one's income increases, the variability of food consumption will increase. A poorer person will spend a rather constant amount by always eating inexpensive food; a wealthier person may occasionally buy inexpensive food and at other times eat expensive meals. Those with higher incomes display a greater variability of food consumption. The income level, or some other measure of personal wealth, is the "ordering quantity".

In Time Series Analysis we usually talk about a *conditional* volatility and a *conditional* heteroscedasticity, as we do not know their *absolute*, "ideal" values. Rather than that, we only know them with respect to the actual initial data.
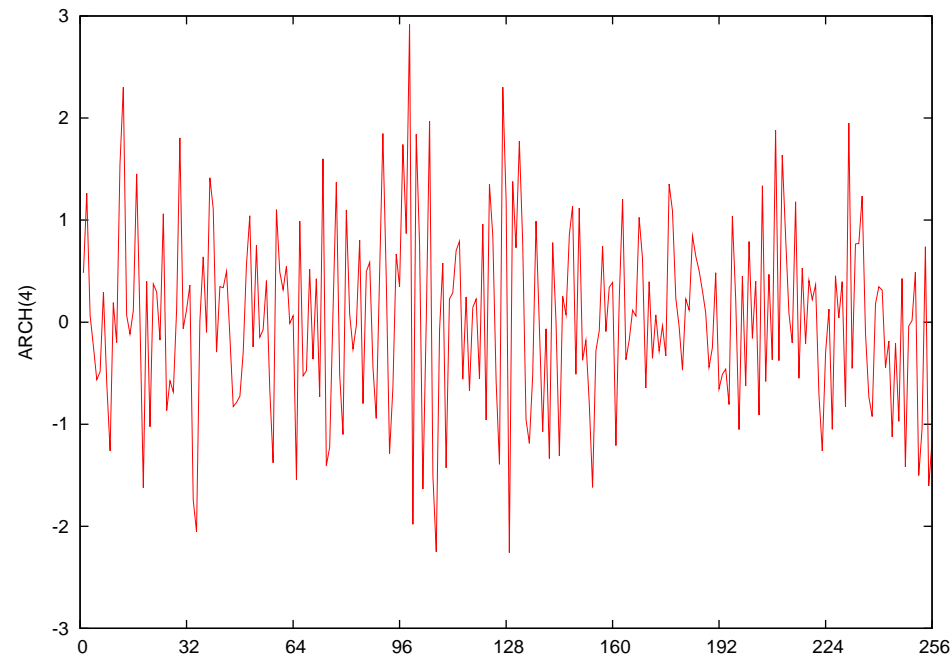
The ARCH model

AutoRegressive Conditional Heteroscedasticity (Robert Engle, 1982). The ARCH(q) model is of the form (1) with

$$h_n = \alpha_0 + \sum_{i=1}^{q} \alpha_i \, \epsilon_{n-i}^2 , \tag{2}$$

where $\alpha_0 > 0$, $\alpha_i \geqslant 0$, $i = 1, 2, \ldots, q-1$, $\alpha_q > 0$. q is the order of ARCH(q) and therefore α_q must be strongly grater than zero.

The question *why* such models describe financial time series correctly is beyond the scope of this lecture.

An example

ARCH(4) with $\alpha_0 = 0.5$, $\alpha_1 = 0.25$, $\alpha_2 = 0.0625$, $\alpha_3 = 0.0$, $\alpha_4 = 0.125$.

Fitting parameters to an ARCH model

1. Find the best fitting AR(q) model of the original time series. Thus

$$y_n = a_0 + \sum_{i=1}^{q} a_i\, y_{n-i} + \epsilon_n\,.$$

2. We now have the order, q, and errors, ϵ_n: errors are differences between what comes from the model and the actual data points. Using least squares, fit parameters $\widehat{\alpha}_i$

$$\epsilon_n^2 = \widehat{\alpha}_0 + \sum_{i=1}^{q} \widehat{\alpha}_i\, \epsilon_{n-i}^2\,.$$

3. Using an χ^2 statistics, verify whether any of the $\widehat{\alpha}_i$'s, $i = 1, \ldots, q$, is significantly greater than zero (these are parameters of the ARCH model), the null hyphothesis being that there are no ARCH contributions.

