Time Series Analysis:

11. Wavelet denoising

Wavelts in image analysis

P. F. Góra

http://th-www.if.uj.edu.pl/zfs/gora/

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The wavelet spectrum

Periodogram — how much power is transported by each Fourier frequency, or at each period.

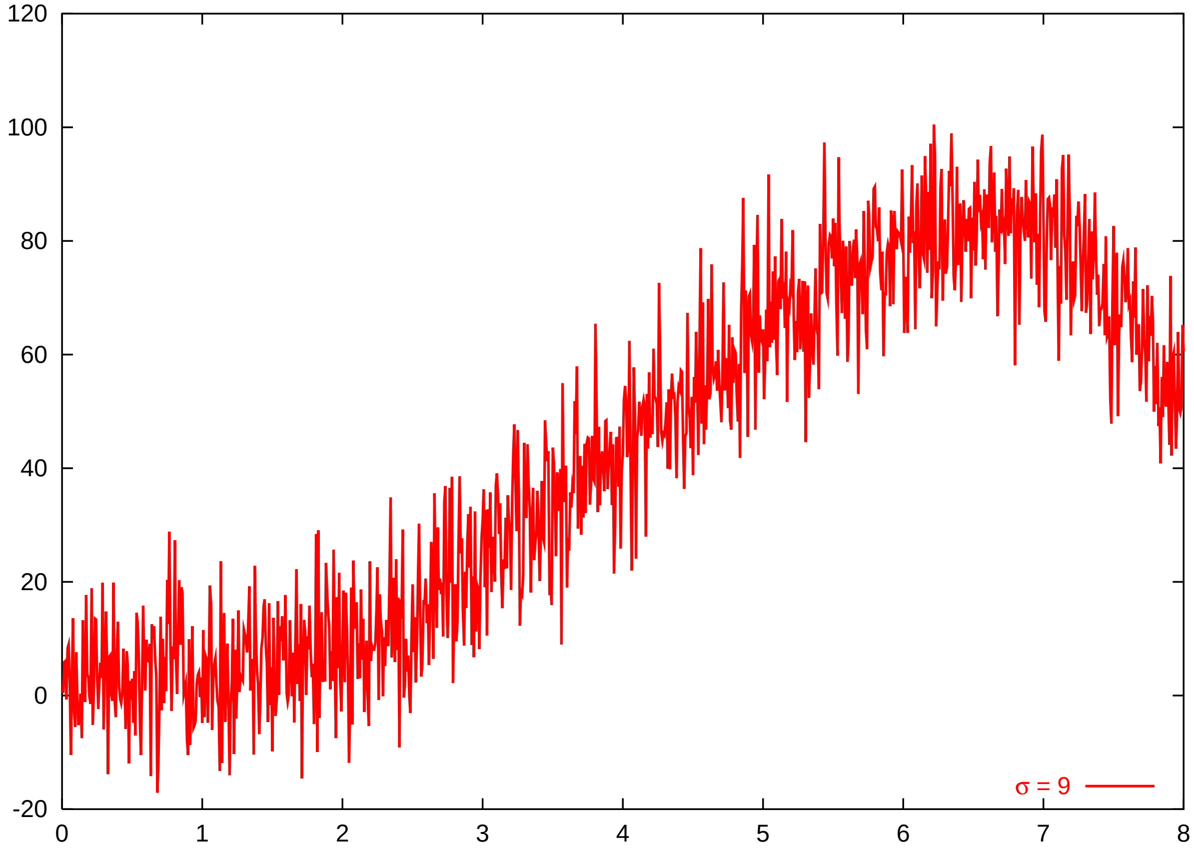
Scalogram — how much power is transformed at each scale.

Let $\{W_i\}$ be the wavelet coefficients of a time series and τ_s be the length of the smallest wavelet in the basis.

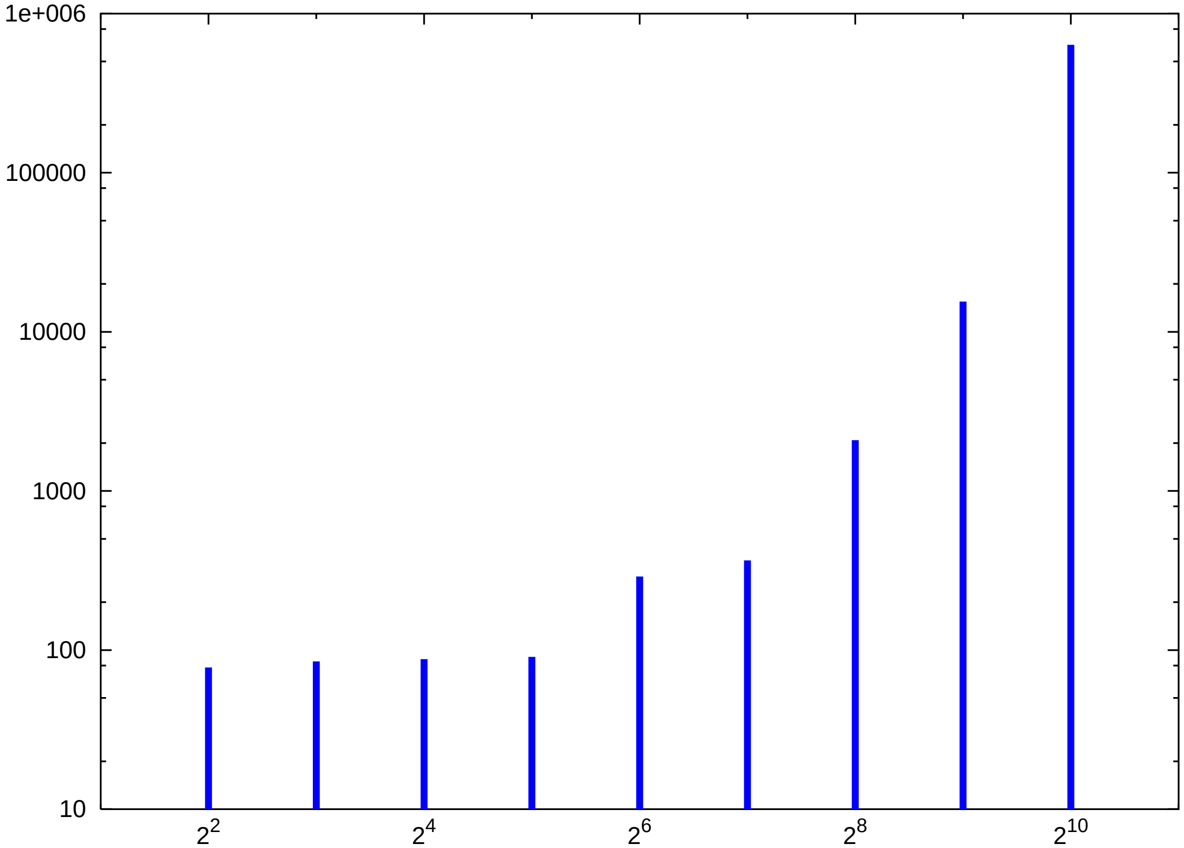
$$v(\tau_j) = \frac{1}{2^{j-1}} \sum_{i=2^{j-1}}^{2^j-1} W_i^2 \qquad j = 1, 2, \ldots, s. \tag{1}$$

$v(\tau_j)$ — the average power transferred at scale τ_j.

A signal. . .



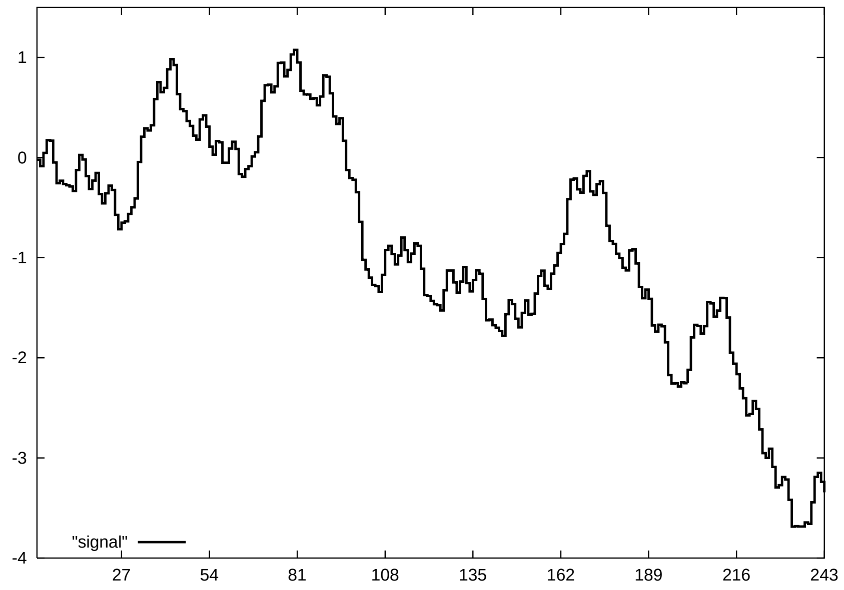
. . . and its scalogram (wavelet spectrum) calculated with DAUB(4)

