Time Series Analysis:

7. Multivariate processes

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All kinds of time series taht have been discussed so far, and some of those that will be discussed in the future, have their multivariate (or vector) counterparts. For example, a process

$$\mathbf{x}_n = \mathbf{A}_1\mathbf{x}_{n-1} + \mathbf{A}_2\mathbf{x}_{n-2} + \cdots + \mathbf{A}_p\mathbf{x}_{n-p} + \mathbf{B}_0\boldsymbol{\eta}_n + \mathbf{B}_1\boldsymbol{\eta}_{n-1} + \cdots + \mathbf{B}_q\boldsymbol{\eta}_{n-q} \tag{1}$$

is a vector autoregressive, moving average process VARMA(p,q). In (1), $\mathbf{x}_n \in \mathbb{R}^m$ is a m-dimensional time series, $\mathbf{x}_{n-k}$ are its past values, $\boldsymbol{\eta}_n$ in a n-dimensional GWN, similar for its past values, and $\mathbf{A}_1, \dots, \mathbf{A}_p, \mathbf{B}_0, \dots, \mathbf{B}_q \in \mathbb{R}^{m\times m}$ are constant, real matrices. It is also possible to consider series in which the dimensionality of the "innovations" $\boldsymbol{\eta}$'s is different from that of the time series; in that case the matrices $\mathbf{B}_j$ are not square, but rectangular.

The need to discuss such processes arises when we observe more than one time series and we expect that they mutually influence each other.

Example

Two processes

$$x_n = \alpha_{11} x_{n-1} + \alpha_{12} y_{n-1} + \sigma_x \eta_{x,n} \tag{2a}$$
$$y_n = \alpha_{21} x_{n-1} + \alpha_{22} y_{n-1} + \sigma_y \eta_{y,n} \tag{2b}$$

together form a VAR(1) process with uncorrelated (independent) noises.

VAR(1)

For simplicity, we shall only deal with processes VAR(1), or of the type (2), or more generally,

$$\mathbf{x}_n = \mathbf{A}\mathbf{x}_{n-1} + \mathbf{\Sigma}\boldsymbol{\eta}_n \tag{3}$$

where $\mathbf{\Sigma} = \mathrm{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_m\}$ meaning that the individual components of the vector noise are uncorrelated.

If the matrix $\mathbf{A}$ in (3) can be diagonalized, i.e. if there exists an invertible matrix $\mathbf{S}$ such that

$$\mathbf{S}^{-1}\mathbf{A}\mathbf{S} = \mathbf{A}_{\textsf{diag}} = \mathrm{diag}\{\lambda_1, \dots, \lambda_m\} \tag{4}$$

the vector process (3) can be "diagonalized", or represented as a collection of series that no longer influence each other. Indeed, multiplying (1) by $\mathbf{S}^{-1}$ from the left, we get

$$\mathbf{z}_n = \mathbf{S}^{-1}\mathbf{x}_n = \mathbf{S}^{-1}\mathbf{A}\mathbf{S}\mathbf{S}^{-1}\mathbf{x}_{n-1} + \mathbf{S}^{-1}\mathbf{\Sigma}\boldsymbol{\eta}_n = \mathbf{A}_{\textsf{diag}}\mathbf{z}_{n-1} + \mathbf{S}^{-1}\mathbf{\Sigma}\boldsymbol{\eta}_n \tag{5}$$

Notes

1. There still may be some interdependence between different components of $\mathbf{z}_n$ as the matrix $\mathbf{S}^{-1}\mathbf{\Sigma}$ is, in general, not diagonal and the noises acting on various components of $\mathbf{z}_n$ get correlated.
2. If the matrix $\mathbf{A}$ in (3) is not symmetrix, the "diagonalized" time series $\mathbf{z}_n$ may become *complex*.
3. For processes of higher orders VAR(p), a "diagonalization" in the spirit of Eq. (5) is possible only if all the matrices $\mathbf{A}_1, \dots, \mathbf{A}_p$ commute.

Embedding in a higher dimension

If we have a general VAR(p) process

$$\mathbf{x}_n = \mathbf{A}_1\mathbf{x}_{n-1} + \mathbf{A}_2\mathbf{x}_{n-2} + \cdots + \mathbf{A}_p\mathbf{x}_{n-p} + \boldsymbol{\Sigma}\boldsymbol{\eta}_n \tag{6}$$

we can formally represent it as a VAR(1) process, but in a space of dimensionality $m \times p$. In block notation,

$$\begin{bmatrix} \mathbf{x}_n \\ \mathbf{x}_{n-1} \\ \vdots \\ \mathbf{x}_{n-p+2} \\ \mathbf{x}_{n-p+1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \cdots & \mathbf{A}_{p-1} & \mathbf{A}_p \\ \mathbb{I} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \mathbb{I} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_{n-1} \\ \mathbf{x}_{n-2} \\ \vdots \\ \mathbf{x}_{n-p+1} \\ \mathbf{x}_{n-p} \end{bmatrix} + \boldsymbol{\Sigma} \begin{bmatrix} \boldsymbol{\eta}_n \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} . \tag{7}$$

Stationarity of VAR(1)

From the "diagonalized" form of a VAR(1) process, we can clearly see that the process is stationary, if and only if all eigenvalues of the matrix $\mathbf{A}$ satisfy

$$\forall i = 1, \ldots, m\colon \quad |\lambda_i| < 1\,, \tag{8}$$

provided these eigenvalues exist. If *any* of the eigenvalues has a modulus that is greater than 1, the process is not stationary and explodes.

Note that the similarity transformation (4) and its inverse do not change the eigenvalues.

Cross-correlations

The most important quantity to analyse while dealing with multvariate series is the *cross-correlation*. Let x_n^j be the j-th component of the vector $\mathbf{x}_n$. Then

$$\rho_{jk}(l) = \frac{1}{\sigma_j \sigma_k} \left\langle \left(x_n^j - \left\langle x_n^j \right\rangle \right) \left(x_{n+l}^k - \left\langle x_n^k \right\rangle \right) \right\rangle \tag{9a}$$

where

$$\sigma_j = \sqrt{\left\langle \left(x_n^j - \left\langle x_n^j \right\rangle \right)^2 \right\rangle}. \tag{9b}$$

Note that in general, $\varrho_{jk}(l) \neq \varrho_{kj}(l)$.

Because in practice we have only a single realization of the process at our disposal, we cannot do the statistical averaging. Therefore, instead of (9) we use

$$\left\langle x_n^j \right\rangle = \frac{1}{N} \sum_{n=1}^{N} x_n^j \tag{10a}$$

$$\sigma_j = \sqrt{\frac{1}{N} \sum_{n=1}^{N} \left(x_n^j - \left\langle x_n^j \right\rangle \right)^2} \tag{10b}$$

$$r_{jk}(l) = \frac{1}{(N-l)\sigma_j \sigma_k} \sum_{n=1}^{N-l} \left(x_n^j - \left\langle x_n^j \right\rangle \right) \left(x_{n+l}^k - \left\langle x_n^k \right\rangle \right) \tag{10c}$$

where N is the length of the time series.