Note: "Significantly greater than zero" does not mean that the value is large, but that the difference form zero is statistically significant.

The GARCH model

Generalized AutoRegressive Conditional Heteroscedasticity (Tim Bollerslev, 1986). The GARCH(p,q) model is again of the form (1) with
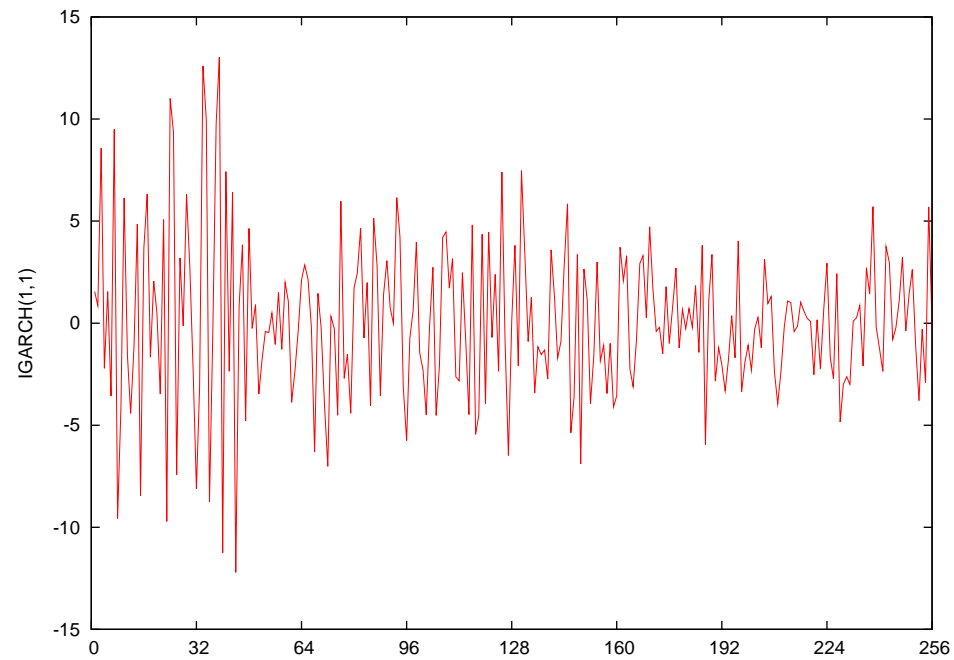
$$h_n = \alpha_0 + \sum_{i=1}^{q} \alpha_i\, y_{n-i}^2 + \sum_{i=1}^{p} \beta_i\, h_{n-i}\,, \quad \alpha_i\,, \beta_i \geqslant 0\,. \tag{3}$$

If

$$\sum_{i=1}^{q} \alpha_i + \sum_{i=1}^{p} \beta_i = 1\,, \tag{4}$$

there can appear persistent trends in the volatility. We call such a model IGARCH.

An example

IGARCH(1,1) with $\alpha_0 = 0.5$, $\alpha_1 = 0.25$, $\beta_1 = 0.75$

FIGARCH

Often the IGARCH model seems to be too restrictive as it implies infinite persistence of a volatility shock. On the other hand, long-range correlations in volatility are frequently present. In such situation a *fractionally integrated* GARCH model comes in handy.

A GARCH(p,q) model has the ARMA(m,p) representation

$$[1 - \alpha(Z) - \beta(Z)]\, \epsilon_n^2 = \alpha_0 + [1 - \beta(Z)]\, \nu_n \,, \tag{5a}$$

where $m = \max(p, q)$, Z is the time-shift operator, $\alpha(Z), \beta(Z)$ are the usual polynomials with roots outside the unit circle and

$$\nu_n = \epsilon_n^2 - h_n \tag{5b}$$

is called a *volatility shock*.