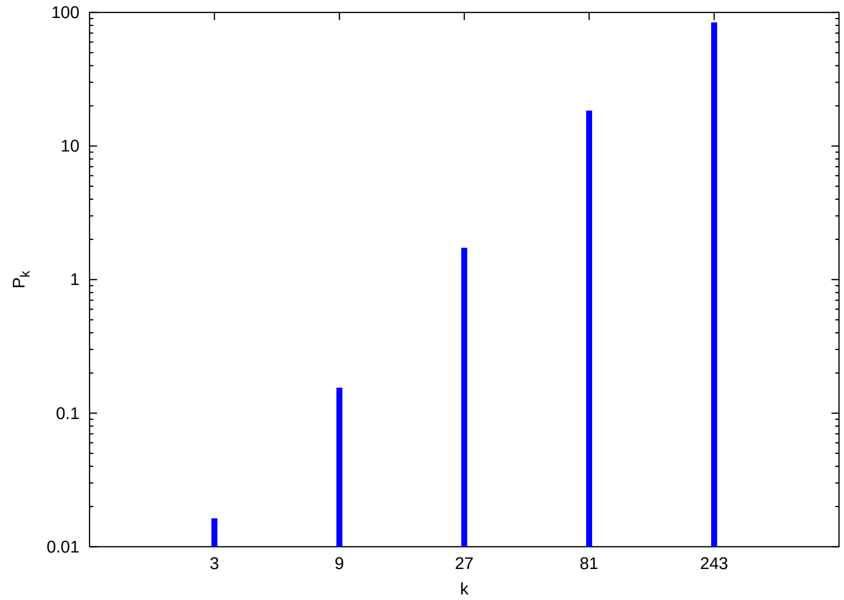


Scalogram and three-point wavelets

Two point wavelets	Three point wavelets
$P_2 = \dfrac{2}{N} \sum\limits_{i=N/2+1}^{N} d_i^2$	$P_3 = \dfrac{3}{2N} \sum\limits_{i=N/3+1}^{N} d_i^2$
$P_4 = \dfrac{4}{N} \sum\limits_{i=N/4+1}^{N/2} D_i^2$	$P_9 = \dfrac{9}{2N} \sum\limits_{i=N/9+1}^{N/3} D_i^2$
$P_8 = \dfrac{8}{N} \sum\limits_{i=N/8+1}^{N/4} \mathcal{D}_i^2$	$P_{27} = \dfrac{27}{2N} \sum\limits_{i=N/27+1}^{N/9} \mathcal{D}_i^2$
$\ldots$	$\ldots$

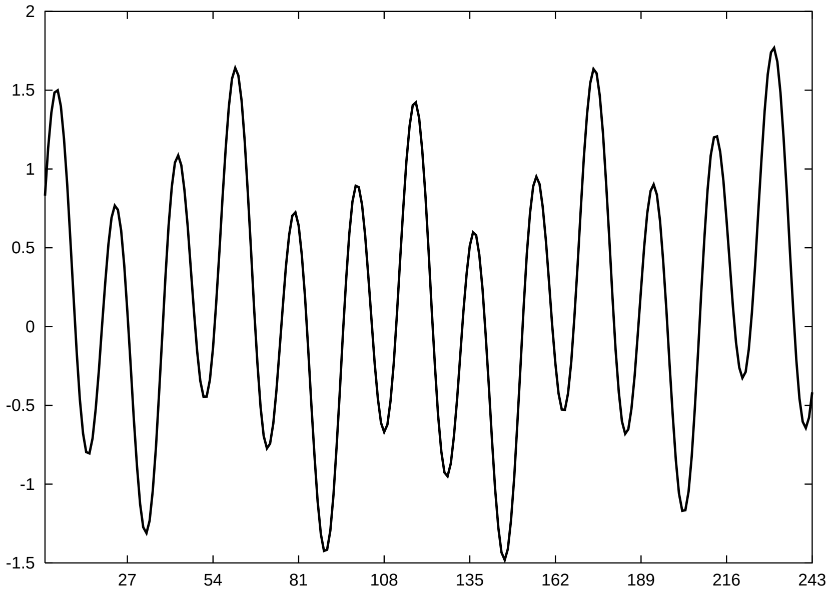
A signal and its scalogram

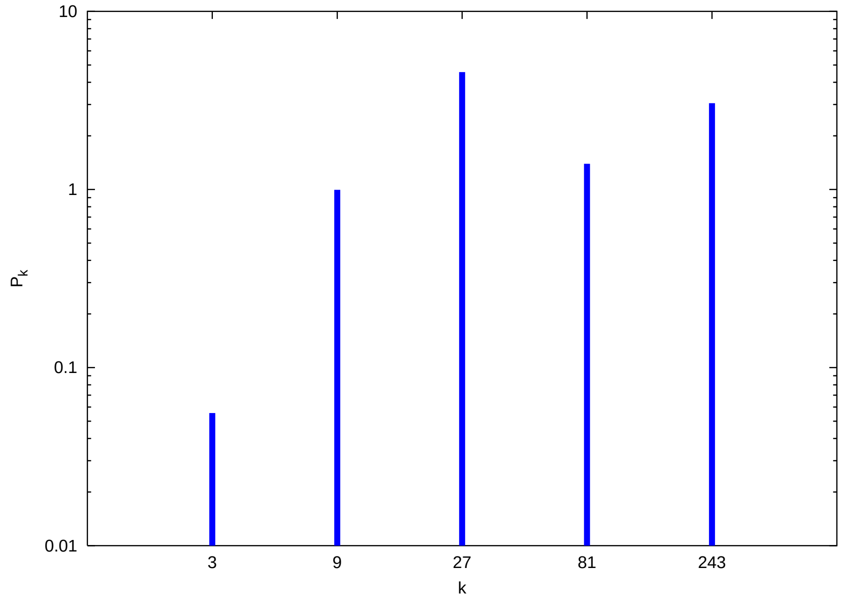




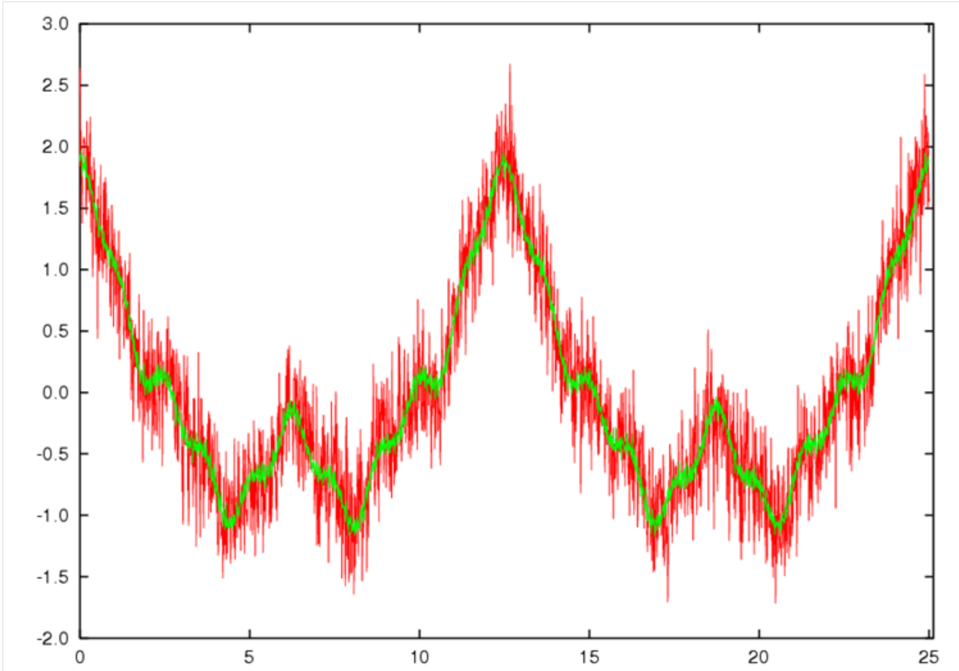
Note that the scalogram is nearly linear. This suggests that the signal is self-similar.

Another signal and its scalogram





Signal (and image) compression



Lossy compression — Genghis Khan rule reversed: survival of the strongest

Wavelet denoising

Better than Wiener filter: can be used with nonstationary signal (with constant noise parameters).

