Time Series Analysis:

10. Wavelets

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Different bases

A time series is “naturally” given in a basis in which we immediately know its value at a given time point, but we do not know its non-local features, like the spectarl decomposition etc. The Fourier basis (DFT, realized with the FFT algorithm) gives the “global” spectral decomposition, but we do not know the local features, like the values at given time points.

There are *inifinitely many* other bases possible; some of them are useful. Bases that try to interpolate between local and non-local features are frequently of particular interest. Such bases are called wavelets. There are infinitely many possible wavelet bases ☺, and some of them are useful.

People familiar with quantum mechanics, may think of the position representation vs. the momentum representation vs. the coherent states.

The wavelet transform

$\psi(x)$ — mother wavelet that satisfies

$$\int_{-\infty}^{\infty} \psi(x)\, dx = 0 \tag{1a}$$

$$\int_{-\infty}^{\infty} \psi^2(x)\, dx = 1 \tag{1b}$$

$$\Psi(f) = \int_{-\infty}^{\infty} \psi(x)\, e^{-2\pi i f x}\, dx \Rightarrow 0 < \int_{0}^{\infty} \frac{|\Psi(f)|^2}{f}\, df < \infty \tag{1c}$$

Discrete wavelet transform (DWT) — a representation of a signal in a basis

$$\psi_{jk}(x) = 2^{-k/2}\psi\left(2^k(x-j)\right) \tag{2}$$

– a rescaled, shifted and digitalized mother wavelet. Note: Usually the wavelet basis is assumed to be orthogonal: $\left\langle \psi_{jk} | \psi_{j'k'} \right\rangle = \delta_{jj'}\delta_{kk'}$.

For ψ to be a wavelet, its support should either be compact, or values of $\psi(x)$ should be negligible outside a certain interval.

A representation of a time series

Consider a time series $X = \{x_n\}_{n=0}^{N-1}$ $(N = 2^p)$. The same information that is carried by this series, is also carried by a series of the sums and differences of neighbouring terms: $Y = \{x_0 + x_1, x_0 - x_1, x_2 + x_3, x_2 - x_3, \dots, x_{N-2} + x_{N-1}, x_{N-2} - x_{N-1}\}$. If we know X, we can calculate Y, and *vice versa*, if we know Y, we can recover X.

The same trick can be applied recursively to the series of sums.

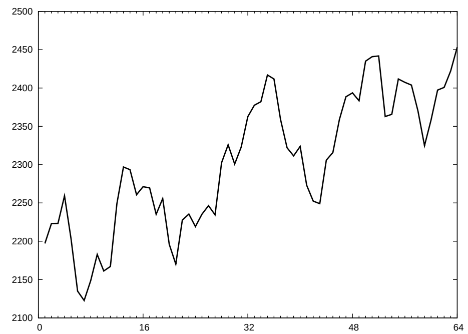
Example

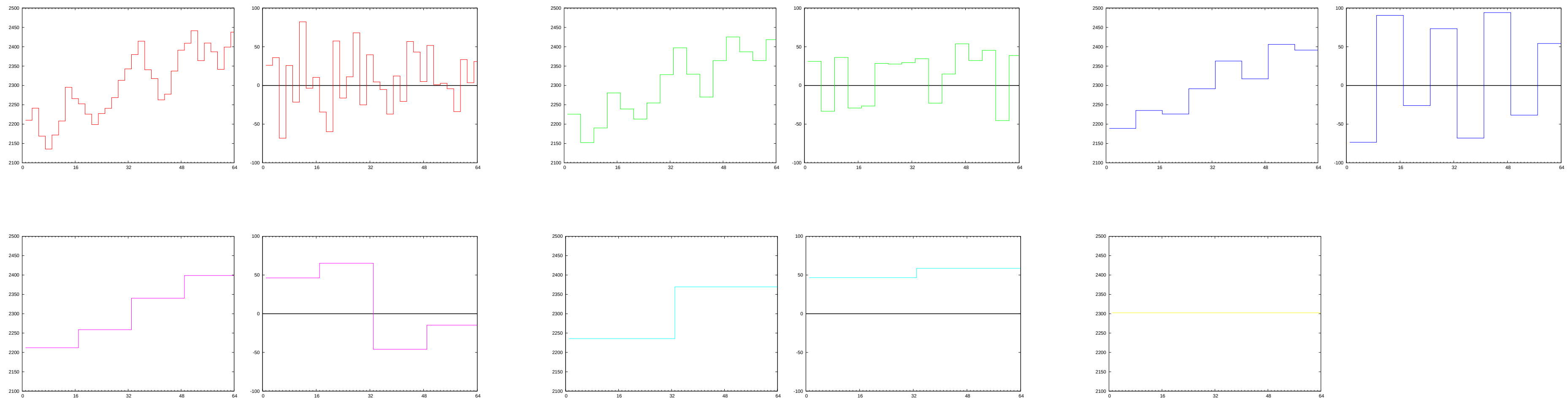
<table>
<tr><td>x_0</td><td>x_1</td><td>x_2</td><td>x_3</td><td>x_4</td><td>x_5</td><td>x_6</td><td>x_7</td><td>. . .</td><td>one-point features</td></tr>
<tr><td colspan="2">$\frac{x_0+x_1}{\sqrt{2}}$
$\frac{x_0-x_1}{\sqrt{2}}$</td><td colspan="2">$\frac{x_2+x_3}{\sqrt{2}}$
$\frac{x_2-x_3}{\sqrt{2}}$</td><td colspan="2">$\frac{x_4+x_5}{\sqrt{2}}$
$\frac{x_4-x_5}{\sqrt{2}}$</td><td colspan="2">$\frac{x_6+x_7}{\sqrt{2}}$
$\frac{x_6-x_7}{\sqrt{2}}$</td><td>. . .
. . .</td><td>two-points features</td></tr>
<tr><td colspan="4">$\frac{x_0+x_1+x_2+x_3}{2}$
$\frac{x_0+x_1-x_2-x_3}{2}$</td><td colspan="4">$\frac{x_4+x_5+x_6+x_7}{2}$
$\frac{x_4+x_5-x_6-x_7}{2}$</td><td>. . .
. . .</td><td>four-points features</td></tr>
<tr><td colspan="8">$\frac{x_0+x_1+x_2+x_3+x_4+x_5+x_6+x_7}{2\sqrt{2}}$
$\frac{x_0+x_1+x_2+x_3-x_4-x_5-x_6-x_7}{2\sqrt{2}}$</td><td>. . .
. . .</td><td>eight-points features</td></tr>
<tr><td colspan="9">. . . . .</td><td></td></tr>
</table>

This procedure shows how the series behaves on *different scales*.

Factors $1/\sqrt{2}$ have been introduced in order to keep the normalization.

Example





We have effectively introduced the...

Haar wavelets

$$\underbrace{\begin{bmatrix} a_1 & a_2 & & & & & \\ b_1 & b_2 & & & & & \\ & & a_1 & a_2 & & & \\ & & b_1 & b_2 & & & \\ & & & & \ddots & & \\ & & & & & a_1 & a_2 \\ & & & & & b_1 & b_2 \end{bmatrix}}_{\mathcal{W}_N,\ N=2^p} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{N-2} \\ x_{N-1} \end{bmatrix} = \begin{bmatrix} s_0 \\ d_0 \\ s_1 \\ d_1 \\ \vdots \\ s_{N/2-1} \\ d_{N/2-1} \end{bmatrix} \xrightarrow[\text{sort}]{} \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{N/2-1} \\ d_0 \\ d_1 \\ \vdots \\ d_{N/2-1} \end{bmatrix} \tag{3}$$

We want $\{b_1, b_2\}$ to form a high-pass filter and $\{a_1, a_2\}$ to form a low-pass filter. $s.$ components have passed through the low-pass filter; $d.$ components have passed through the high-pass filter.