In this notation, an IGARCH is represented as

$$[1 - \alpha(Z) - \beta(Z)] \, (1 - Z)\epsilon_n^2 = \alpha_0 + [1 - \beta(Z)] \, \nu_n \,. \qquad (6)$$

If we replace the first difference operator $1 - Z$ by its fractional counterpart, we get the FIGARCH(p,d,q):

$$[1 - \alpha(Z) - \beta(Z)] \, (1 - Z)^d \epsilon_n^2 = \alpha_0 + [1 - \beta(Z)] \, \nu_n \qquad (7)$$

with $0 < d < 1$. Note: we calculate h_n from (7) and then use Eq. (1) which then becomes a nonlinear equation for ϵ_n.

More information of FIGARCH can be found for example in Maryam Tayefi and T. V. Ramanathan, *An Overview of FIGARCH and Related Time Series Models*, Austrian Journal of Statistics, **41**, 175–196 (2012).

EGARCH

All of the models: ARCH, GARCH, IGARCH, FIGARCH are *symmetric*: As they on ϵ_n^2, they react the same to negative and positive innovations of the same magnitude. However, it is known that vaolatilities after a negative innovation increase more than after a positive innovation of the same magnitude. To accomodate for this type of behaviour, the EGARCH, or exponential GARCH model has been introduced. In this model the $\ln h_n$ instead of pure h_n is used. In its simples form an EGARCH(1,1) is:

$$\ln h_n = \alpha_0 + g_1(\eta_{n-1}) + \beta_1 \ln h_{n-1} \tag{8a}$$

where

$$g_1(\eta_{n-1}) = \alpha_1 \eta_{n-1} + \gamma_1 \left(|\eta_{n-1}| - \langle |\eta_{n-1}| \rangle \right). \tag{8b}$$

$\langle|\eta_{n-1}|\rangle$ is the *weighted innovation* or the average of $|\eta_j|$ up to η_{n-1}. $\alpha_0, \alpha_1, \beta_1, \gamma_1$ are constants (no restrictions on the sign of these constants because the *logarithm* of h_n is calculated from the model). The model responds differently to "good news" and "bad news". A positive shock results in $(\alpha_1 + \gamma_1)\eta_{n-1}$ and a negative in $(\alpha_1 - \gamma_1)\eta_{n-1}$.

The model can be easily extended to contain more past values of $\epsilon_{n-j}, \ln h_{n-j}$.

More complicated models like FIEGARCH[*] or models with seasonalities can be built.

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[*]Hojatallah Goudarzi, *Modeling Long Memory in The Indian Stock Market using Fractionally Integrated Egarch Model*, International Journal of Trade, Economics and Finance, **1**, 2010-023X (2010).

Example
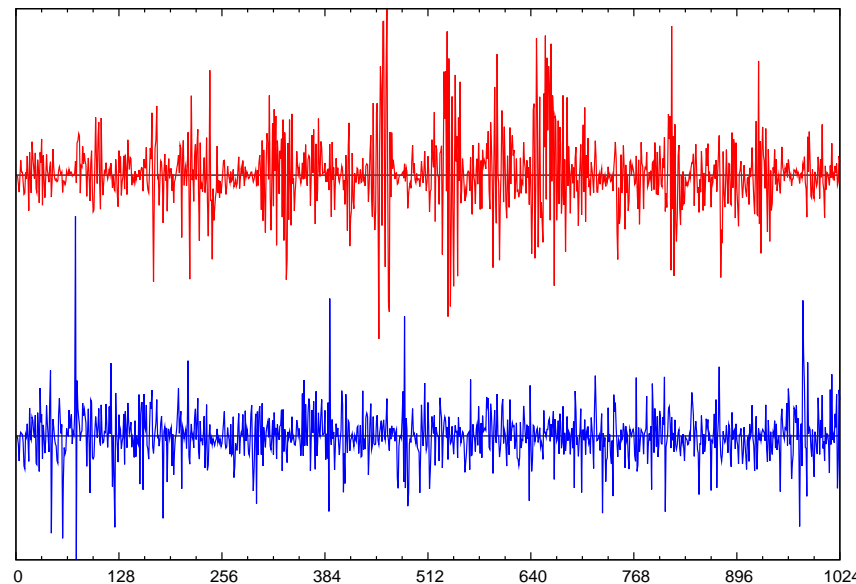
Two EGARCH(1,1), Eq. (8), processes.
Upper curve: $\alpha_0 = 0.25, \alpha_1 = 0.25, \beta_1 = 0.85, \gamma_1 = +0.95$.
Lower curve: $\alpha_0 = 0.25, \alpha_1 = 0.25, \beta_1 = 0.85, \gamma_1 = -0.95$.
The underlying GWN noises are **identical** in both cases. The curves have been shifted vertically for better visibility.