Assume that the signal $\{x_i\}_{i=0}^{N-1}$ has the form

$$x_i = f(i) + \sigma\,\eta_i \tag{2}$$

where $f(i)$ is a "deterministic" component and η_i is a GWN, uncorrelated with the signal. Like in the Wiener filter, we kill (zero) the noise-dominated components. Because the noise is Gaussian, we can estimate the *threshold* as

$$\delta = \sqrt{2\log_2 N}\,\sigma \tag{3}$$

Compression rules

Let $\{W_i\}$ be the wavelet coefficients. We use one of the three rules:

Hard rule:

$$\widetilde{W_i} = \begin{cases} W_i & |W_i| \geqslant \delta \\ 0 & |W_i| < \delta \end{cases} \tag{4}$$

The after thresholding wavelet coefficients can exhibit jumps, which is not welcome.

Soft rule:

$$\widetilde{W_i} = \begin{cases} \text{sgn}(W_i)\left(|W_i| - \delta\right) & |W_i| \geqslant \delta \\ 0 & |W_i| < \delta \end{cases} \tag{5}$$

This leads to an unwelcome reduction in *large* wavelet coefficients.

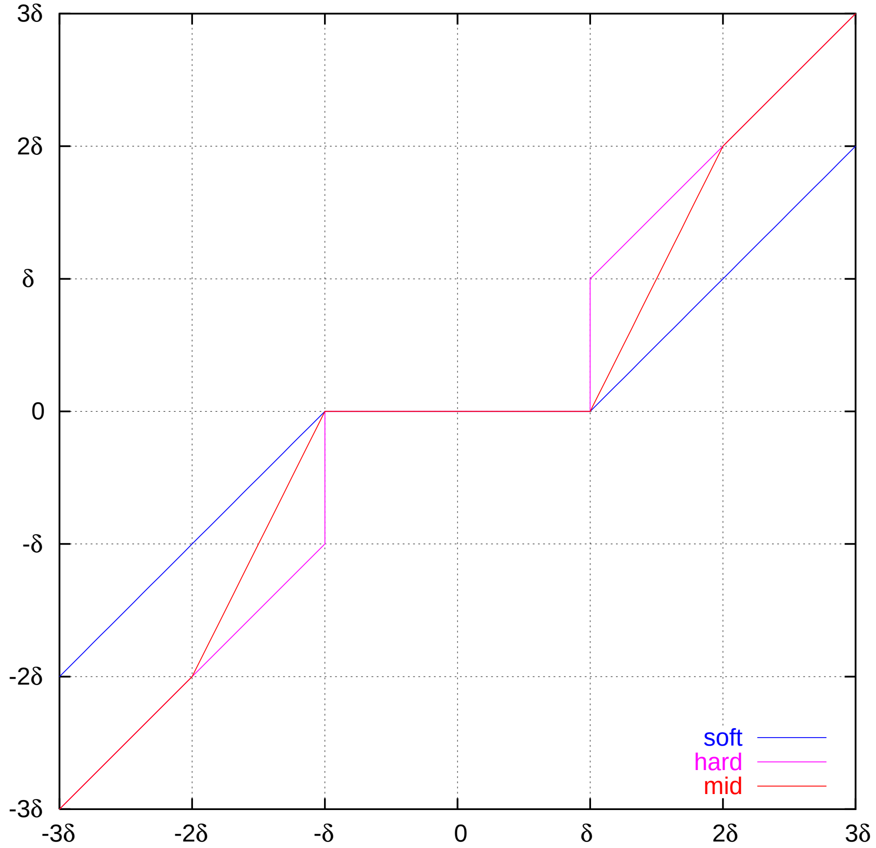
Mid rule:

$$\widetilde{W_i} = \begin{cases} W_i & |W_i| \geqslant 2\delta \\ 2\,\mathrm{sgn}(W_i)\,(|W_i| - \delta) & \delta \leqslant |W_i| < 2\delta \\ 0 & |W_i| < \delta \end{cases} \tag{6}$$

Large coefficients are not affected, the small are killed, the intermediate are reduced. More complicated formulas, with smooth changes, can also be used.

Note that each method leaves *the same number* of non-zero coefficients $\widetilde{W_i}$.

Compression rules illustrated



How do we assess the noise level?

Usually the noise level, σ, is unknown. How to assess it based on the noisy signal alone?