The next stage

$$
\begin{bmatrix} \mathcal{W}_{N/2} & \\ & \mathbb{I}_{N/2\times N/2} \end{bmatrix}
\begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_{N/2-1} \\ d_0 \\ d_1 \\ \vdots \\ d_{N/2-1} \end{bmatrix}
=
\begin{bmatrix} S_0 \\ D_0 \\ S_1 \\ D_1 \\ \vdots \\ S_{N/4-1} \\ D_{N/4-1} \\ d_0 \\ d_1 \\ \vdots \\ d_{N/2-1} \end{bmatrix}
\underset{\text{sort}}{\longrightarrow}
\begin{bmatrix} S_0 \\ S_1 \\ \vdots \\ S_{N/4-1} \\ D_0 \\ D_1 \\ \vdots \\ D_{N/4-1} \\ d_0 \\ d_1 \\ \vdots \\ d_{N/2-1} \end{bmatrix}
\tag{4}
$$

The following stage

$$
\begin{bmatrix}
\mathcal{W}_{N/4} & & \\
& \mathbb{I}_{N/4\times N/4} & \\
& & \mathbb{I}_{N/2\times N/2}
\end{bmatrix}
\begin{bmatrix}
S_0 \\ \vdots \\ S_{N/4-1} \\ D_0 \\ \vdots \\ D_{N/4-1} \\ d_0 \\ \vdots \\ d_{N/2-1}
\end{bmatrix}
\xrightarrow[\text{sort}]{}
\begin{bmatrix}
\mathcal{S}_0 \\ \vdots \\ \mathcal{S}_{N/8-1} \\ \mathcal{D}_0 \\ \vdots \\ \mathcal{D}_{N/8-1} \\ D_0 \\ \vdots \\ D_{N/4-1} \\ d_0 \\ \vdots \\ d_{N/2-1}
\end{bmatrix}
\tag{5}
$$

etc. . .

The pyramid algorithm

$$\begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{bmatrix} \longrightarrow \begin{bmatrix} \text{slow}_{N/2} \\ \\ \text{detailed}_{N/2} \end{bmatrix} \longrightarrow \begin{bmatrix} slow_{N/4} \\ \\ detailed_{N/4} \\ \\ \text{detailed}_{N/2} \end{bmatrix} \longrightarrow \begin{bmatrix} Slow_{N/8} \\ \\ Detailed_{N/8} \\ \\ detailed_{N/4} \\ \\ \text{detailed}_{N/2} \end{bmatrix} \longrightarrow \cdots \tag{6}$$

Because the transformation matrices are sparse, and because at each stage only a half of the previously transformed entries undergo a further transformation, the pyramid algorithm *is faster than FFT* (the computational complexity is $O(N \log N)$ but the coefficient is better than in FFT). If the matrix $\mathcal{W}_N$ is orthogonal, the pyramid algorithm realizes an *orthogonal transformation*.

The Haar matrix

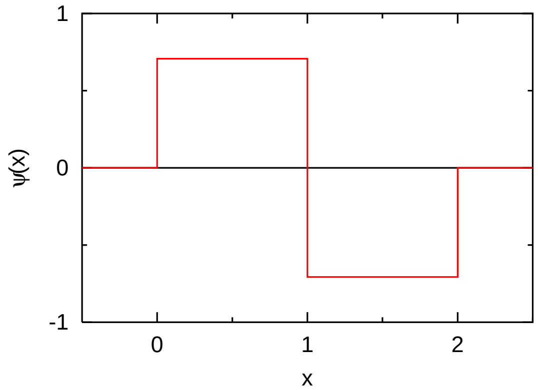
$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \Rightarrow$$

orthogonal
b: a high-pass filter

$$\begin{aligned} a_1^2 + a_2^2 &= 1 \\ b_1^2 + b_2^2 &= 1 \\ a_1 b_1 + a_2 b_2 &= 0 \\ b_1 + b_2 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2a_1^2 &= 1 \\ 2b_1^2 &= 1 \\ a_1 &= a_2 \\ b_1 &= -b_2 \end{aligned}$$

The only nontrivial solution:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \qquad \psi(x) = \begin{cases} \frac{1}{\sqrt{2}} & 0 \leqslant x \leqslant 1 \\ -\frac{1}{\sqrt{2}} & 1 < x \leqslant 2 \\ 0 & \text{otherwise} \end{cases}$$



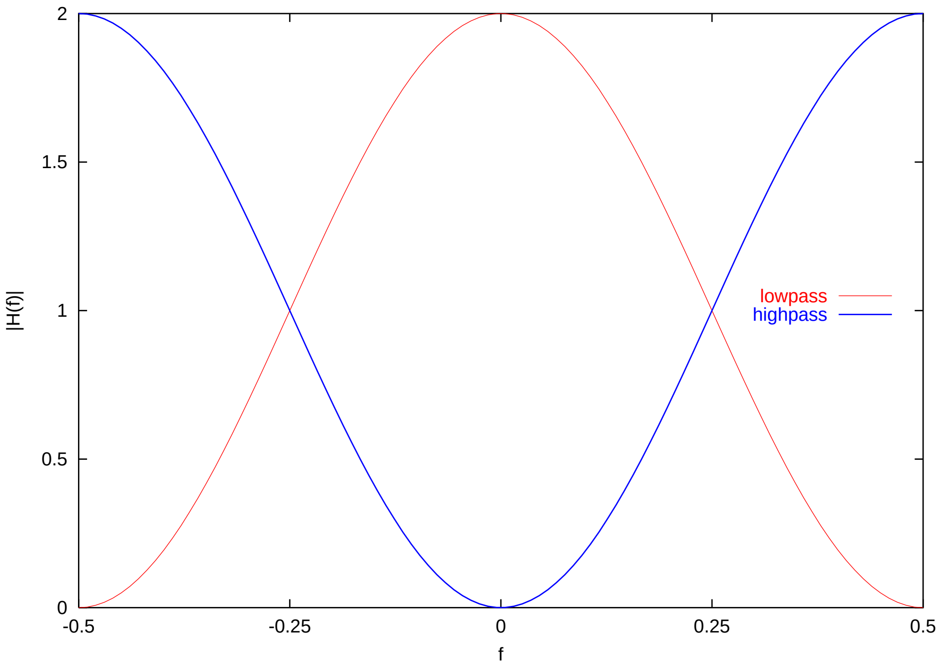
mother Haar wavelet

The scaling function is constant.