Formal expressions for correlations of VAR(1)

Multiplying Eq. (3) from the right by $\boldsymbol{\eta}_n^T$ and taking the statistical average, we get

$$\left\langle \mathbf{x}_n \boldsymbol{\eta}_n^T \right\rangle = \mathbf{A} \left\langle \mathbf{x}_{n-1} \boldsymbol{\eta}_n^T \right\rangle + \boldsymbol{\Sigma} \left\langle \boldsymbol{\eta}_n \boldsymbol{\eta}_n^T \right\rangle \tag{11}$$

The first average on the right-hand side of Eq. (11) vanishes as the process is causal and cannot depend on future noises. The other average gives the unit matrix, $\mathbb{I}$. Therefore,

$$\left\langle \mathbf{x}_n \boldsymbol{\eta}_n^T \right\rangle = \boldsymbol{\Sigma} \tag{12}$$

Now let

$$\boldsymbol{\Gamma}(l) = \left\langle \mathbf{x}_n \mathbf{x}_{n-l}^T \right\rangle = \left\langle \mathbf{x}_n \mathbf{x}_{n+l}^T \right\rangle \tag{13}$$

where the last equality holds by virtue of stationarity.

$$\begin{aligned}
\boldsymbol{\Gamma}(0) &= \left\langle \mathbf{x}_n \mathbf{x}_n^T \right\rangle \\
&= \mathbf{A} \left\langle \mathbf{x}_{n-1} \mathbf{x}_n^T \right\rangle + \boldsymbol{\Sigma} \left\langle \boldsymbol{\eta}_n \mathbf{x}_n^T \right\rangle \\
&= \mathbf{A}\, \boldsymbol{\Gamma}(1)^T + \boldsymbol{\Sigma}\boldsymbol{\Sigma}^T
\end{aligned} \tag{14}$$

On the other hand,

$$
\begin{aligned}
\mathbf{\Gamma}(1) &= \left\langle \mathbf{x}_n \mathbf{x}_{n-1}^T \right\rangle \\
&= \mathbf{A} \left\langle \mathbf{x}_{n-1} \mathbf{x}_{n-1}^T \right\rangle + \mathbf{\Sigma} \left\langle \boldsymbol{\eta}_n \mathbf{x}_{n-1}^T \right\rangle \\
&= \mathbf{A}\mathbf{\Gamma}(0) \qquad (15) \\
\mathbf{\Gamma}(2) &= \mathbf{A}\mathbf{\Gamma}(1) \qquad (16) \\
\mathbf{\Gamma}(3) &= \mathbf{A}\mathbf{\Gamma}(2) \qquad (17) \\
&\dots
\end{aligned}
$$

where we have used stationarity and causality.

Therefore,

$$\mathbf{\Gamma}(l) = \mathbf{A}^l \mathbf{\Gamma}(0) \qquad (18)$$

Finally, to calculate $\mathbf{\Gamma}(0)$, we can combine (14) and (15). First, we transpose (15)

$$\mathbf{\Gamma}(1)^T = \mathbf{\Gamma}(0)\mathbf{A}^T \tag{19}$$

as $\mathbf{\Gamma}(0)$ is symmetric.

Then

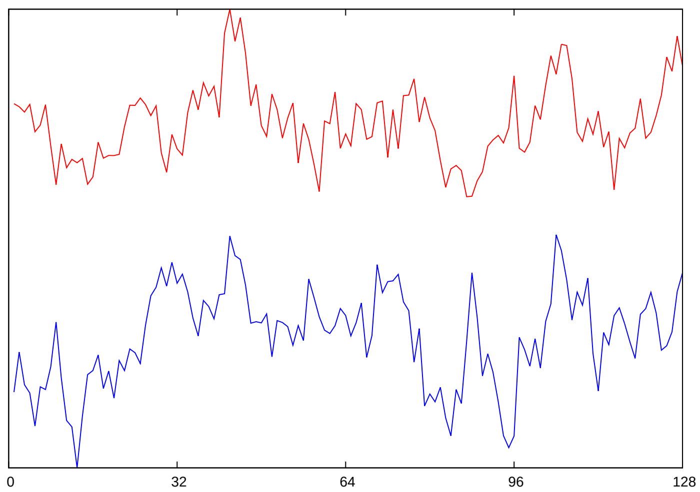
$$\begin{aligned} \mathbf{\Gamma}(0) &= \mathbf{A}\,\mathbf{\Gamma}(1)^T + \mathbf{\Sigma}\mathbf{\Sigma}^T \\ &= \mathbf{A}\mathbf{\Gamma}(0)\mathbf{A}^T + \mathbf{\Sigma}\mathbf{\Sigma}^T \end{aligned} \tag{20}$$

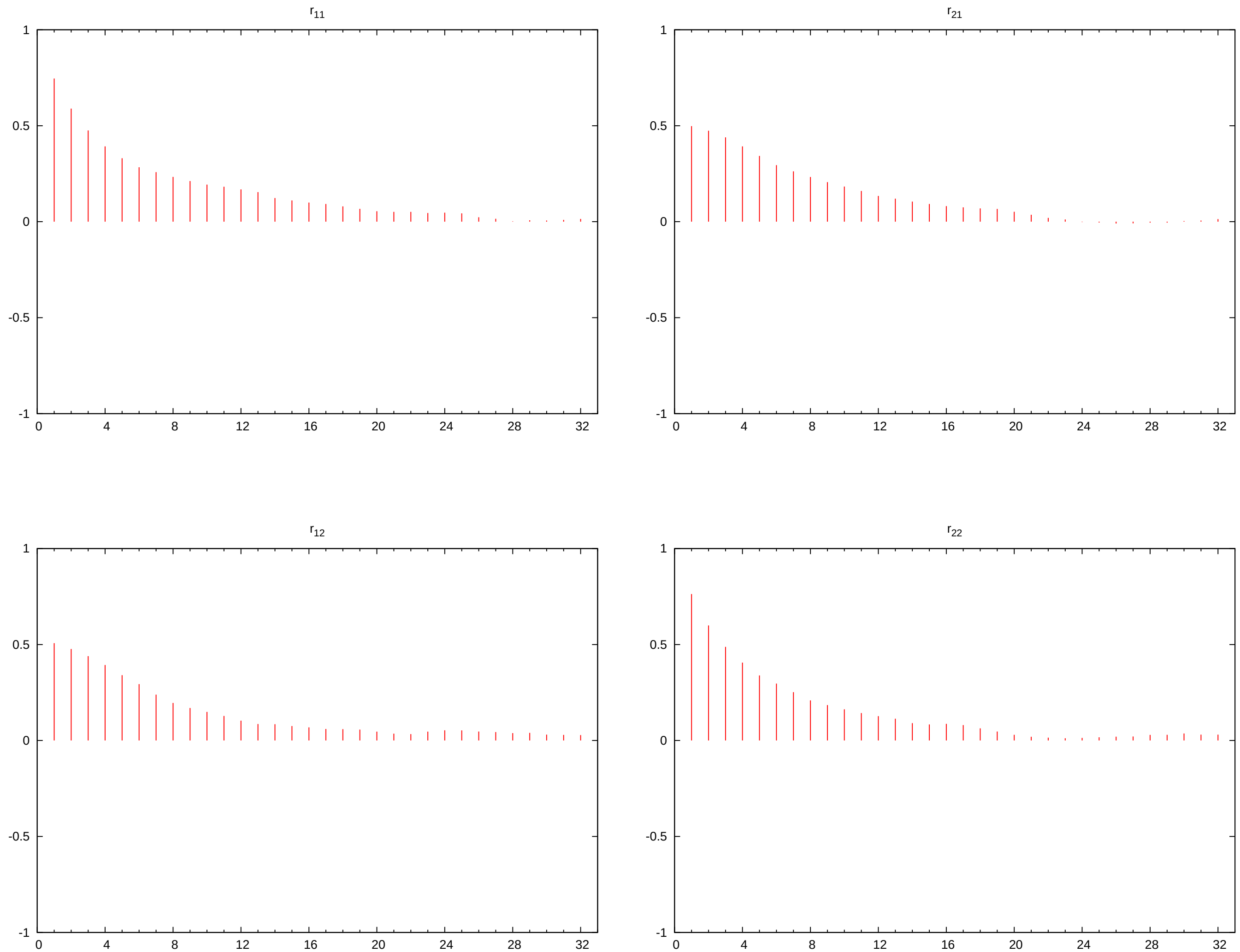
which can be solved for elements of $\mathbf{\Gamma}(0)$. For stationary VAR(1) processes the solution exists.

We can see from Eq. (20) that if $\mathbf{A}$ is not diagonal, $\mathbf{\Gamma}(l > 0)$ is not diagonal, either.