1. White noise is “fast”, so we assume that it brings contributions in the smallest scale mostly.

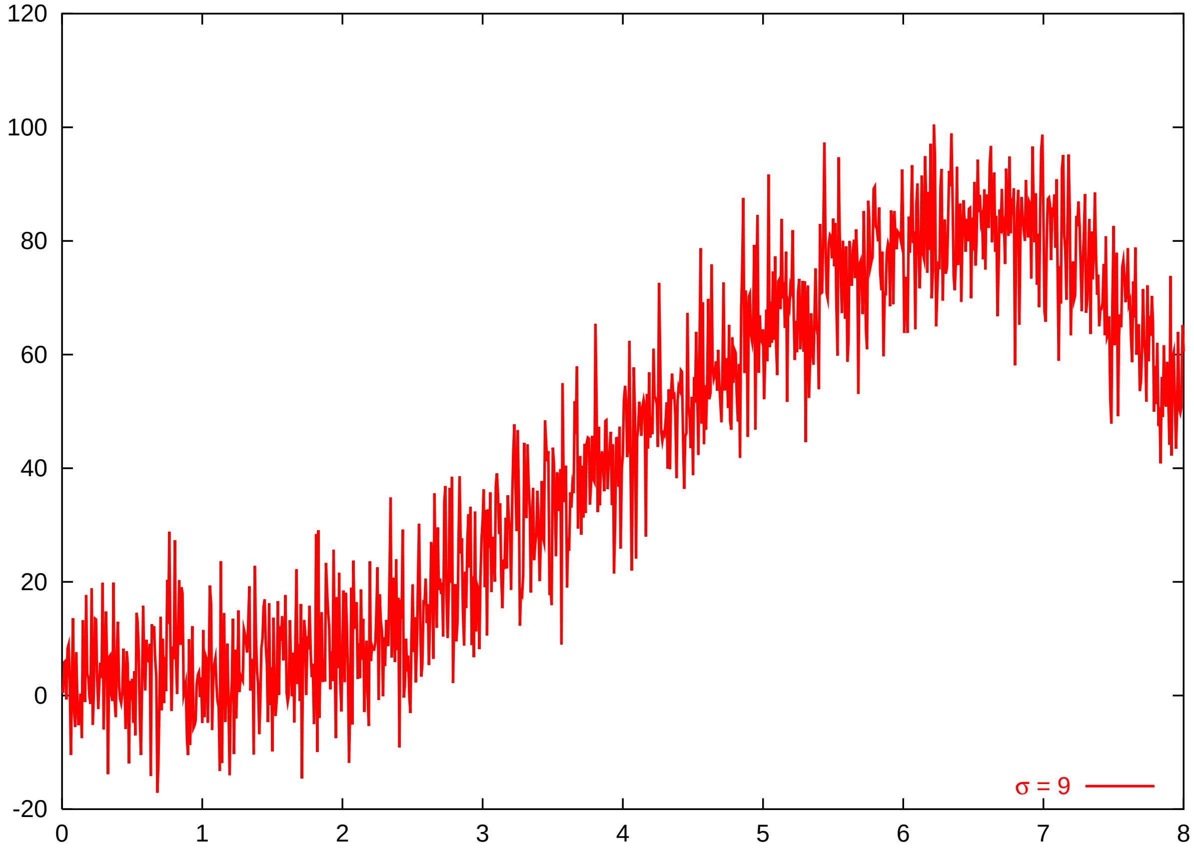
2. Therefore, we assess

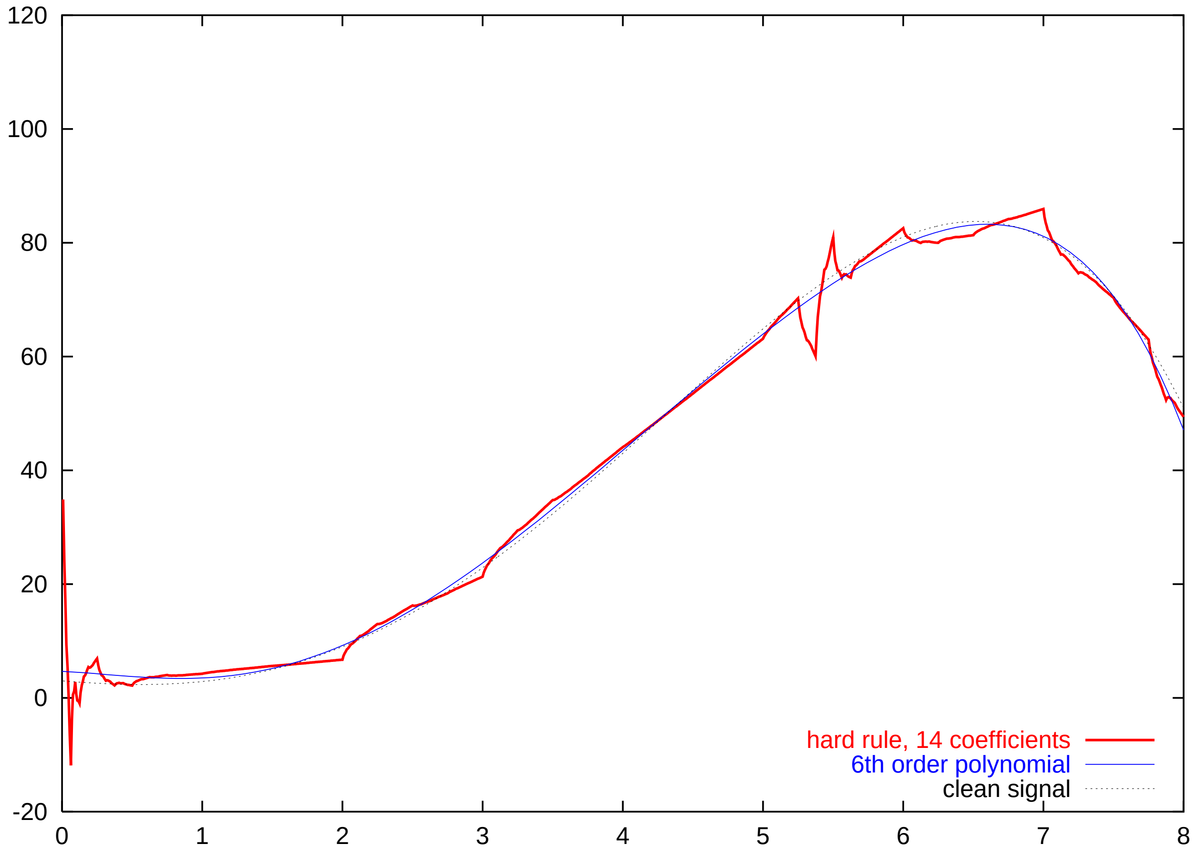
$$\sigma = \frac{1}{0.6745} \text{median}\left\{ \left| W_{N/2} \right|, \left| W_{N/2+1} \right|, \left| W_{N/2+2} \right|, \dots, \left| W_{N-1} \right| \right\} \quad (7)$$

3. When compressing, we *never* change the W_0 and W_1 components (the ones with largest support), as they contain the long-wave information on the signal, in particular, the average.

Similarly, if we do a partial wavelet transform only, we never change the untransformed (slowly varying) part. Compress the fast varying part only and add to the uncompressed slowly varying part.

A noisy signal





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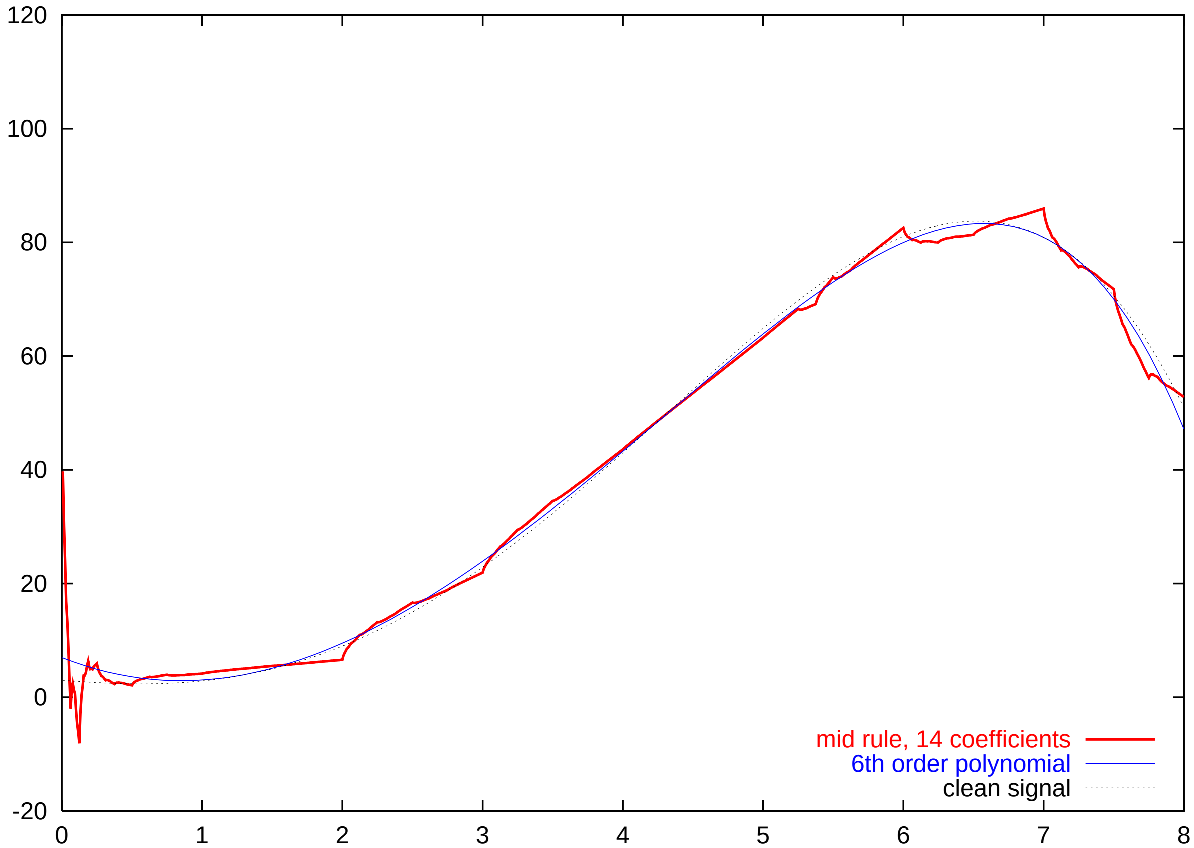
hard rule, 14 coefficients
6th order polynomial
clean signal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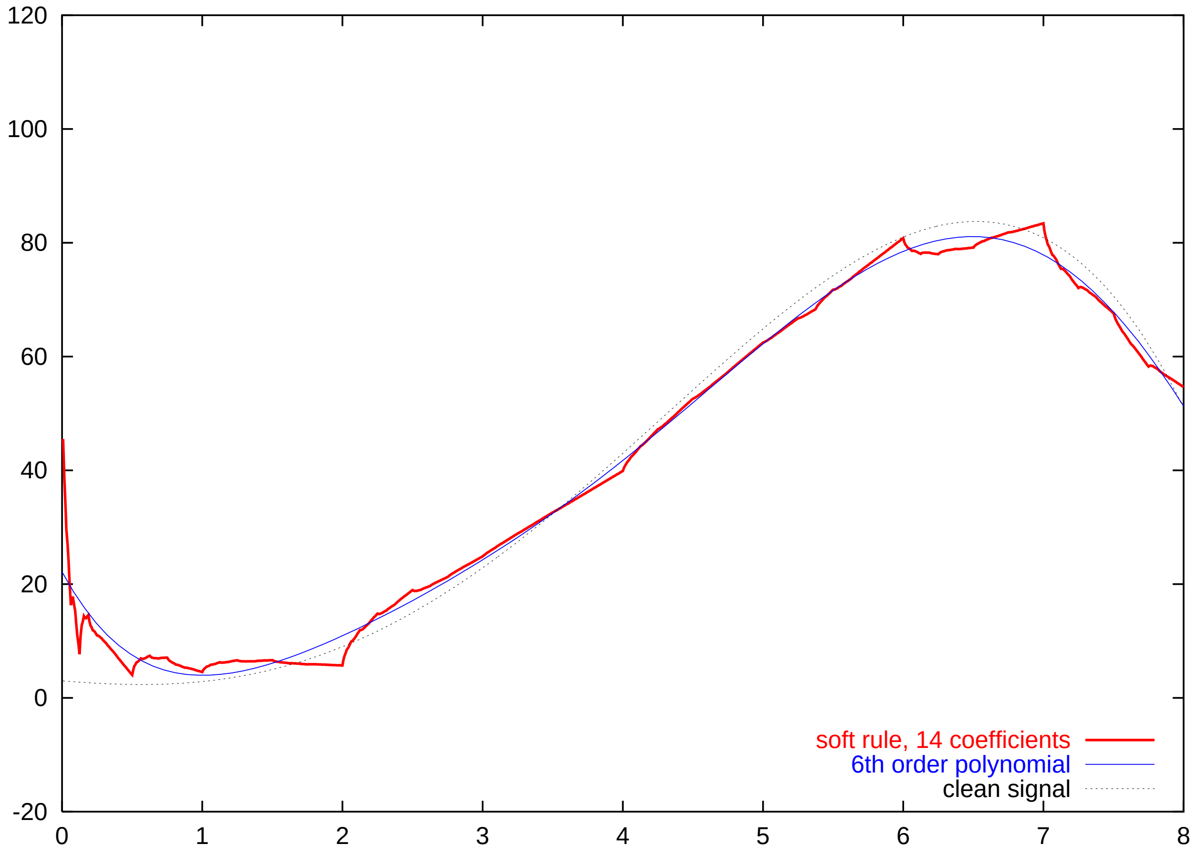