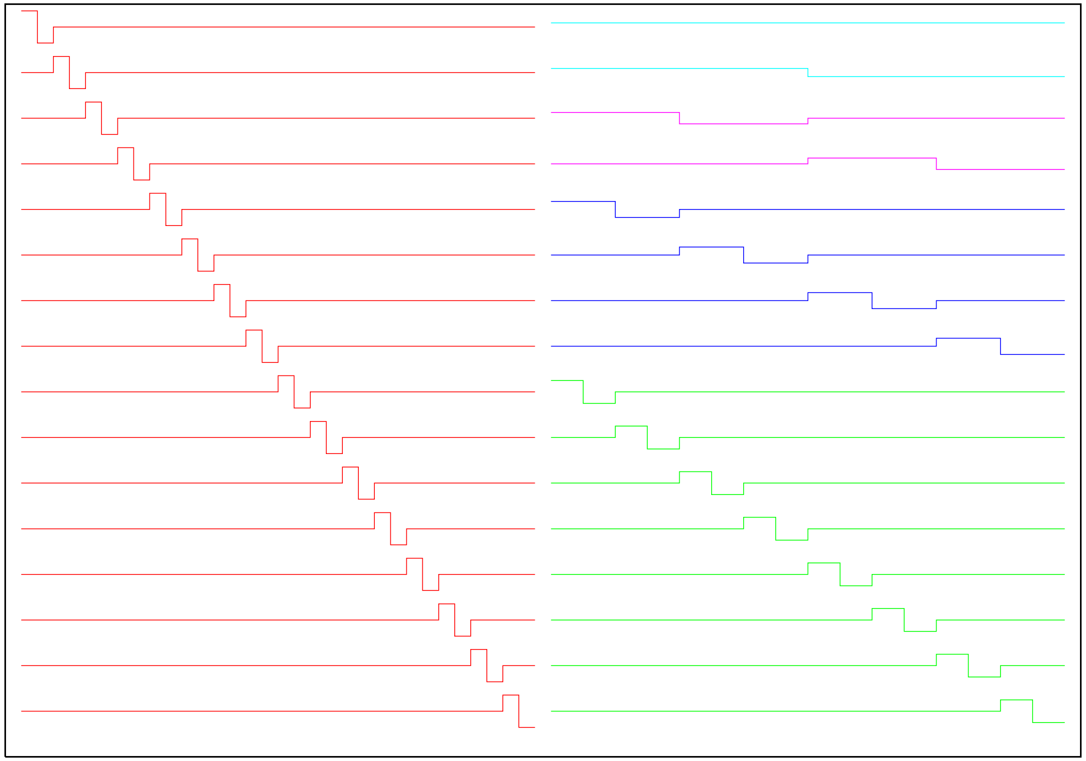
Transfer functions:

Low pass: $H(f) = \frac{1}{\sqrt{2}}(1 + \cos 2\pi f) + \frac{i}{\sqrt{2}} \sin 2\pi f$

High pass: $H(f) = \frac{1}{\sqrt{2}}(1 - \cos 2\pi f) - \frac{i}{\sqrt{2}} \sin 2\pi f$