A current example

As of April 28, 2021, the Google Scholar database lists **more than 2,000 papers** with the key word GARCH, published in 2021 alone. One of these papers: Viviane Naimy, Omar Haddad, Gema Fernández-Avilés, Rim El Khoury, The predictive capacity of GARCH-type models in measuring the volatility of crypto and world currencies, PLoS ONE **16**(1), e0245904, 2021.
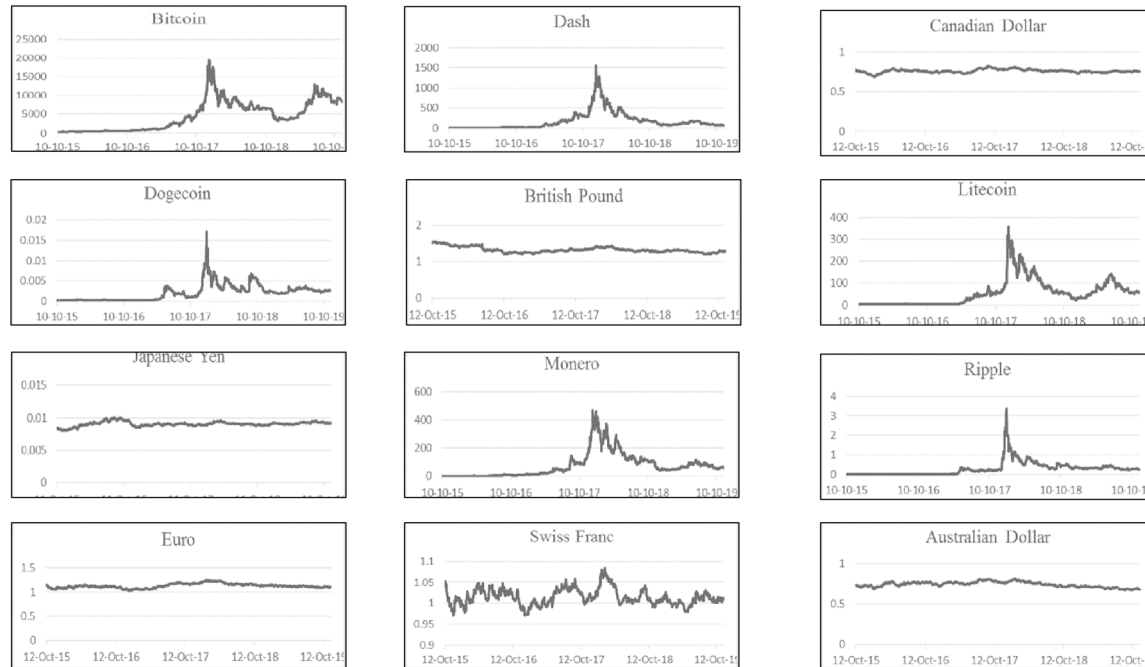
Time series plot of the daily prices of the six cryptocurrencies and exchange rates between October 10, 2015 and November 18, 2019