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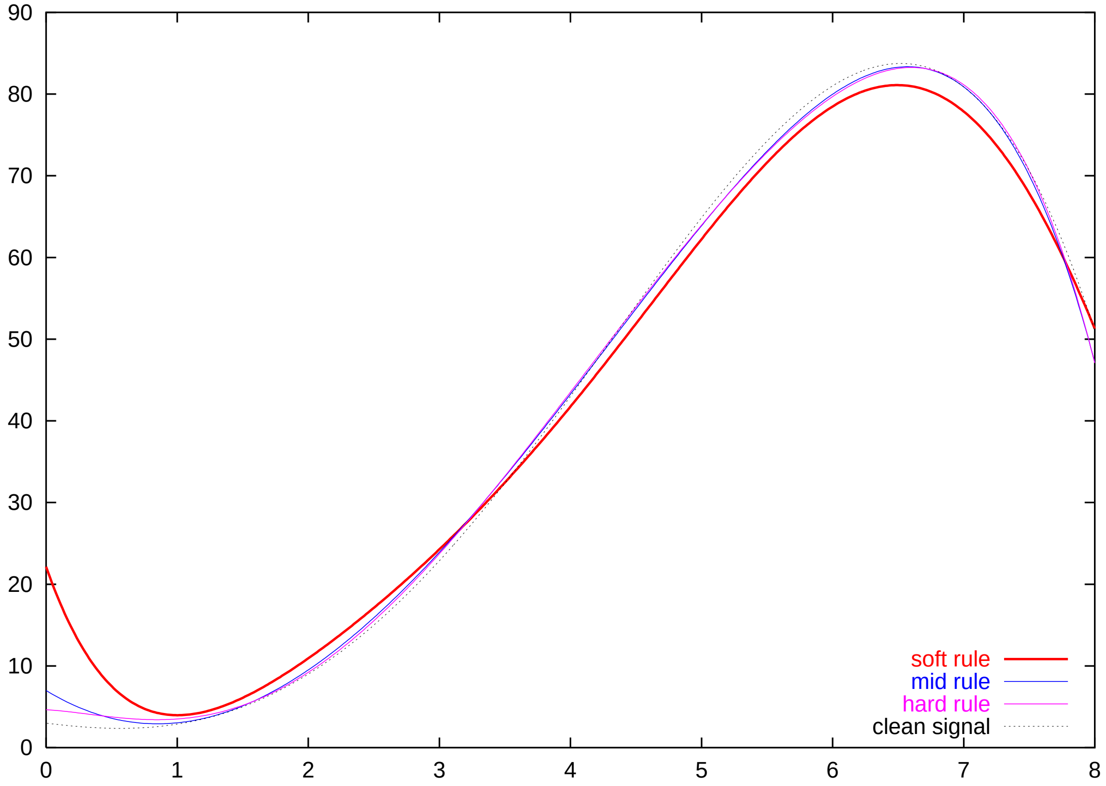
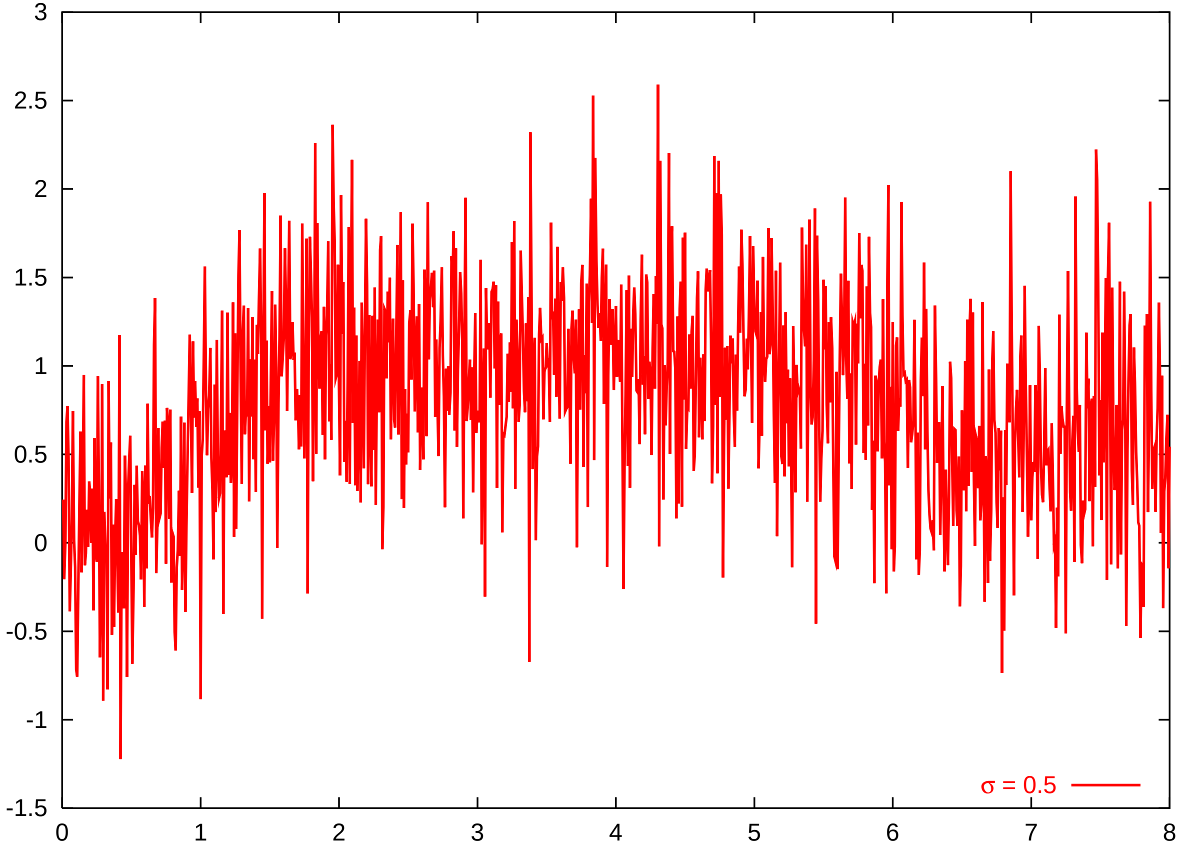
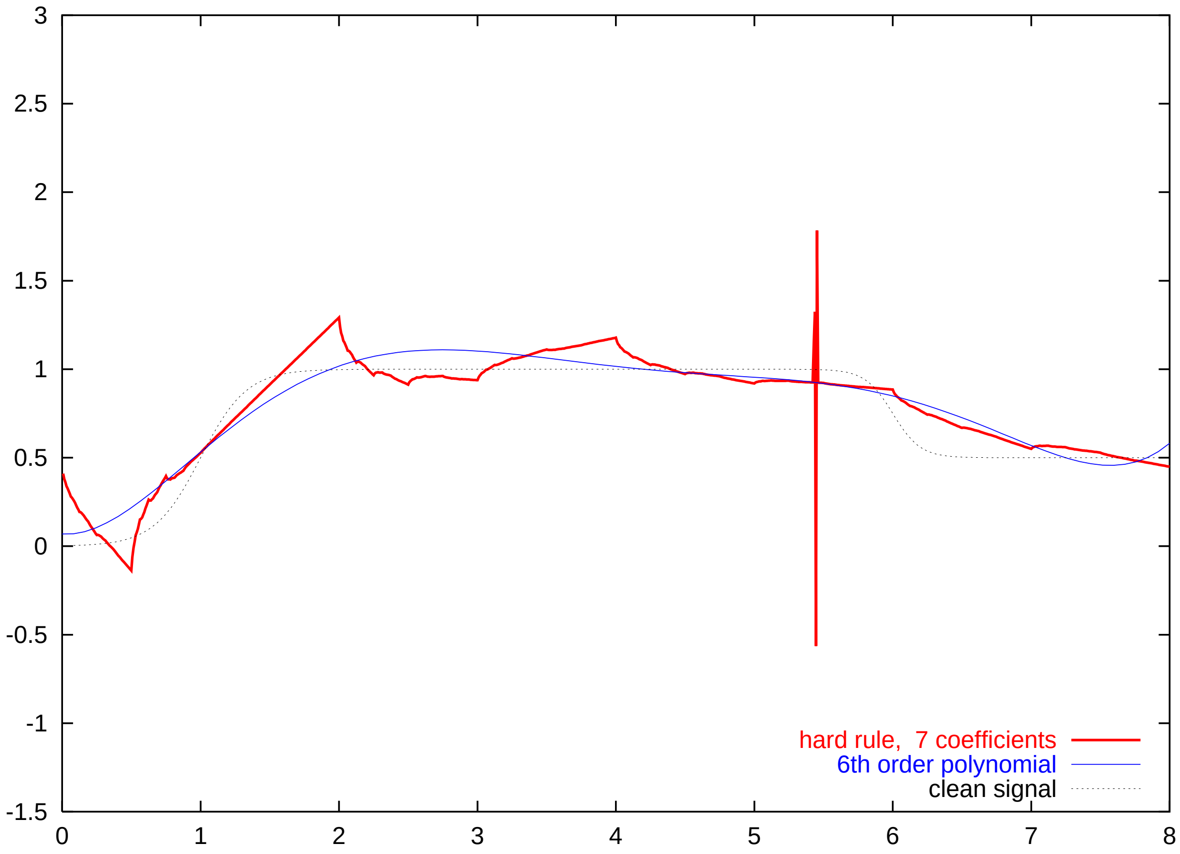
Comparison of the fitted polynomials



Boundary effects are clearly visible!