Example 1

$$\begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{5} \\ \frac{1}{5} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_{n-1}^1 \\ x_{n-1}^2 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} \eta_n^1 \\ \eta_n^2 \end{bmatrix} \tag{21}$$

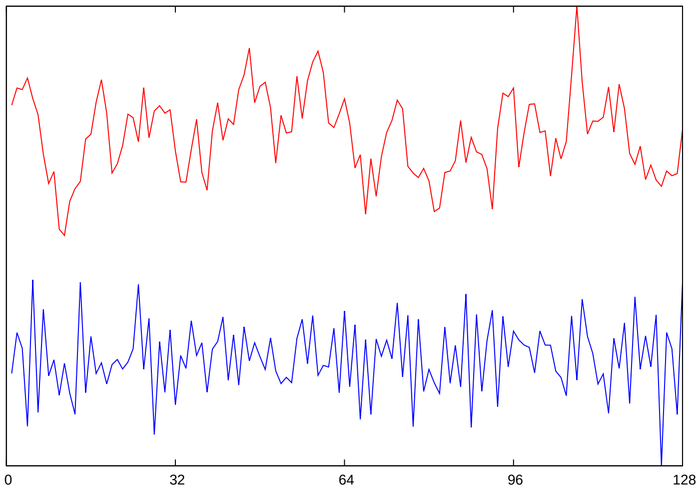


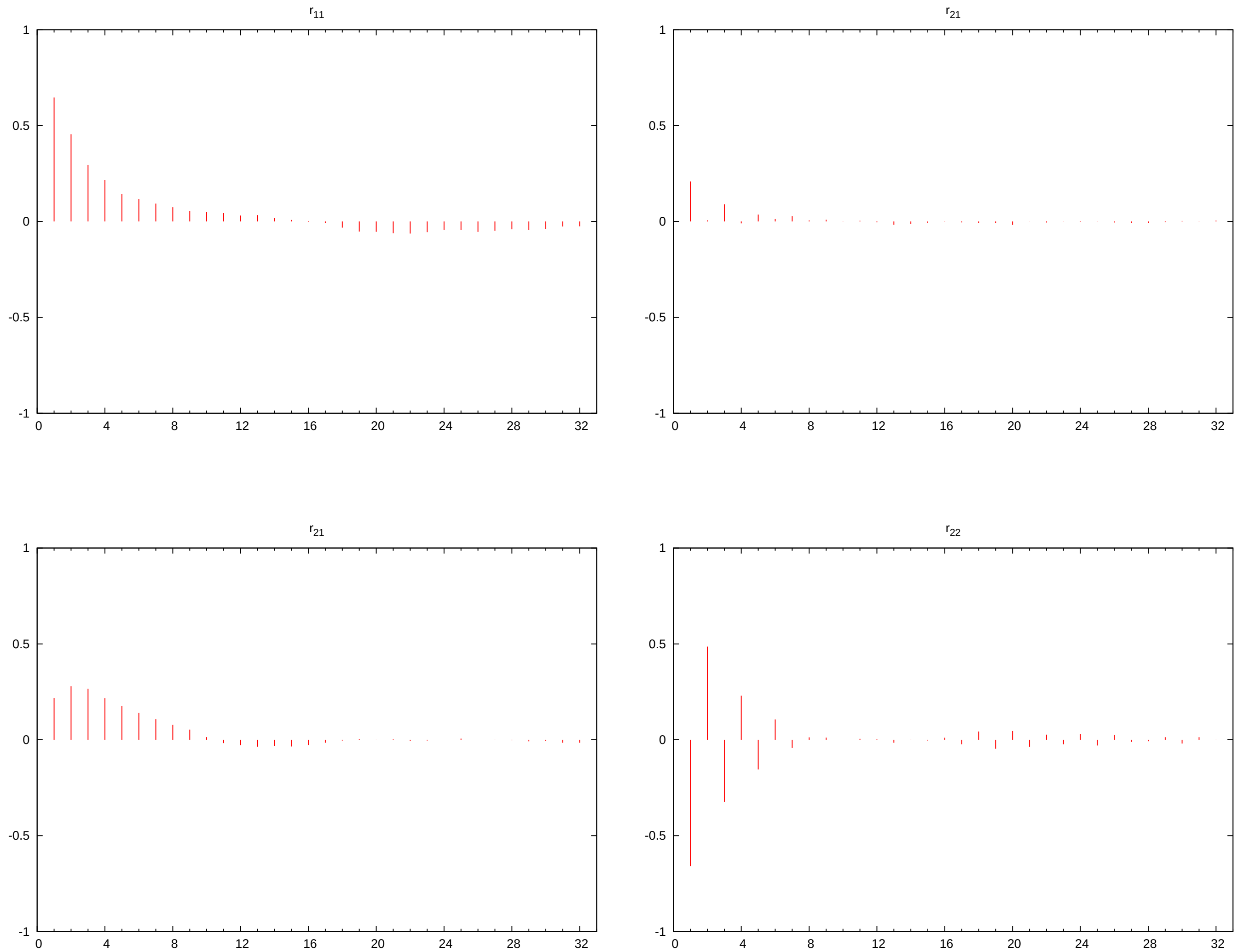


r_{11}
r_{21}
r_{12}
r_{22}

Example 2

$$\begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{5} \\ \frac{1}{5} & -\frac{2}{3} \end{bmatrix} \begin{bmatrix} x_{n-1}^1 \\ x_{n-1}^2 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} \eta_n^1 \\ \eta_n^2 \end{bmatrix} \tag{22}$$