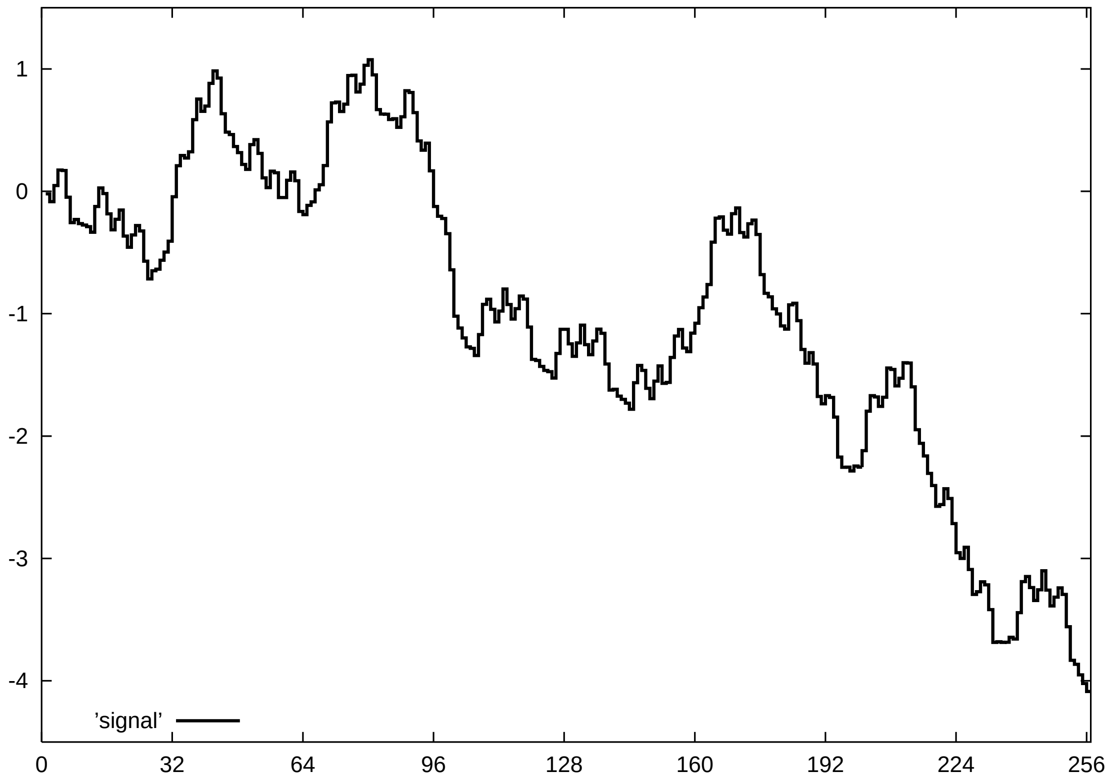


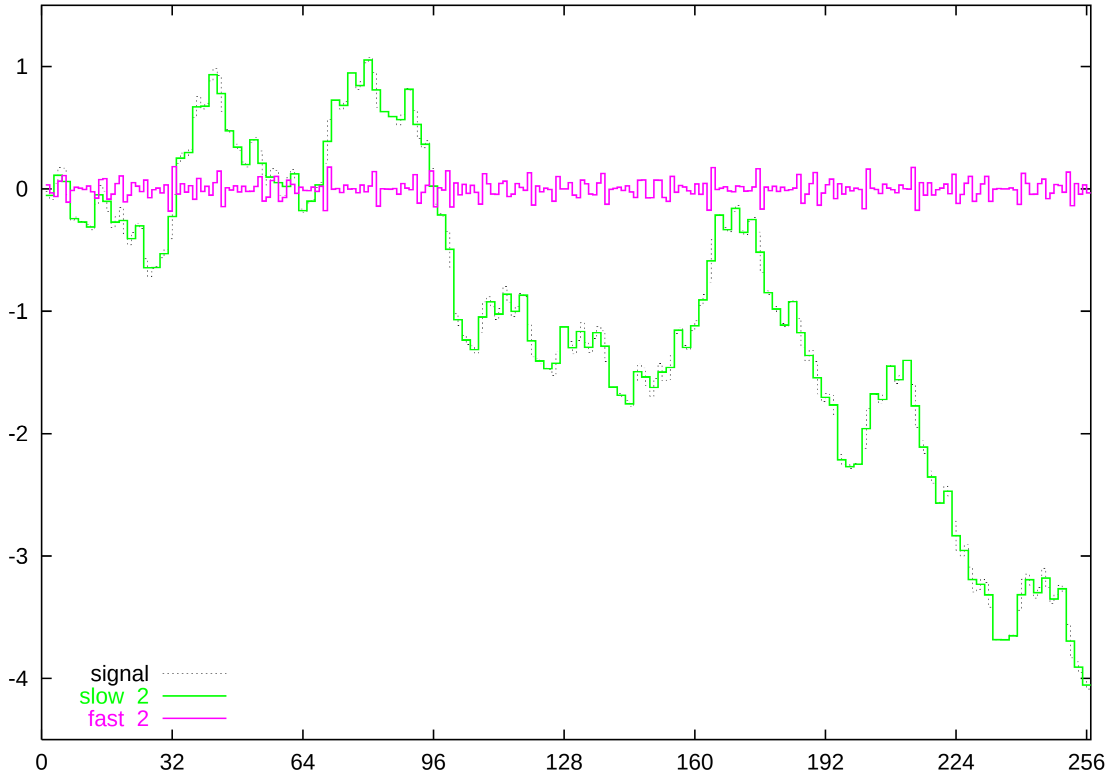
The Haar basis



32-point Haar wavelets

Multiresolution analysis

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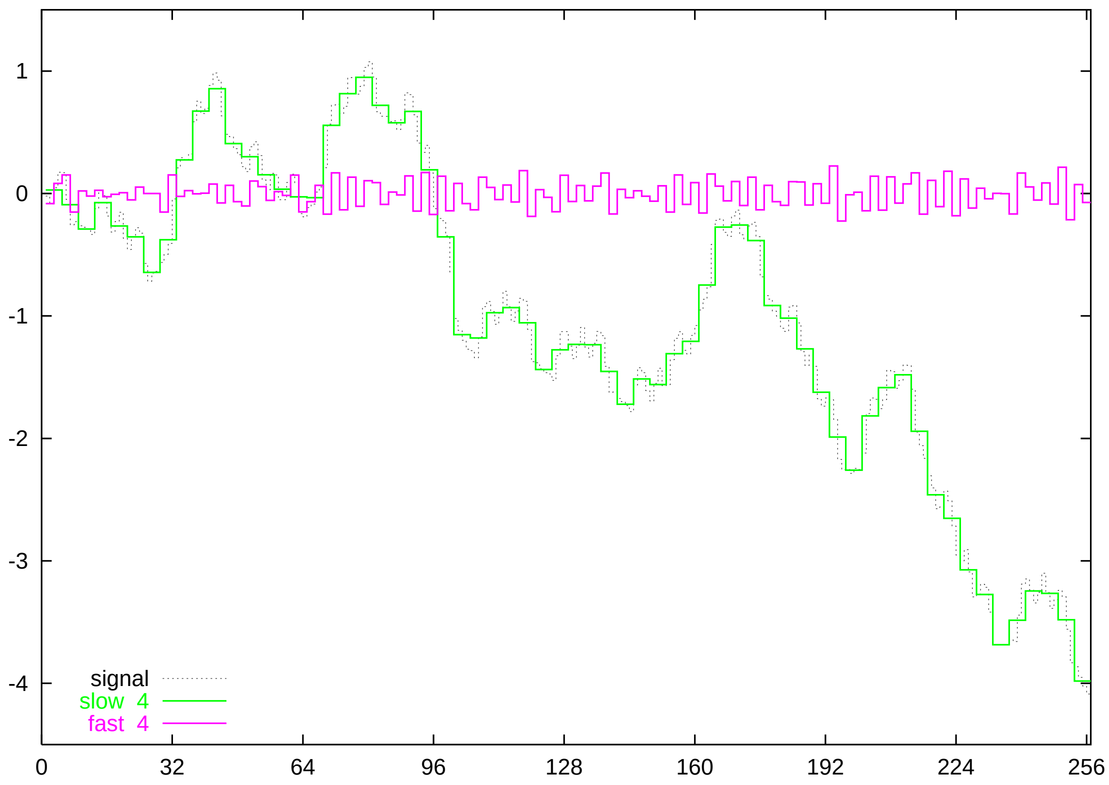

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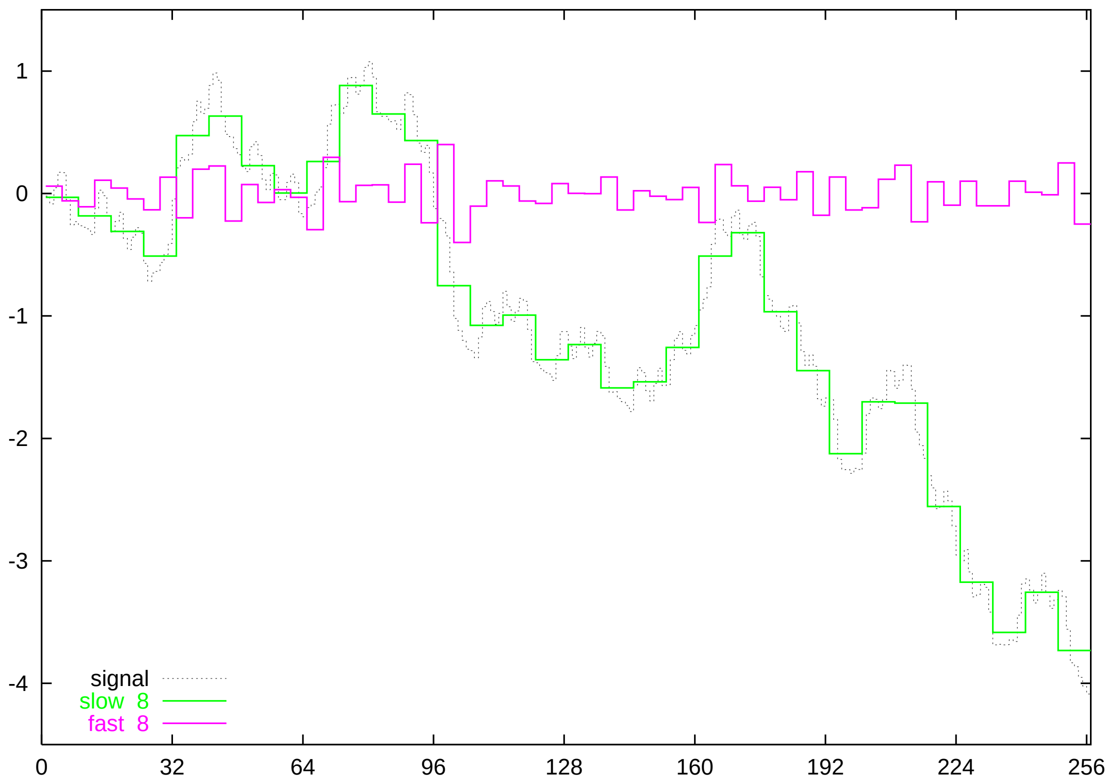