A noisy “nonpolynomial” signal





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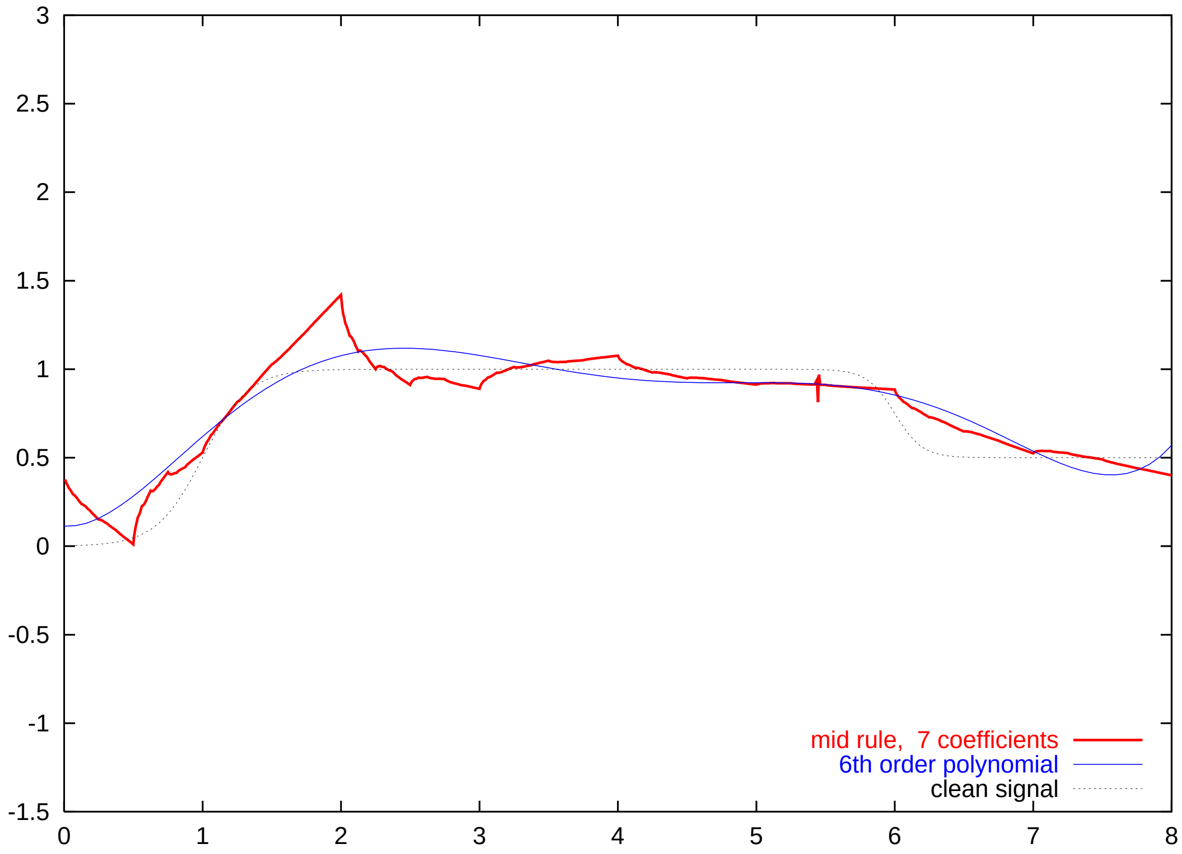
hard rule, 7 coefficients
6th order polynomial
clean signal



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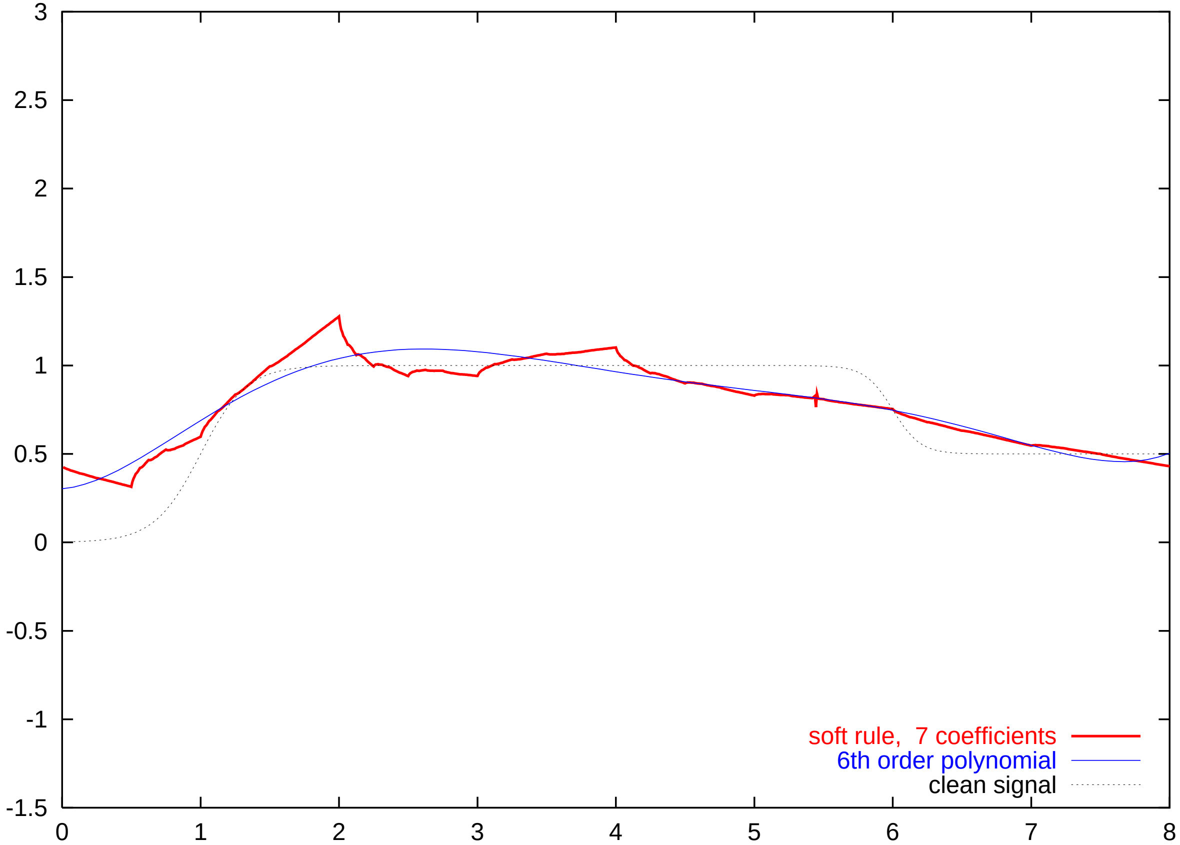