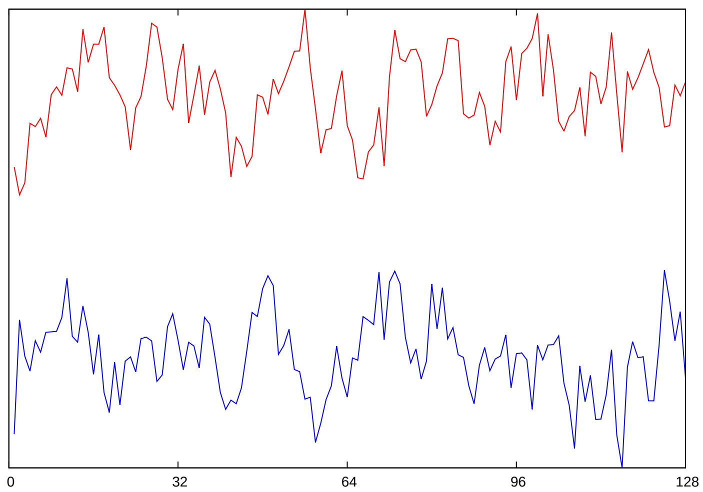


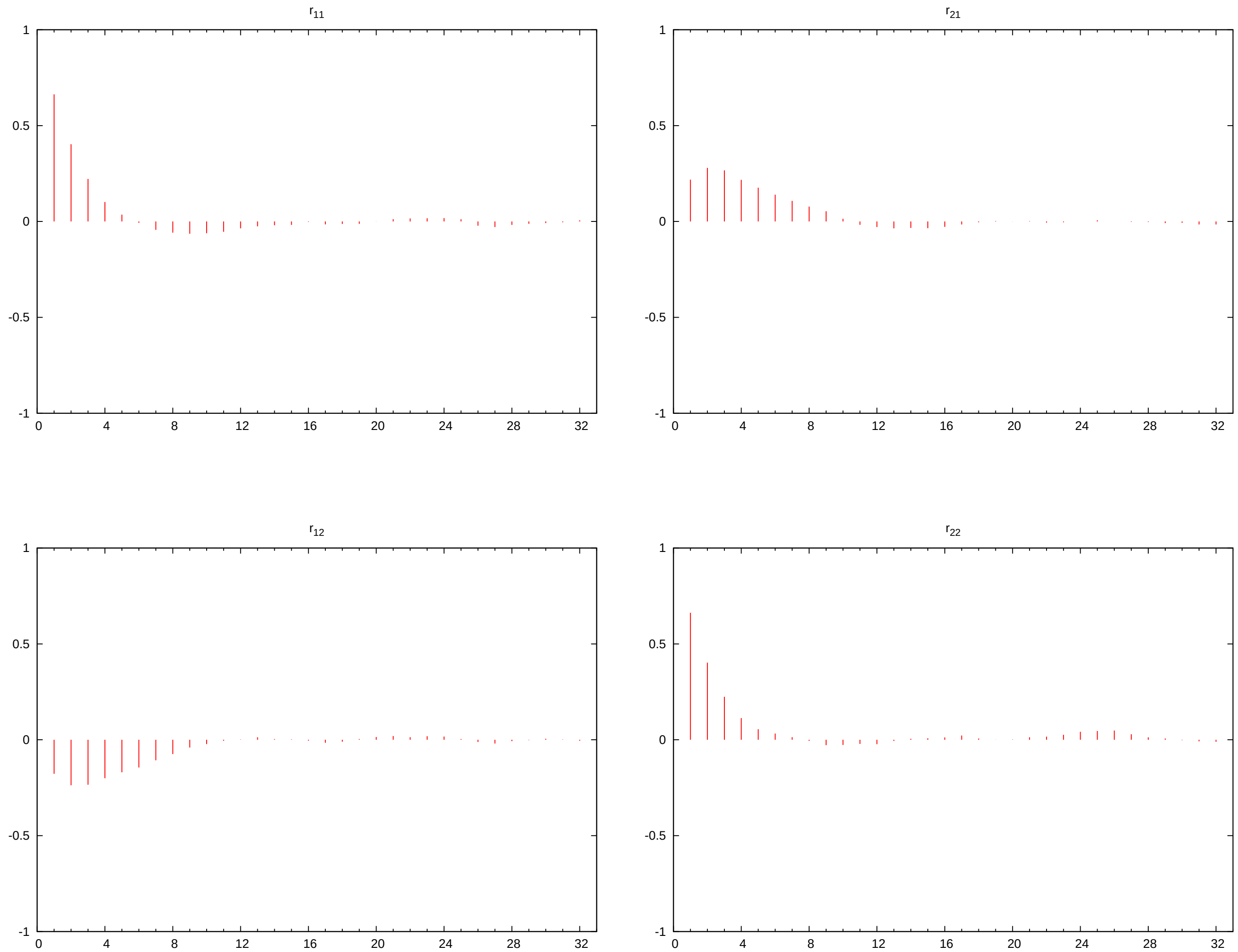


r_{11}
r_{21}
r_{21}
r_{22}

Example 3

$$\begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{5} \\ -\frac{1}{5} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_{n-1}^1 \\ x_{n-1}^2 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} \eta_n^1 \\ \eta_n^2 \end{bmatrix} \tag{23}$$

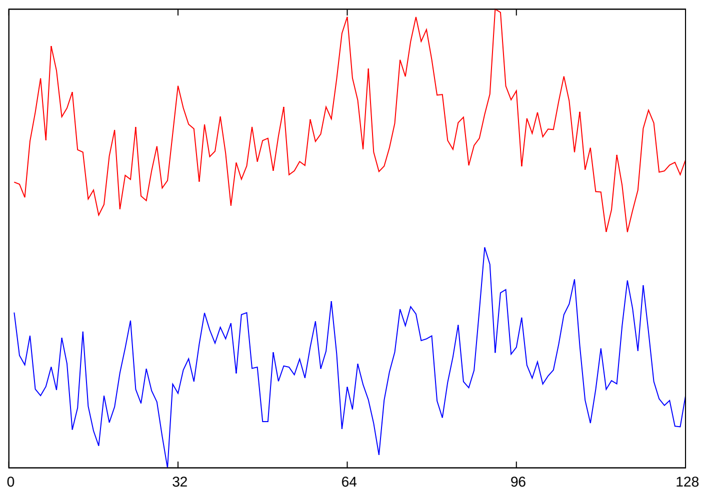


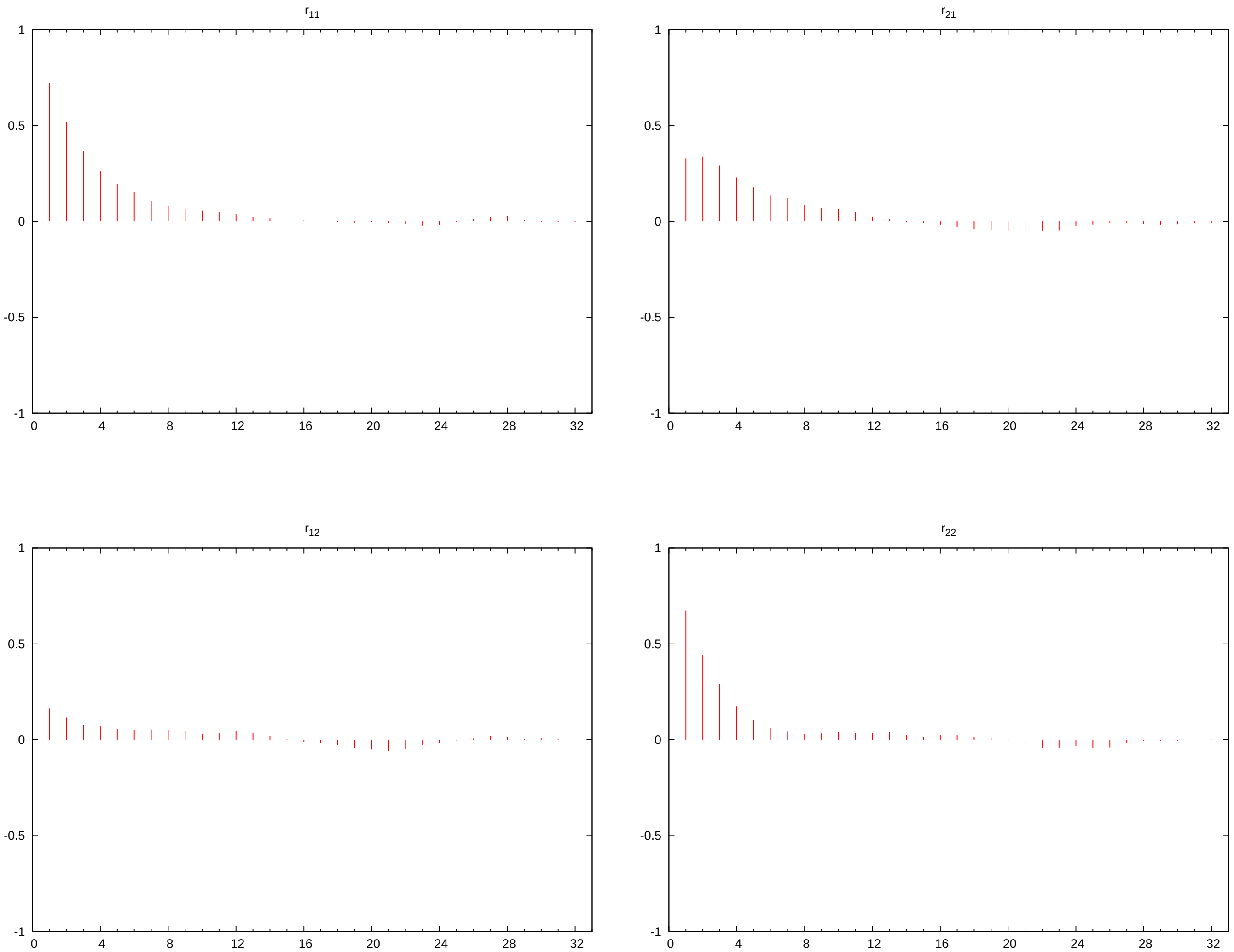


r_{11}
r_{21}
r_{12}
r_{22}
1
0.5
0
-0.5
-1
0
4
8
12
16
20
24
28
32

Example 4 — one process drives another

$$\begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{5} \\ 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_{n-1}^1 \\ x_{n-1}^2 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} \eta_n^1 \\ \eta_n^2 \end{bmatrix} \tag{24}$$