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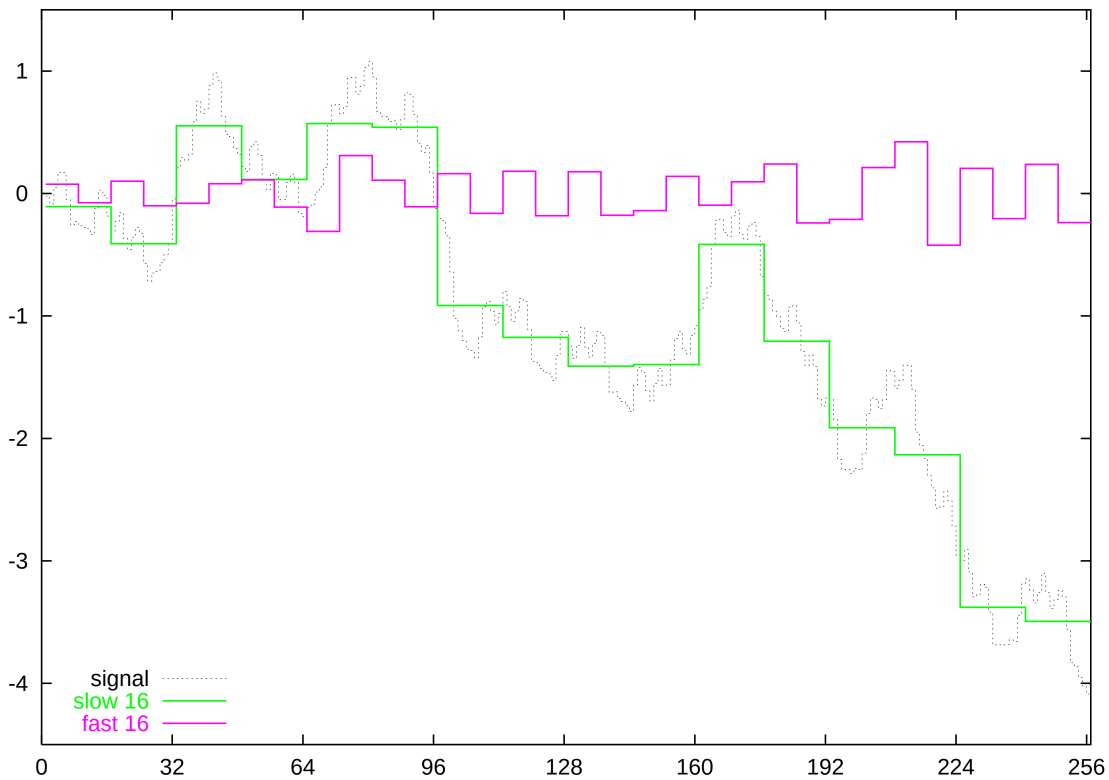

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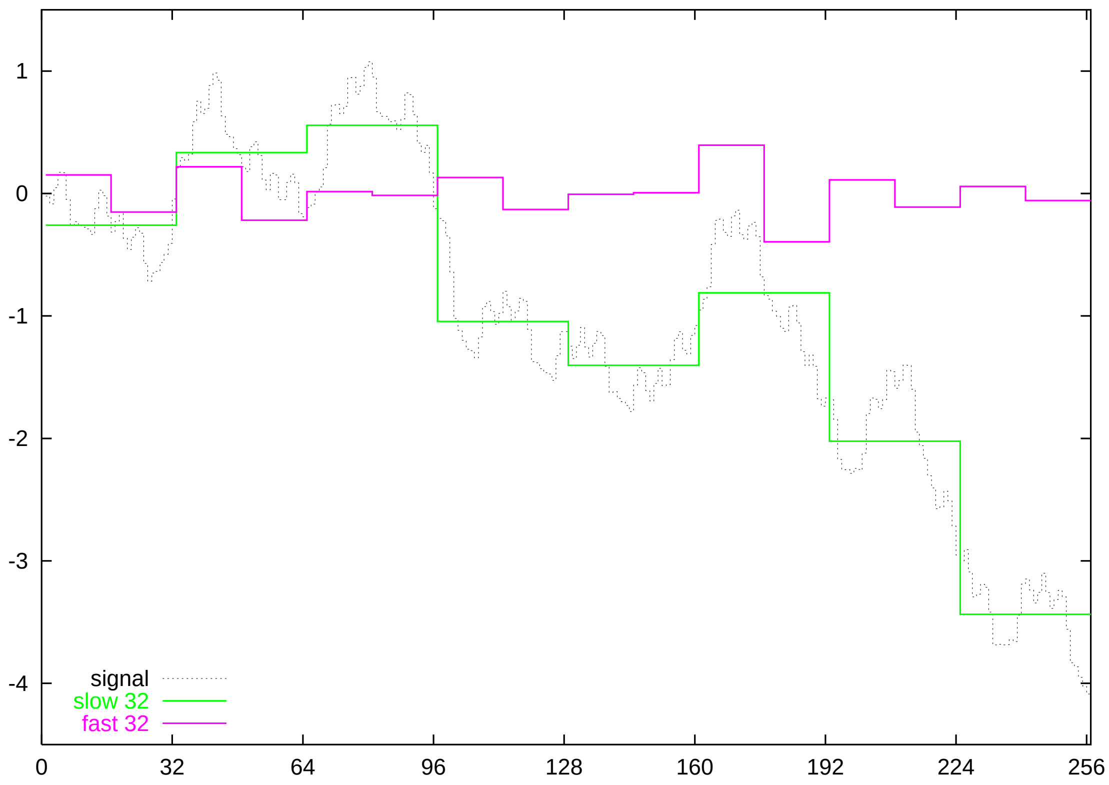

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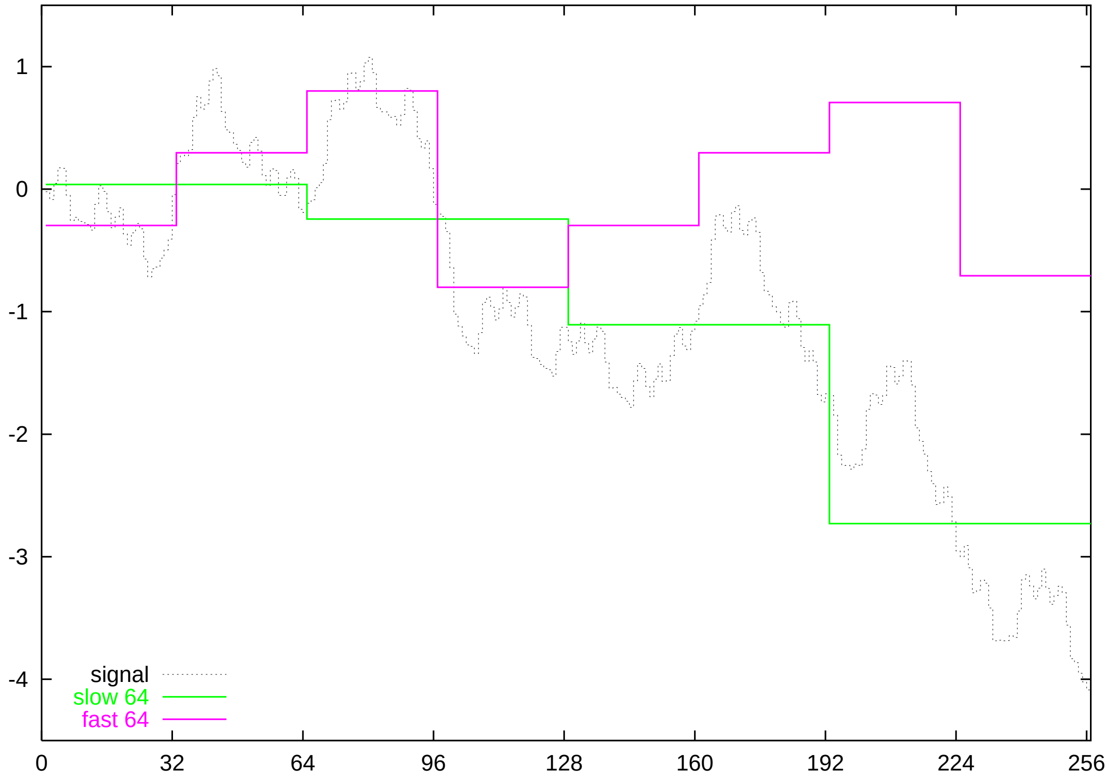