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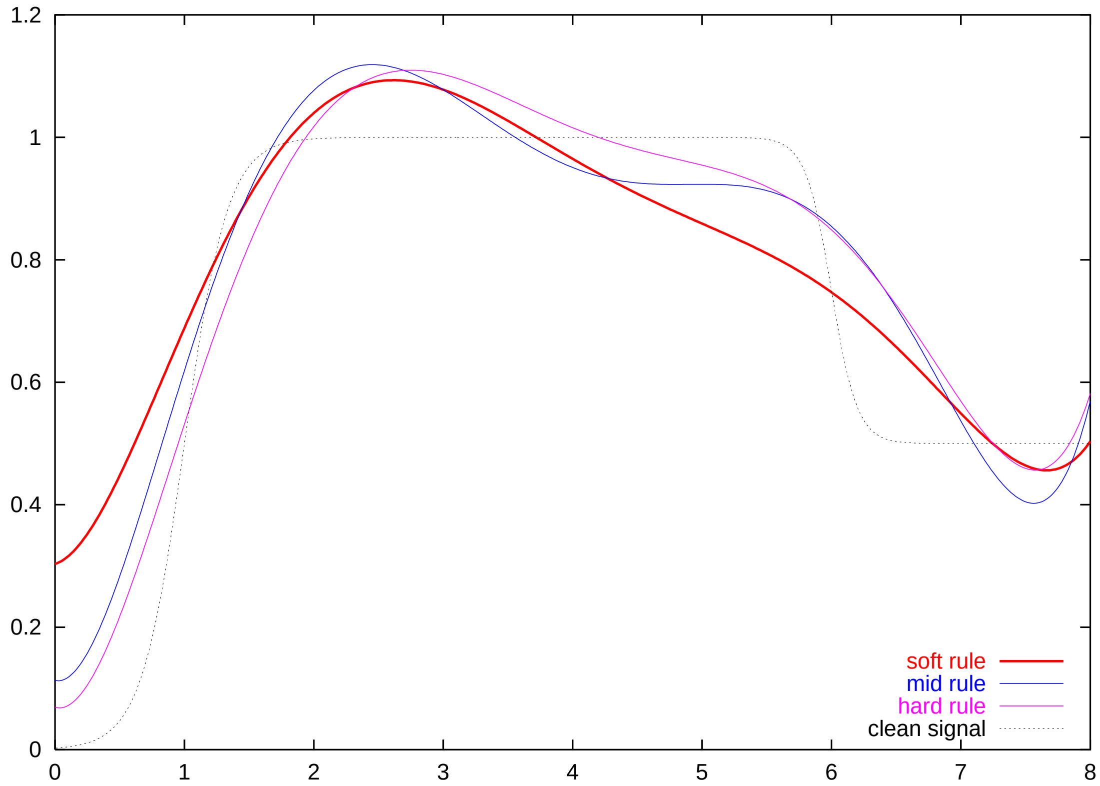


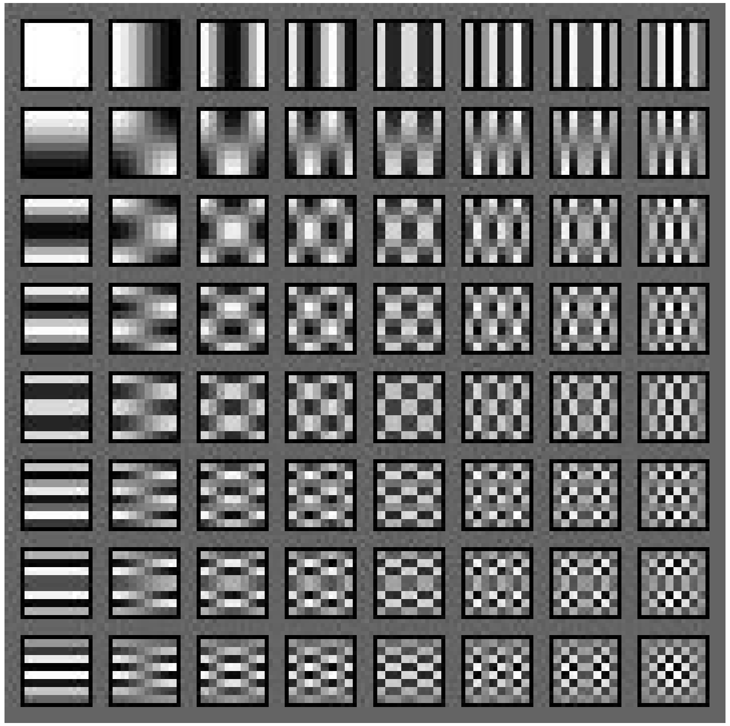
A nonpolynomial (for example, rational) fit would be better!

Digital image analysis

Digitalization (and quantization): raster graphics

- Representation in a certain orthonormal basis
- The trigonometric basis — Fast Fourier Transform
- Numerical cost $O(N \log N)$
- Trigonometric functions are “global”!
- The JPEG (Joint Photographers Expert Group) Standard — the image is divided into 8×8 pixel blocks, 2d Fast Cosine Transform in each block