r11
r21
r12
r22
1
0.5
0
-0.5
-1
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12
16
20
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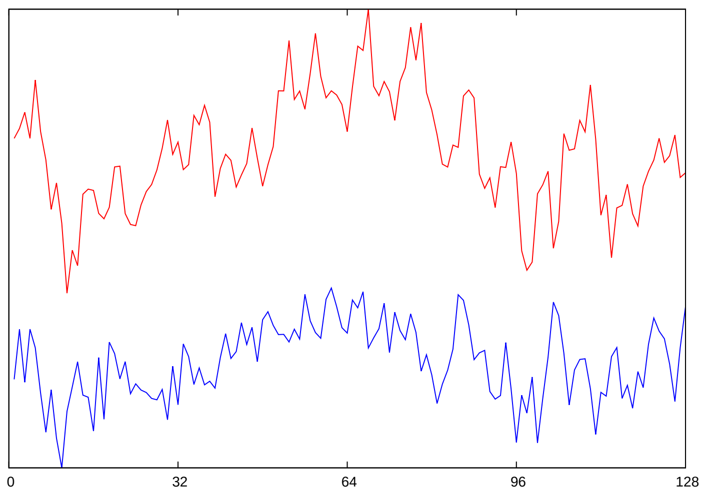
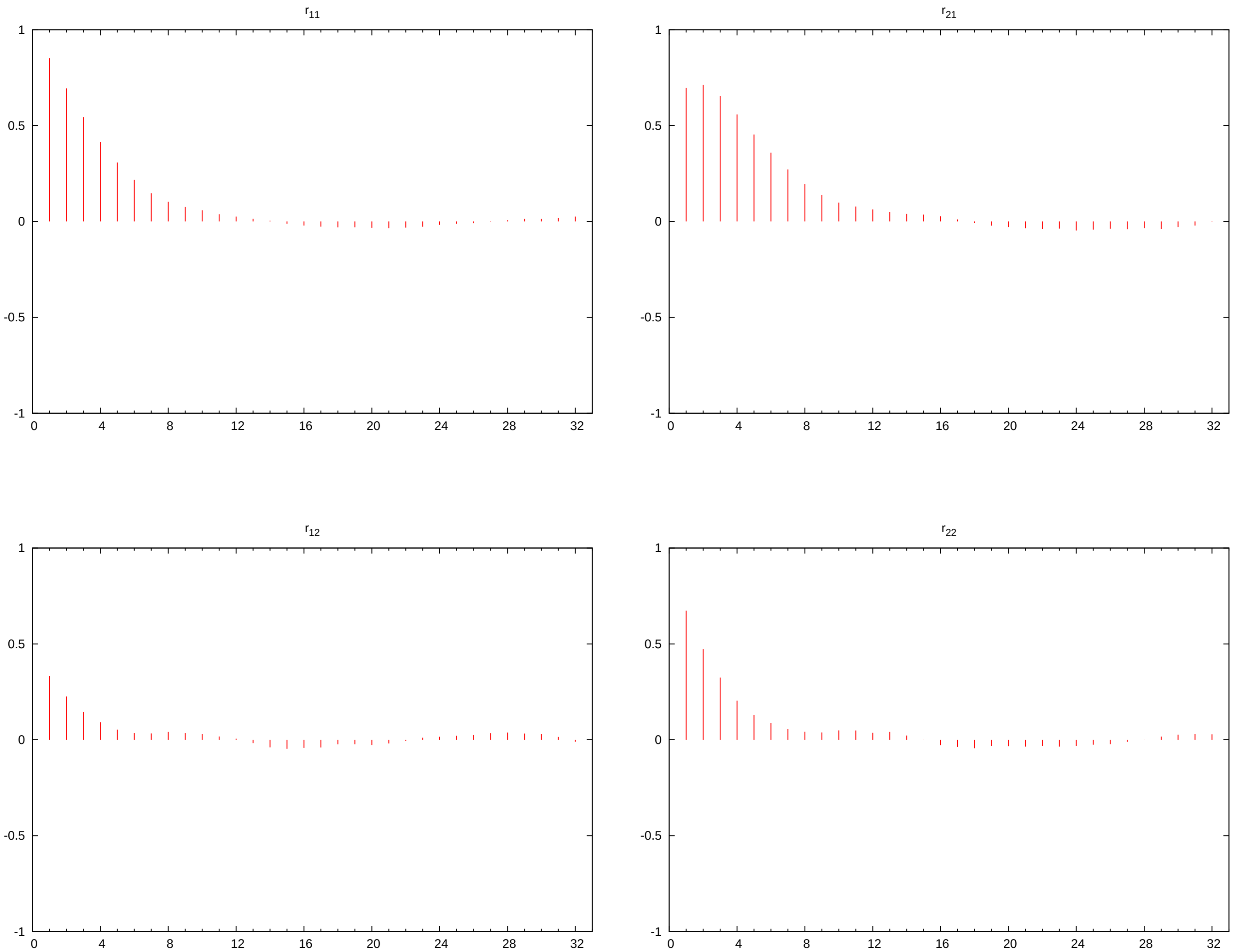
28
32

Example 5 — “non-diagonalizable” process

$$\begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_{n-1}^1 \\ x_{n-1}^2 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} \eta_n^1 \\ \eta_n^2 \end{bmatrix} \tag{25}$$




r11
r21
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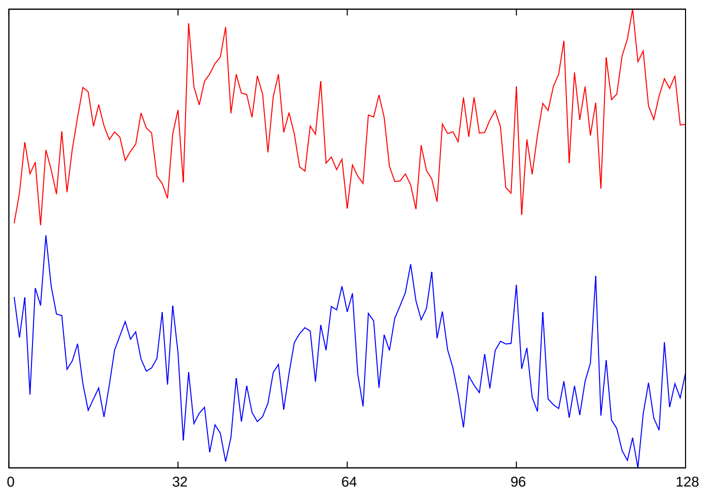
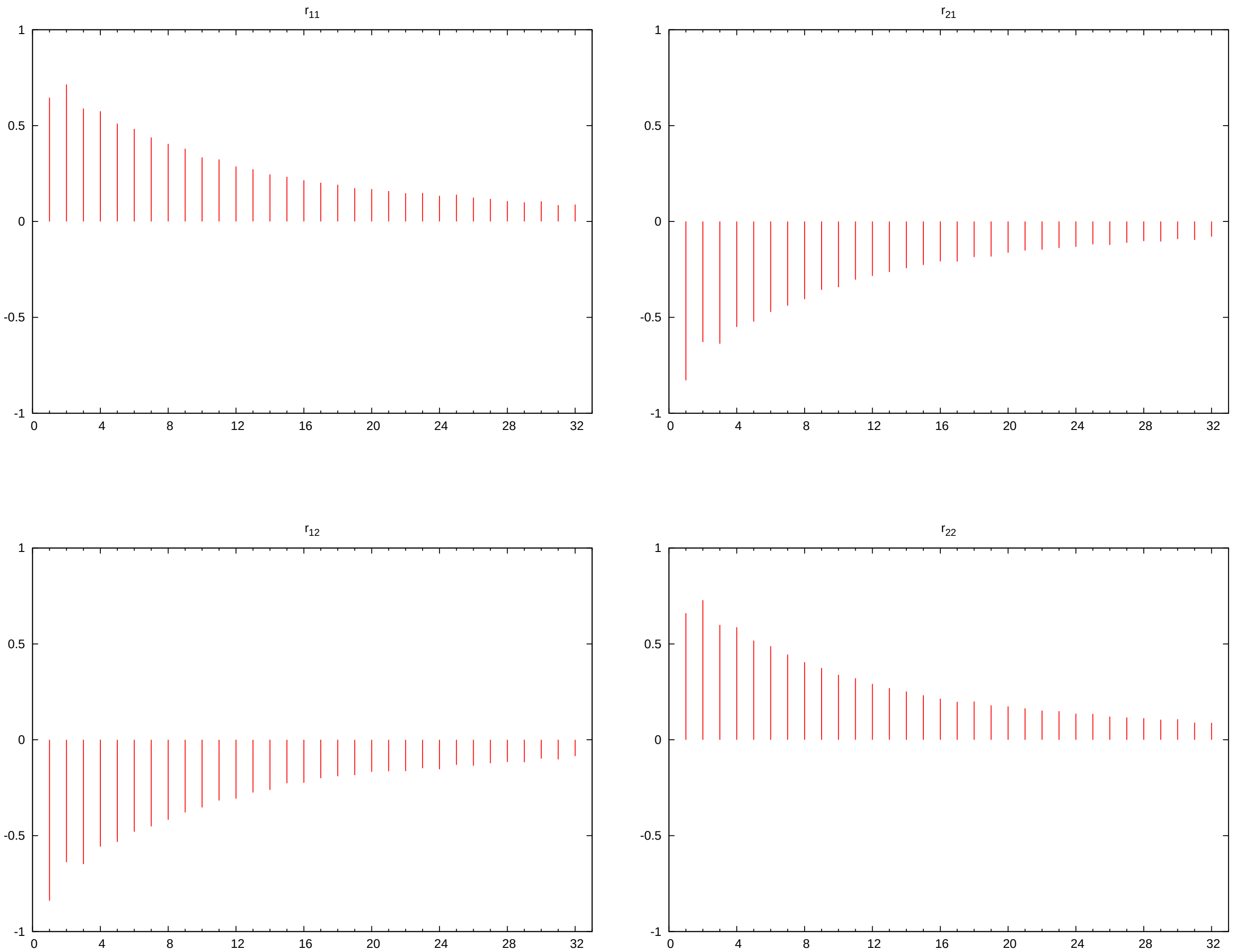
1
0.5
0
-0.5
-1
0 4 8 12 16 20 24 28 32