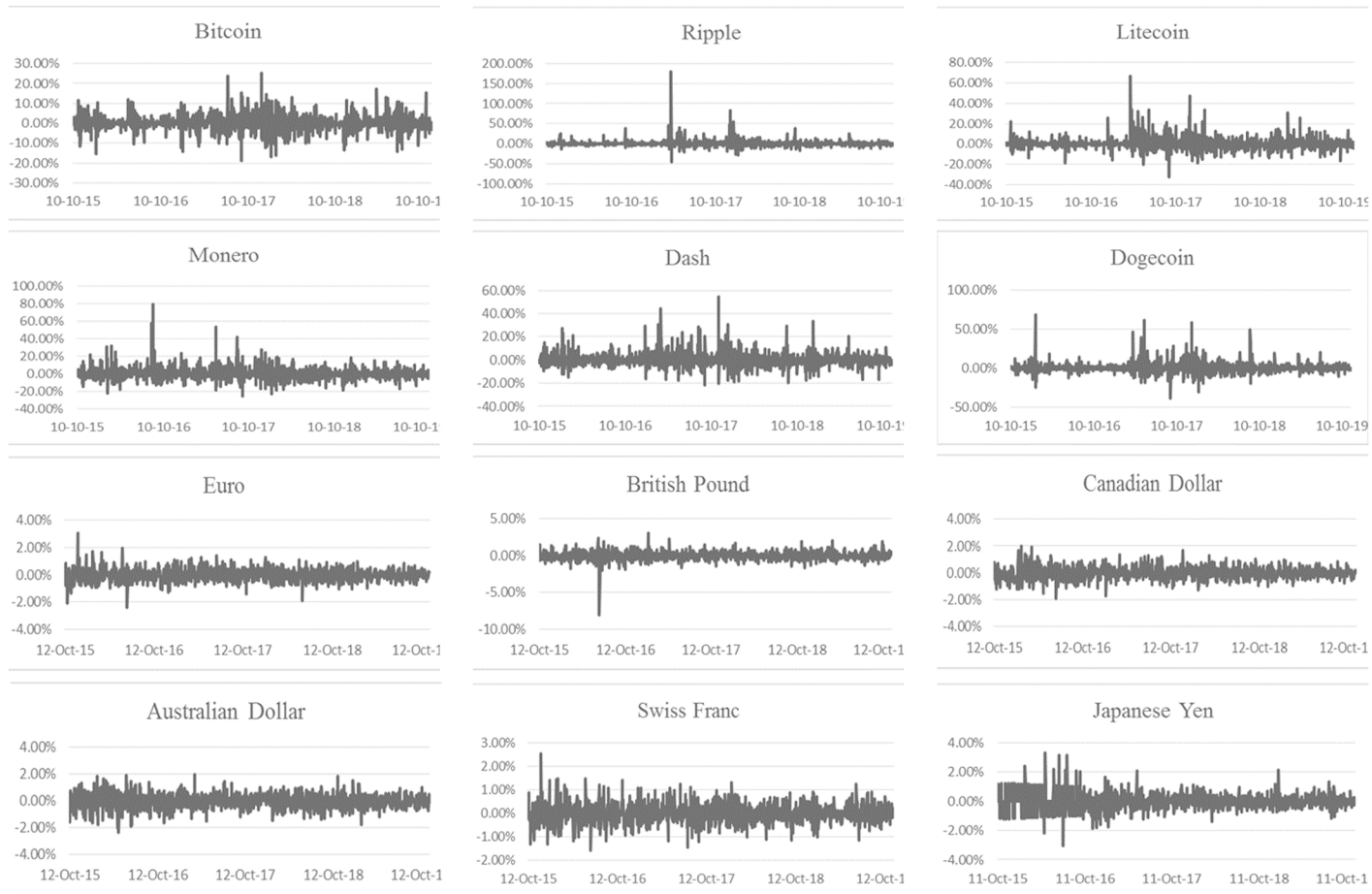
Time series plot of the daily simple returns of the six cryptocurrencies and exchange rates

The authors fit various models of the GARCH family to the data.

However, the outcomes were far from being uniform for the selected cryptocurrencies: convincing for Dash and Dogcoin, acceptable for Litecoin and Monero and unconvincing for Bitcoin and Ripple, where the (optimal) model was not rejected only at the 99% confidence level