The JPEG basis functions

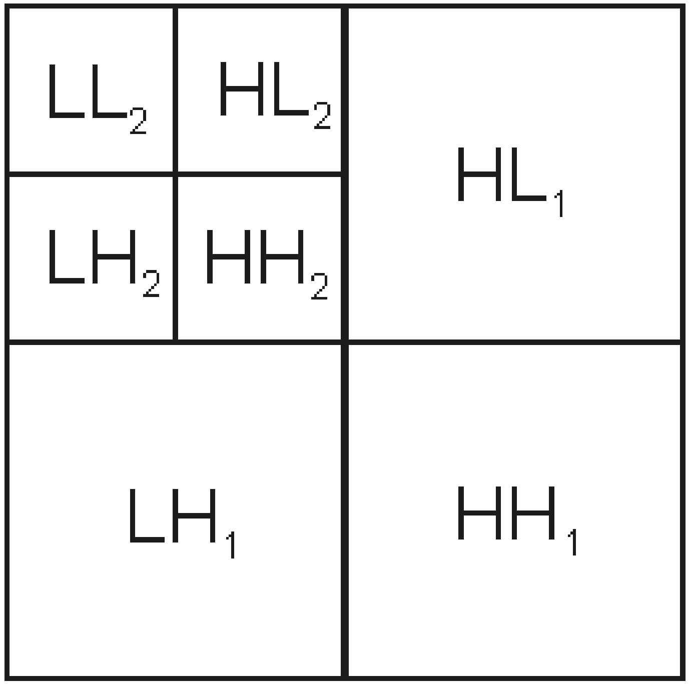


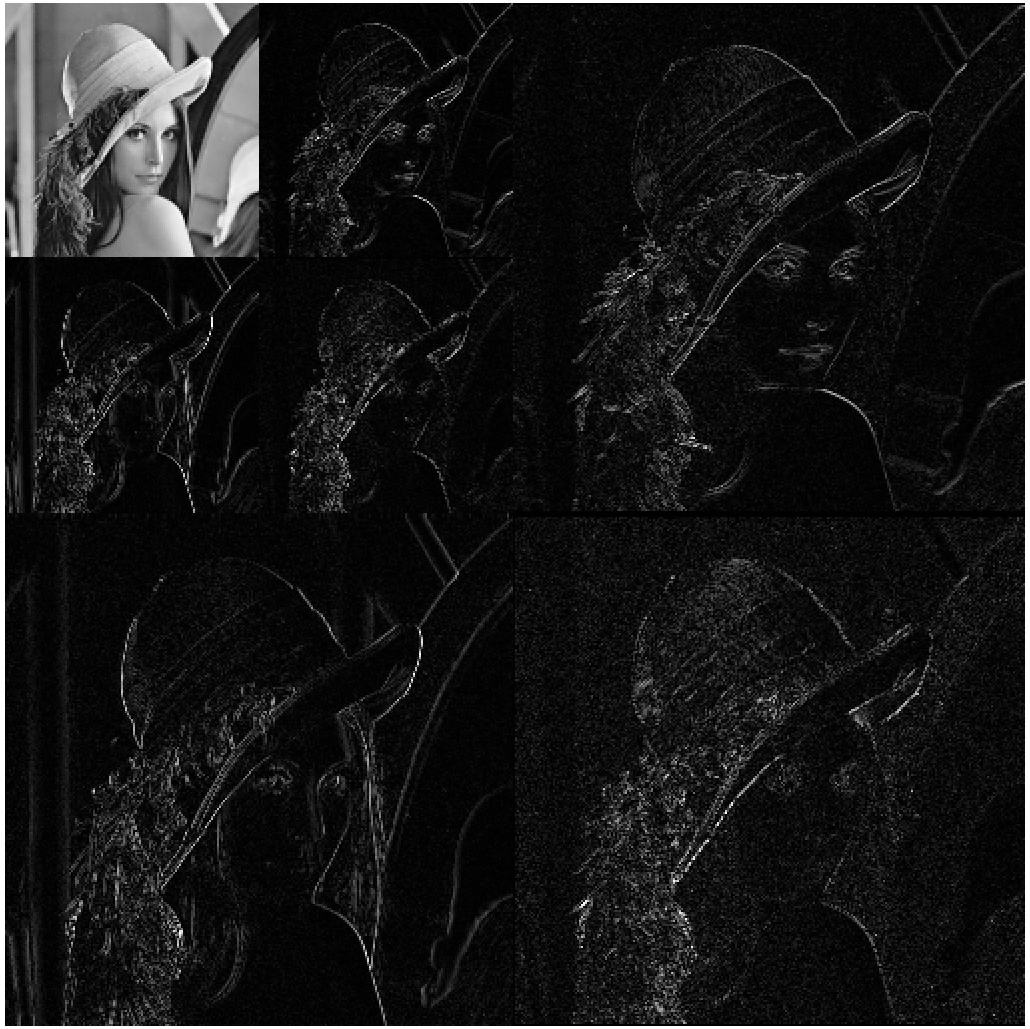
 

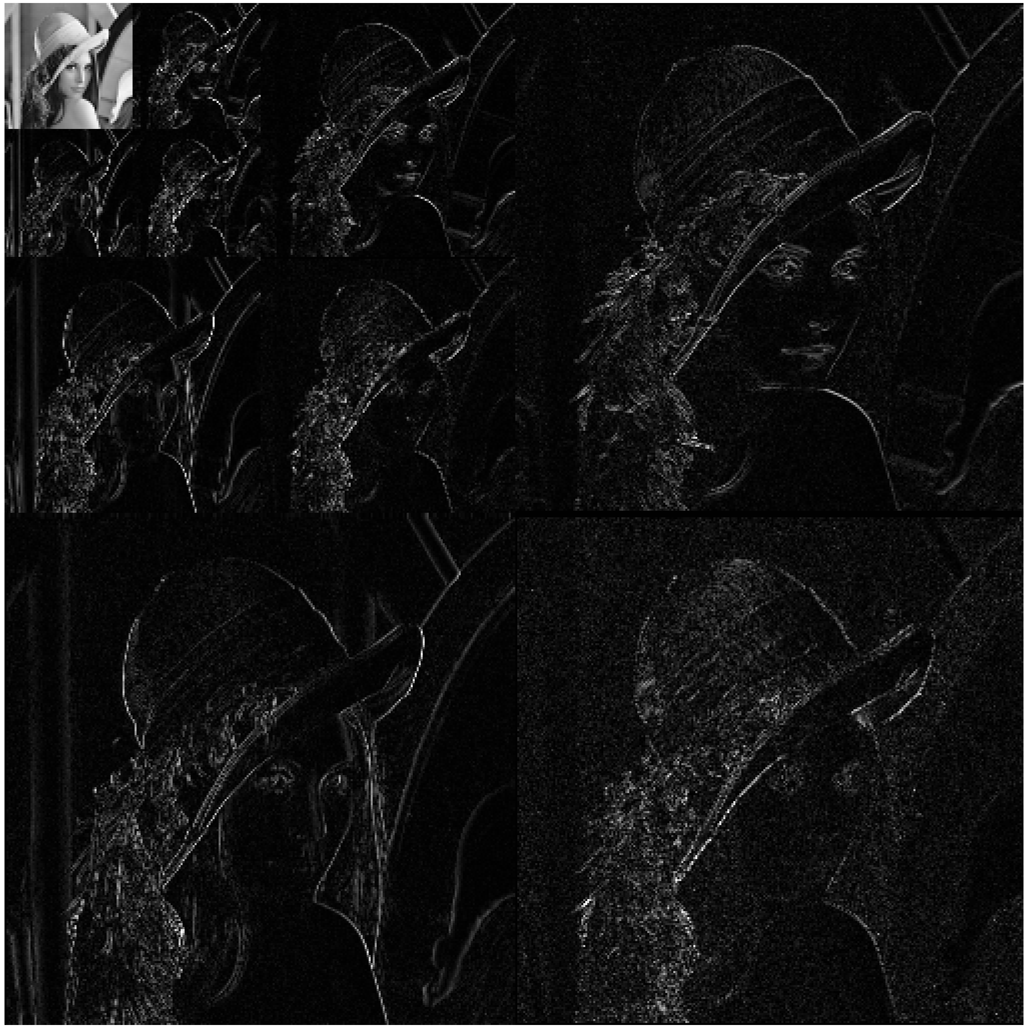
2d wavelet transform

Let $\mathbf{W}$ be a wavelet transform matrix for one stage.

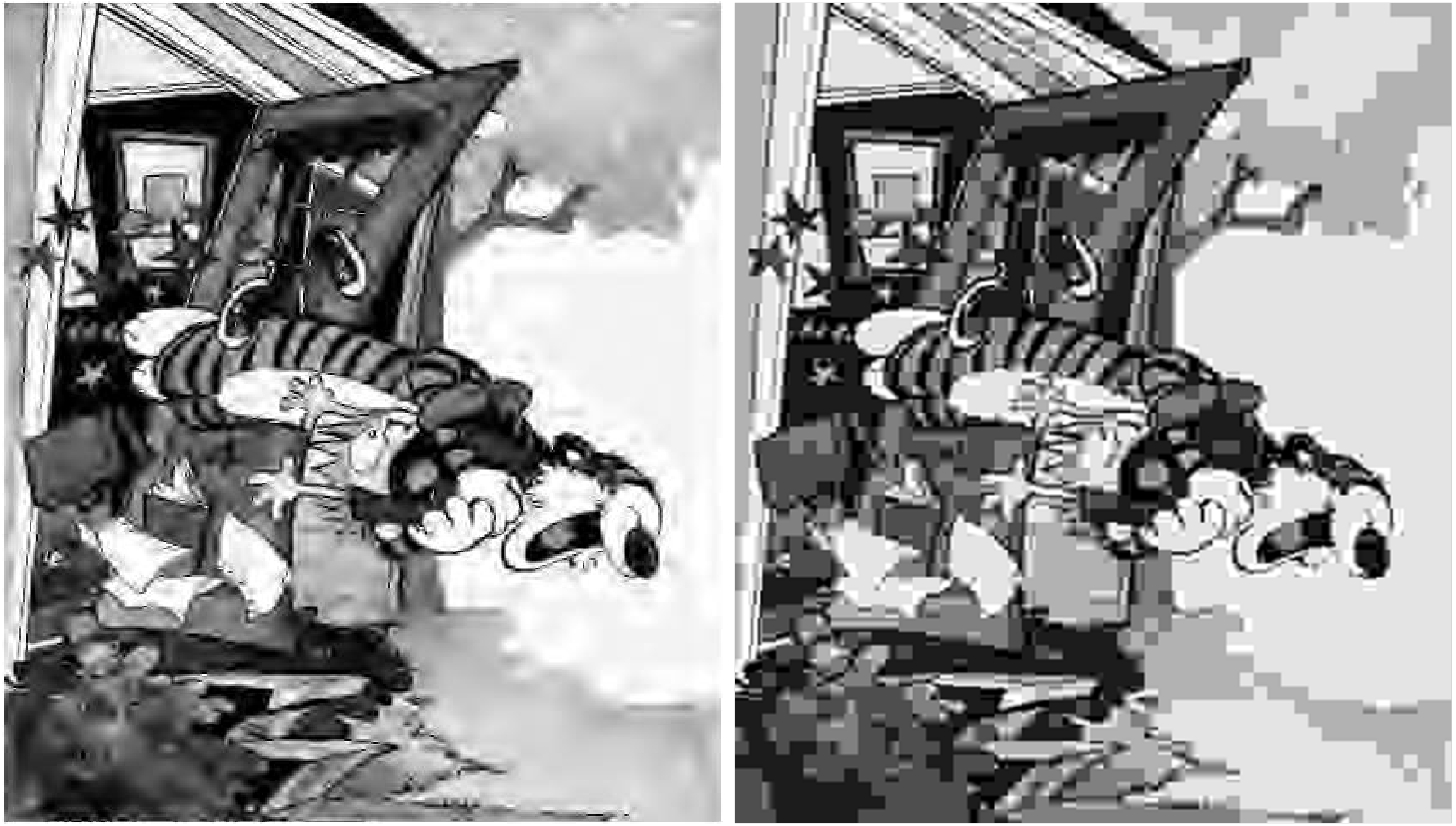
$$\mathbf{W}\,\mathbf{A}\,\mathbf{W}^T \xrightarrow[\text{sort}]{}$$



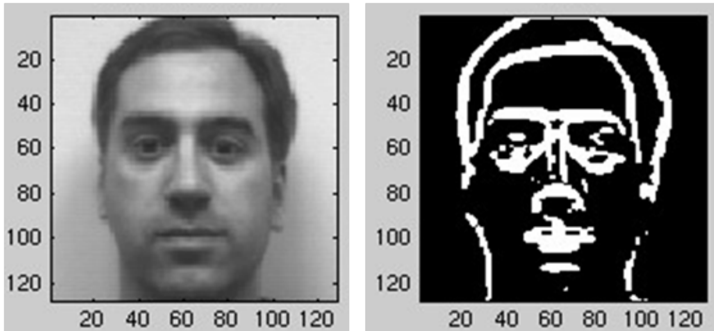




Wavelet compression — the wavelet transform (up to a certain order), “survival of the strongest”, quantization, the inverse wavelet transform



Edge detection with wavelets



Speckles removal