Example 6 — non-symmetric matrix, negative cross-correlations

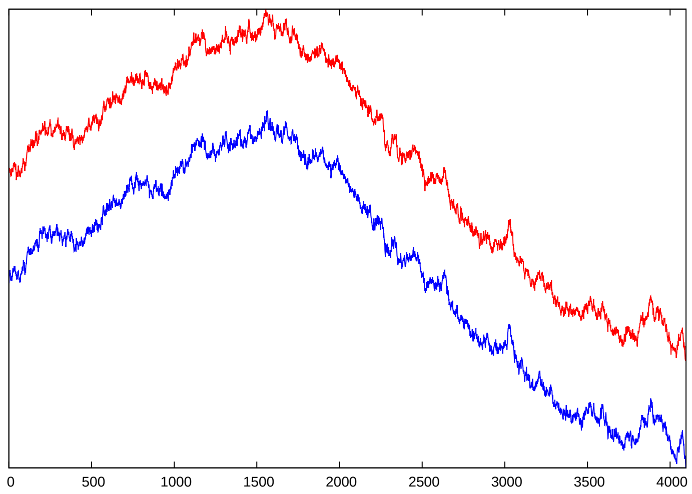
$$\begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & -\frac{2}{3} \\ -\frac{3}{4} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} x_{n-1}^1 \\ x_{n-1}^2 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} \eta_n^1 \\ \eta_n^2 \end{bmatrix} \tag{26}$$



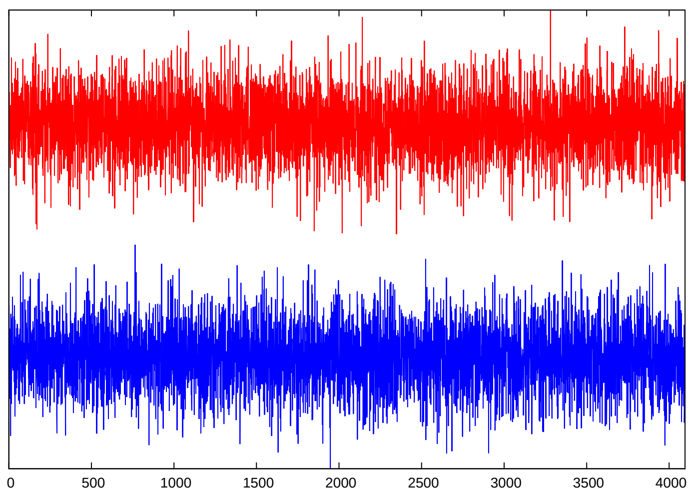

r_{11}
r_{21}
r_{12}
r_{22}
1
0.5
0
-0.5
-1
0 4 8 12 16 20 24 28 32

Example 7 — a linear trend

$$\begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} = \begin{bmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix} \begin{bmatrix} x_{n-1}^1 \\ x_{n-1}^2 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} \eta_n^1 \\ \eta_n^2 \end{bmatrix} \tag{27}$$



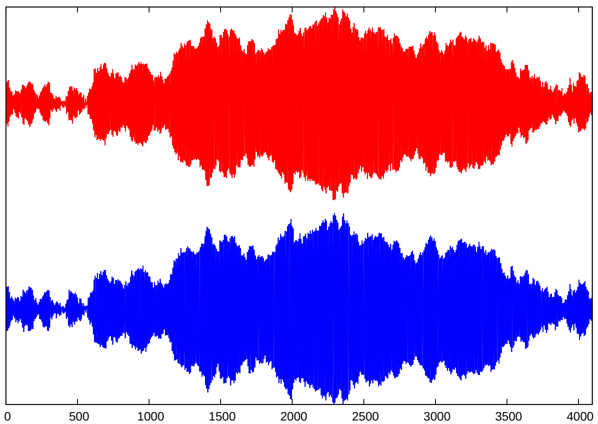
The matrix in (27) has eigenvalues $1, \frac{1}{2}$. The unit eigenvalue causes a linear trend. The series of first differences, $x^1_{n+1} - x^1_n, x^2_{n+1} - x^2_n$ are stationary.



Example 8 — another kind of nonstationarity

$$\begin{bmatrix} x_n^1 \\ x_n^2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_{n-1}^1 \\ x_{n-1}^2 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} \eta_n^1 \\ \eta_n^2 \end{bmatrix} \tag{28}$$

This matrix has eigenvalues $\lambda_{1,2} = \frac{1}{\sqrt{2}}(1 \pm i)$, $\left|\lambda_{1,2}\right| = 1$.



Noise induced correlations

If the noises are correlated, components of a multivariate process can be correlated even though there is no *direct* interactions between them — see note below Eq. (20). Consider process (3) but with a *diagonal* $\mathbf{A} = \mathtt{diag}\{\alpha_1, \dots, \alpha_m\}$ and a *not diagonal* $\mathbf{\Sigma}$. In Eqns. (18),(18) $\mathbf{A}^l$ are diagonal, but as $\mathbf{\Sigma}\mathbf{\Sigma}^T$ is not diagonal, $\mathbf{\Gamma}(l)$ has off-diagonal terms corresponding to cross-correlations.