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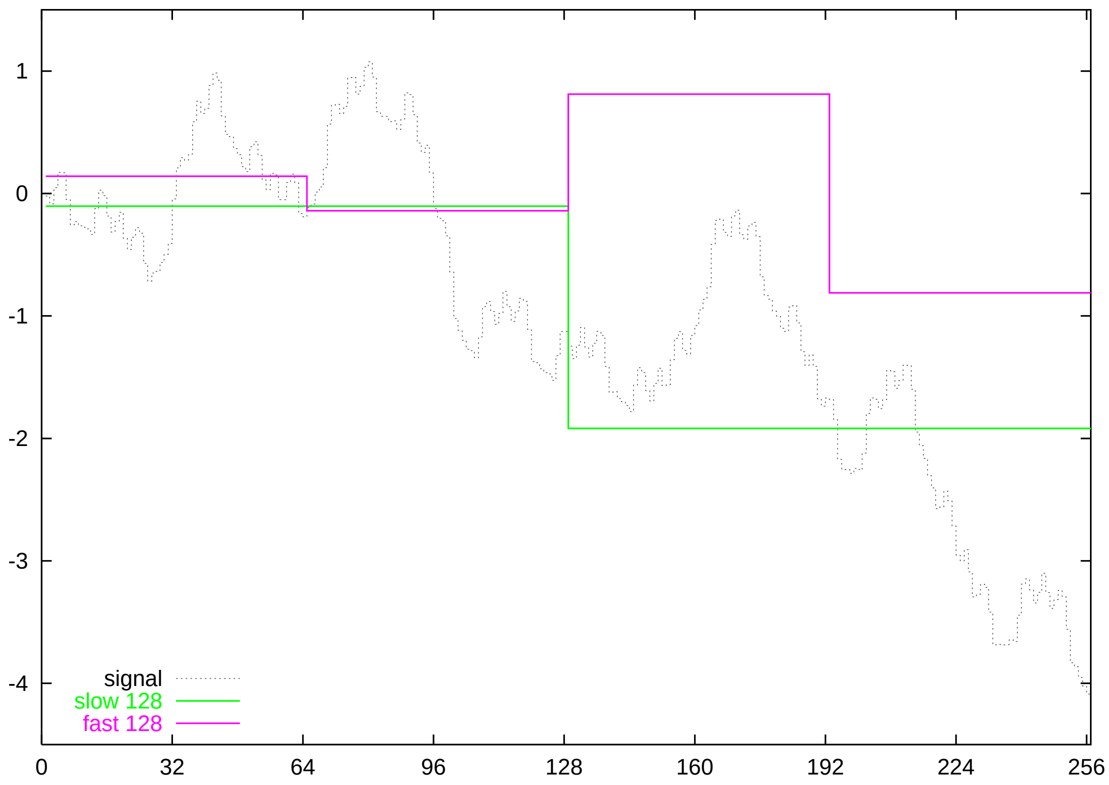
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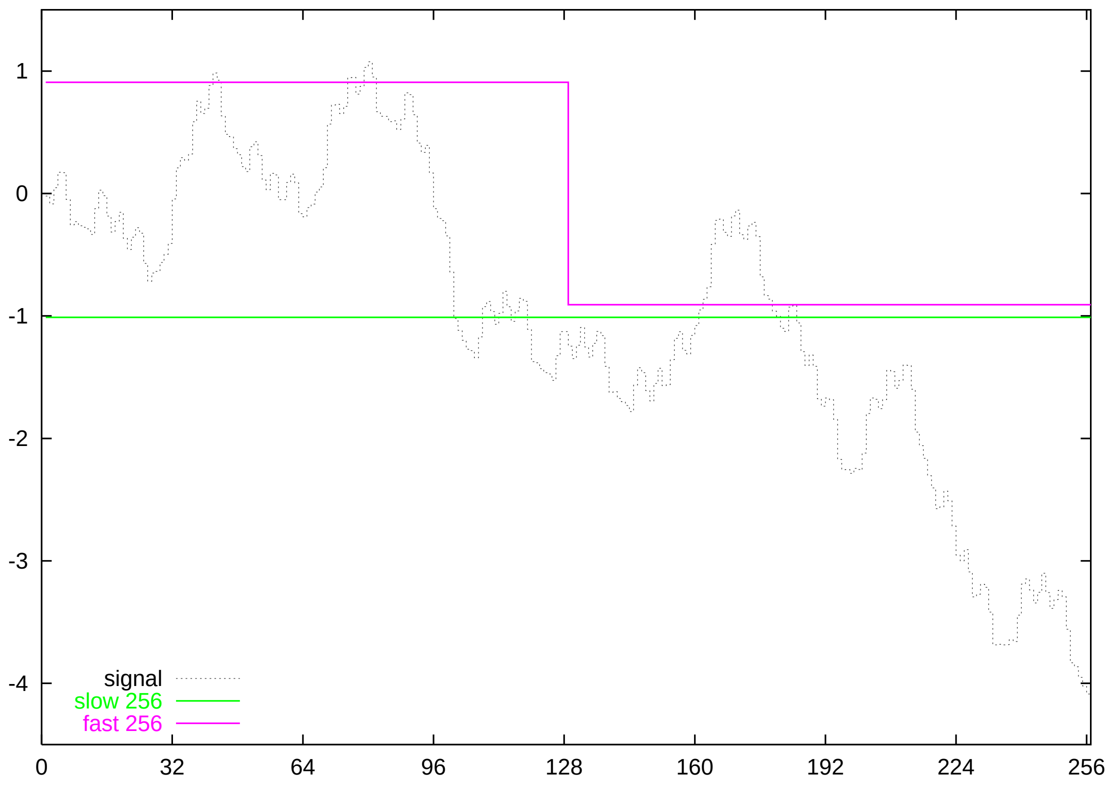
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Daubechies wavelets of order 4

$$
\begin{bmatrix}
c_0 & c_1 & c_2 & c_3 & & & & & & \\
c_3 & -c_2 & c_1 & -c_0 & & & & & & \\
 & & c_0 & c_1 & c_2 & c_3 & & & & \\
 & & c_3 & -c_2 & c_1 & -c_0 & & & & \\
 & & & & & & \ddots & & & \\
 & & & & & & c_0 & c_1 & c_2 & c_3 \\
 & & & & & & c_3 & -c_2 & c_1 & -c_0 \\
c_2 & c_3 & & & & & & & c_0 & c_1 \\
c_1 & -c_0 & & & & & & & c_3 & -c_2
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_{N-3} \\ x_{N-2} \\ x_{N-1} \\ x_N
\end{bmatrix}
=
\begin{bmatrix}
s_1 \\ d_1 \\ s_2 \\ d_2 \\ \vdots \\ s_{N/2-1} \\ d_{N/2-1} \\ s_{N/2} \\ d_{N/2}
\end{bmatrix}
\qquad (7)
$$

Etc...