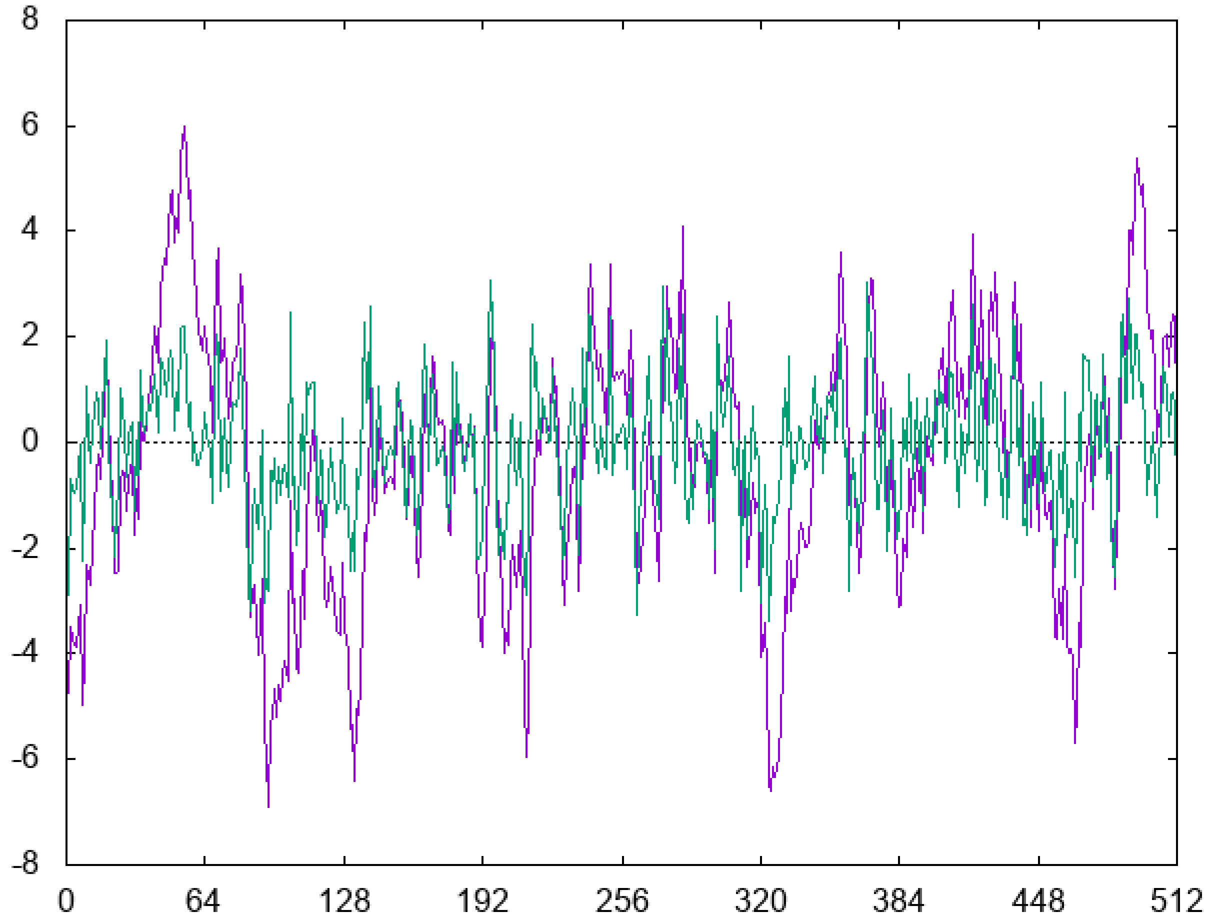
Example 9

The simples case of noise-induce correlations occurs when the noises acting on all components of a multivariate process are identical. Consider

$$x_n^1 = \alpha_1 x_{n-1}^1 + \sigma\eta_n \tag{29a}$$

$$x_n^2 = \alpha_2 x_{n-1}^2 + \sigma\eta_n \tag{29b}$$

with $\alpha_1 = 0.9$, $\alpha_2 = 0.5$, $\sigma = 1$.



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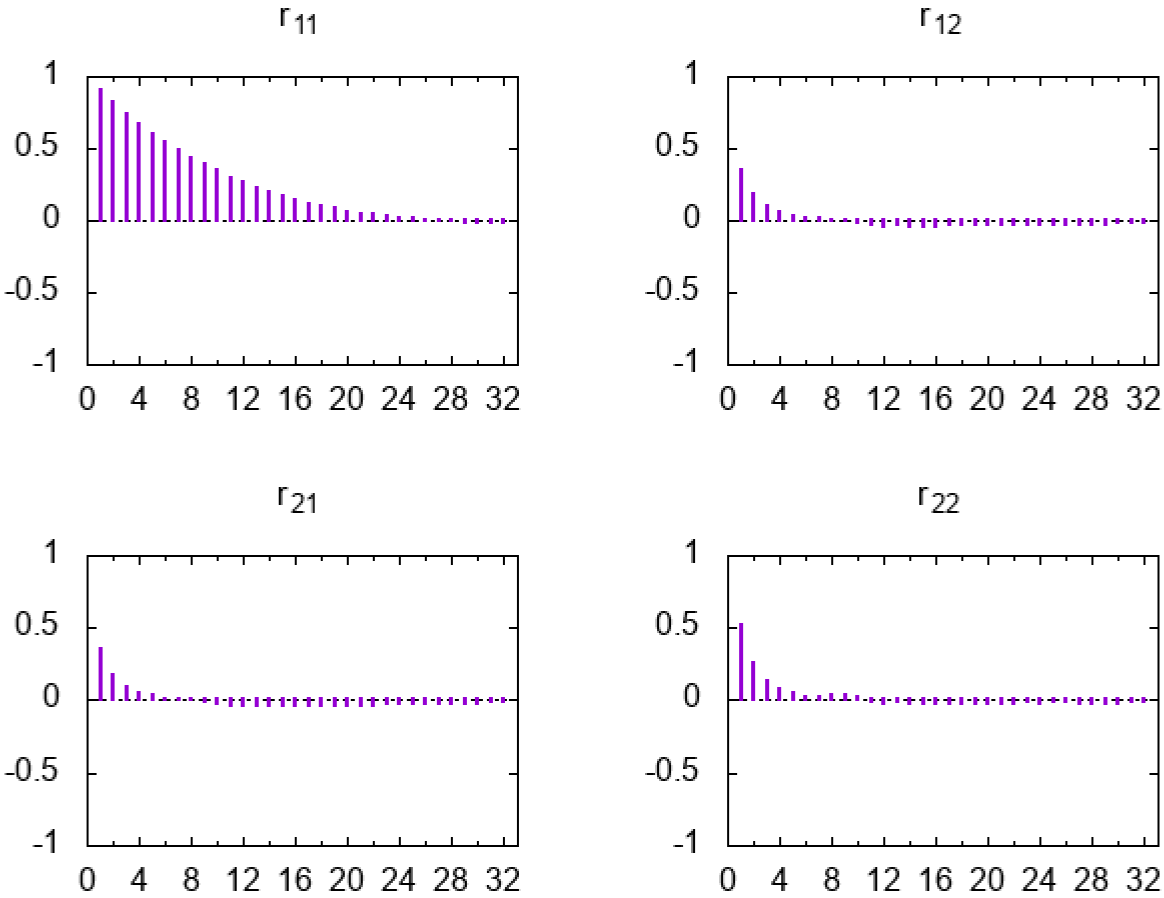
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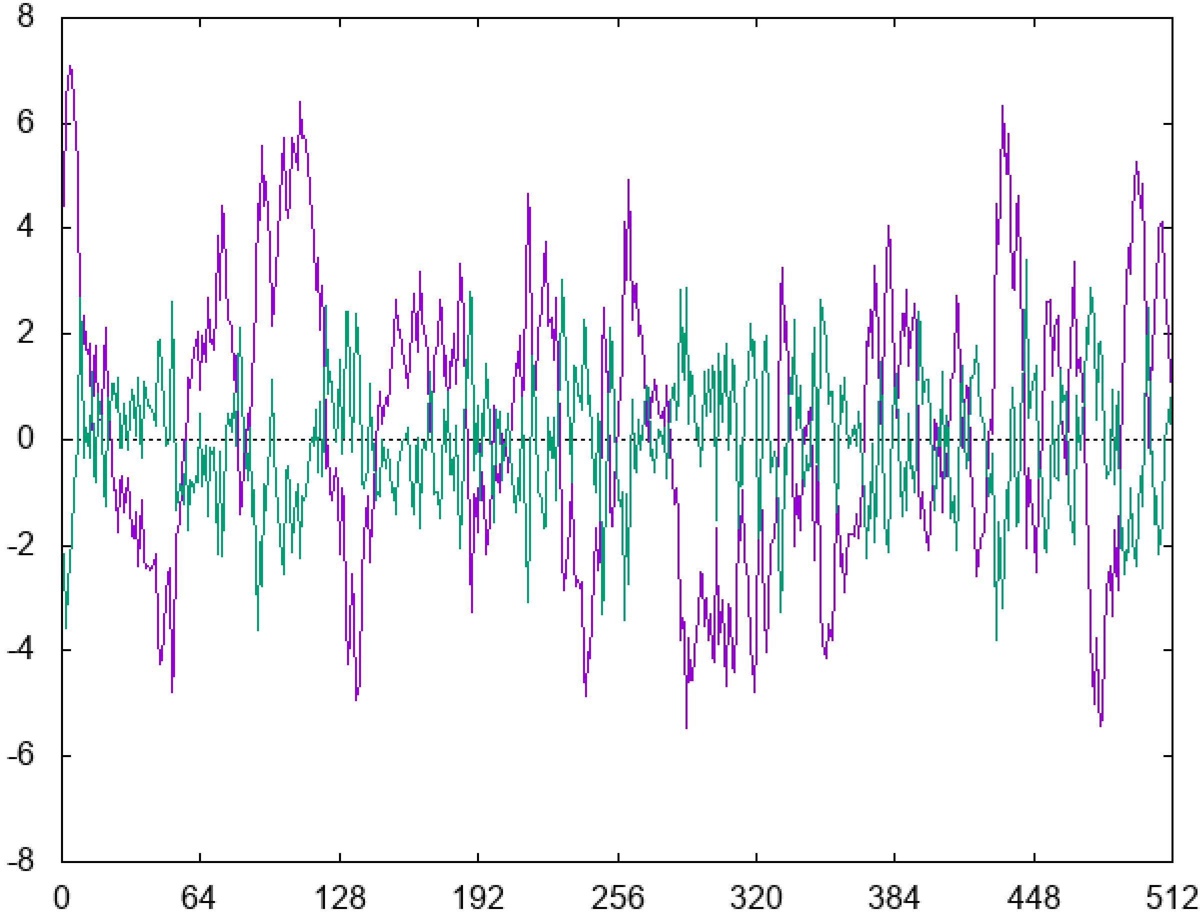
Example 10

$$x_n^1 = \alpha_1 \, x_{n-1}^1 + \sigma\eta_n \tag{30a}$$
$$x_n^2 = \alpha_2 \, x_{n-1}^2 - \sigma\eta_n \tag{30b}$$

with $\alpha_1 = 0.9$, $\alpha_2 = 0.5$, $\sigma = 1$.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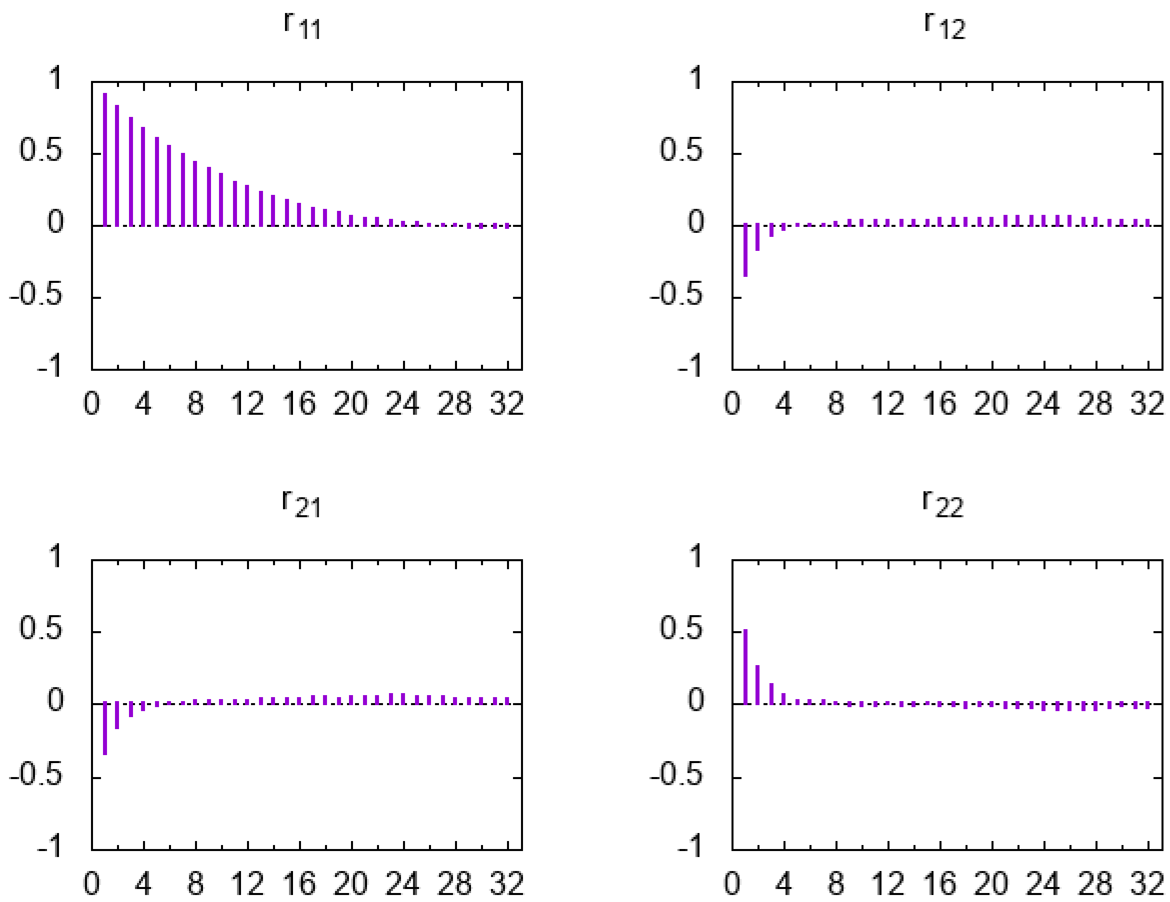
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Fitting parameters to a VAR(1) model

The following procedure works for stationary VAR(1) processes only. If components of a multivariate process display apparent nonstaionarity, like trends or seasonalities, we should detrend them first by taking series of first differences, or of differences between terms offset by some $k > 1$ in case seasonalities, and fit parameters to the stationary series that results.

We replace exact values of $\mathbf{\Gamma}(l)$ by their "experimental" approximations, calculated form the multivariate series at hand. Then we use Eq. (15) to calculate elements of $\mathbf{A}$:

$$\mathbf{\Gamma}(1) = \mathbf{A}\mathbf{\Gamma}(0) \tag{31}$$

This is a set of linear equations for elements of $\mathbf{A}$. It is quite easy to solve, but the approximations obtained can possibly carry some error. Therefore, we sometimes extend our set of equations to

$$
\begin{aligned}
\boldsymbol{\Gamma}(1) &= \mathbf{A}\boldsymbol{\Gamma}(0) && \text{(32a)} \\
\boldsymbol{\Gamma}(2) &= \mathbf{A}\boldsymbol{\Gamma}(1) && \text{(32b)} \\
\boldsymbol{\Gamma}(3) &= \mathbf{A}\boldsymbol{\Gamma}(2) && \text{(32c)} \\
&\cdots \\
\boldsymbol{\Gamma}(p) &= \mathbf{A}\boldsymbol{\Gamma}(p-1) && \text{(32d)}
\end{aligned}
$$

The set of linear equations (32) for elements of the matrix $\mathbf{A}$ is overdetermined and we cannot solve it exactly. We can, however, find its *best approximate* (in the sense of least squares) solution by using *SVD* or by a direct minimisation.

Finally, having calculated $\mathbf{A}$, we can use Eq. (14) to calculate elements od $\mathbf{\Sigma}\mathbf{\Sigma}^T$. One could think of using Eq. (12) to calculate elements of $\mathbf{\Sigma}$. Unfortunatelly, we usually do not know the noise.