Note the wrap-around of the input vector!

$$
\begin{aligned}
c_0^2 + c_1^2 + c_2^2 + c_3^2 &= 0 \qquad (8a)\\
c_2 c_0 + c_3 c_1 &= 0 \qquad (8b)\\
c_3 - c_2 + c_1 - c_0 &= 0 \qquad (8c)\\
0 \cdot c_3 - 1 \cdot c_2 + 2 \cdot c_1 - 3 \cdot c_0 &= 0 \qquad (8d)
\end{aligned}
$$

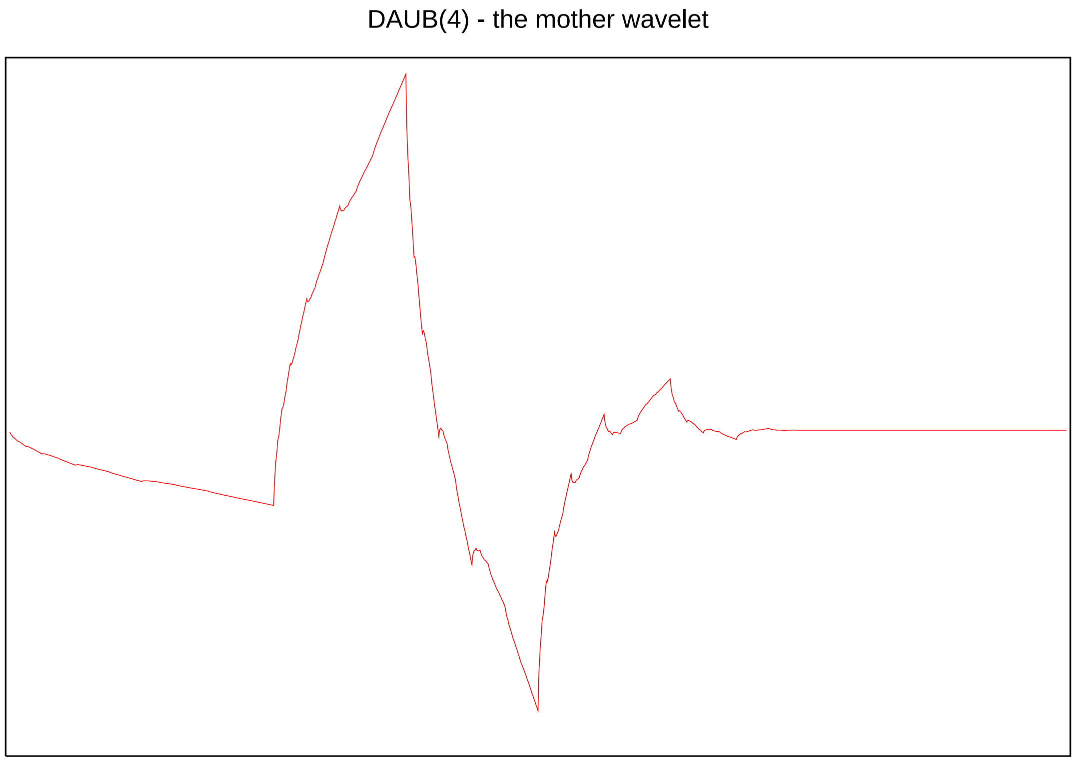
Condition (8c): kill the constant part
Condition (8d): kill a local linear trend

The solution:

$$
c_0 = \frac{1+\sqrt{3}}{4\sqrt{2}} \qquad c_1 = \frac{3+\sqrt{3}}{4\sqrt{2}}
$$

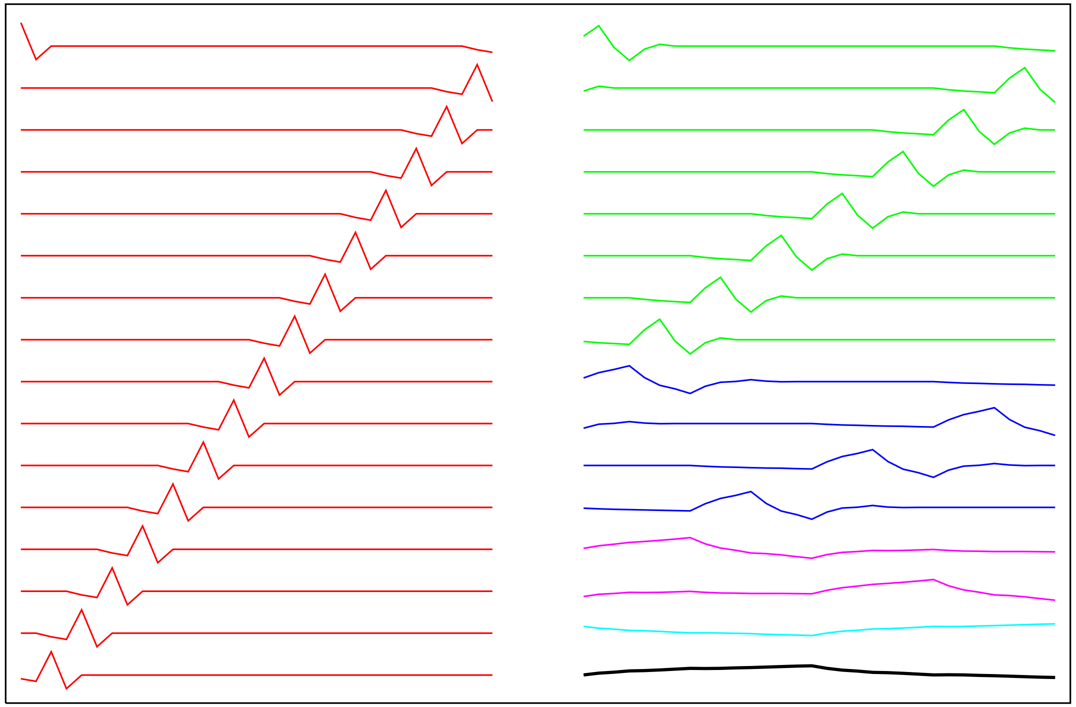
$$
c_2 = \frac{3-\sqrt{3}}{4\sqrt{2}} \qquad c_1 = \frac{1-\sqrt{3}}{4\sqrt{2}}
$$

Mother DAUB(4) wavelet

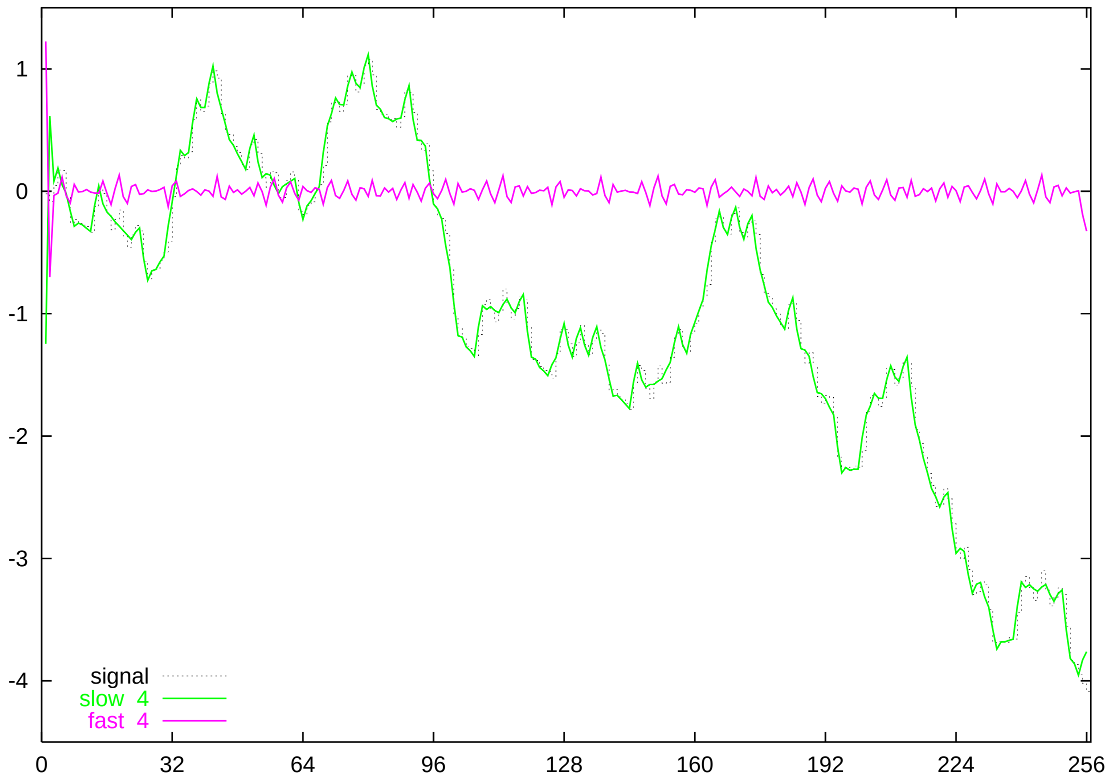


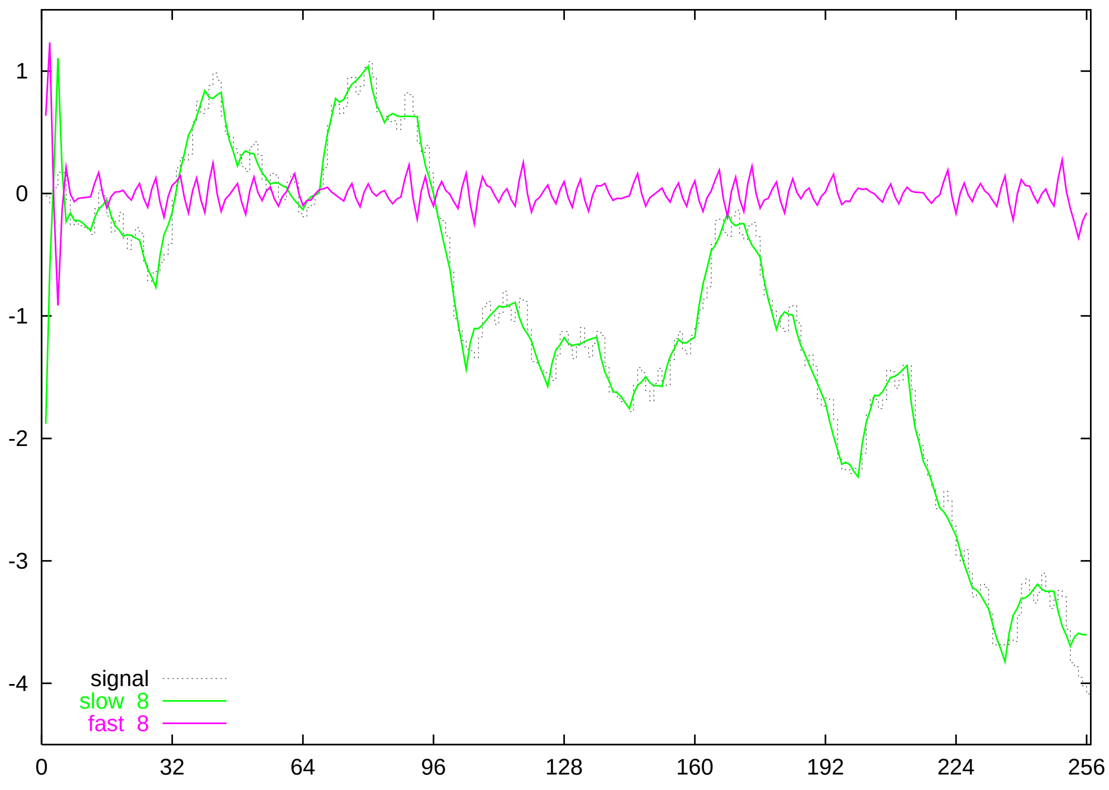
DAUB(4) basis

DAUB(4)



Multiresolution analysis with DAUB(4)





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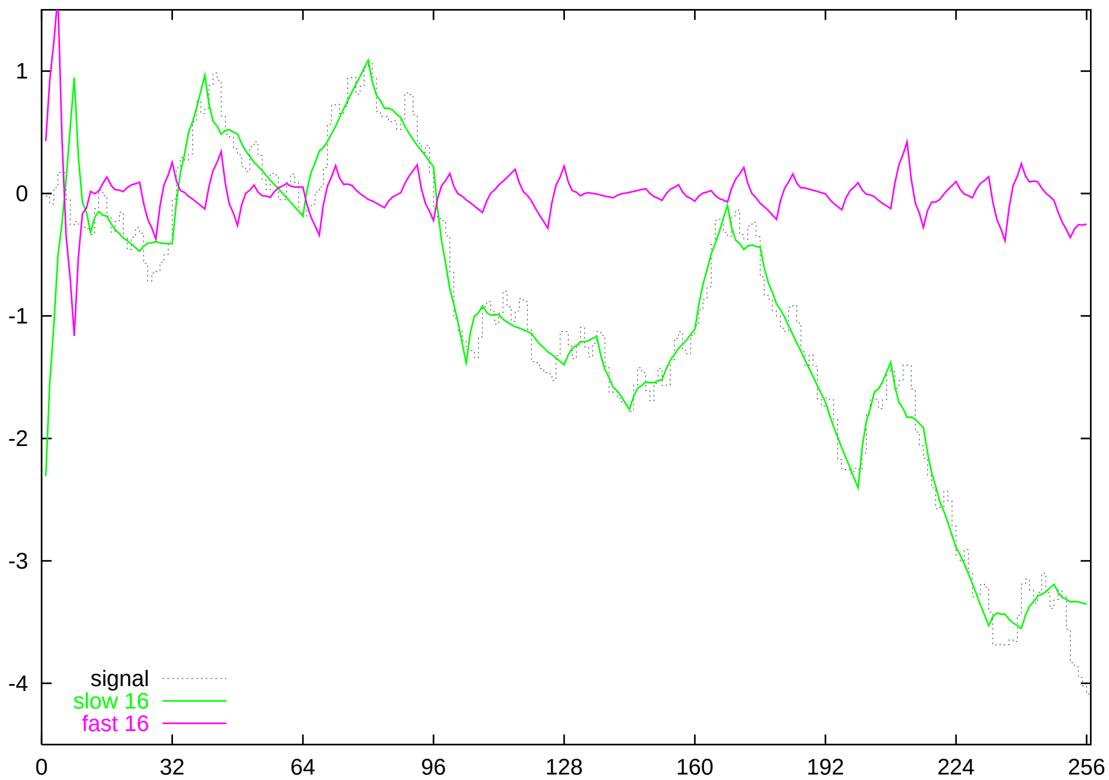

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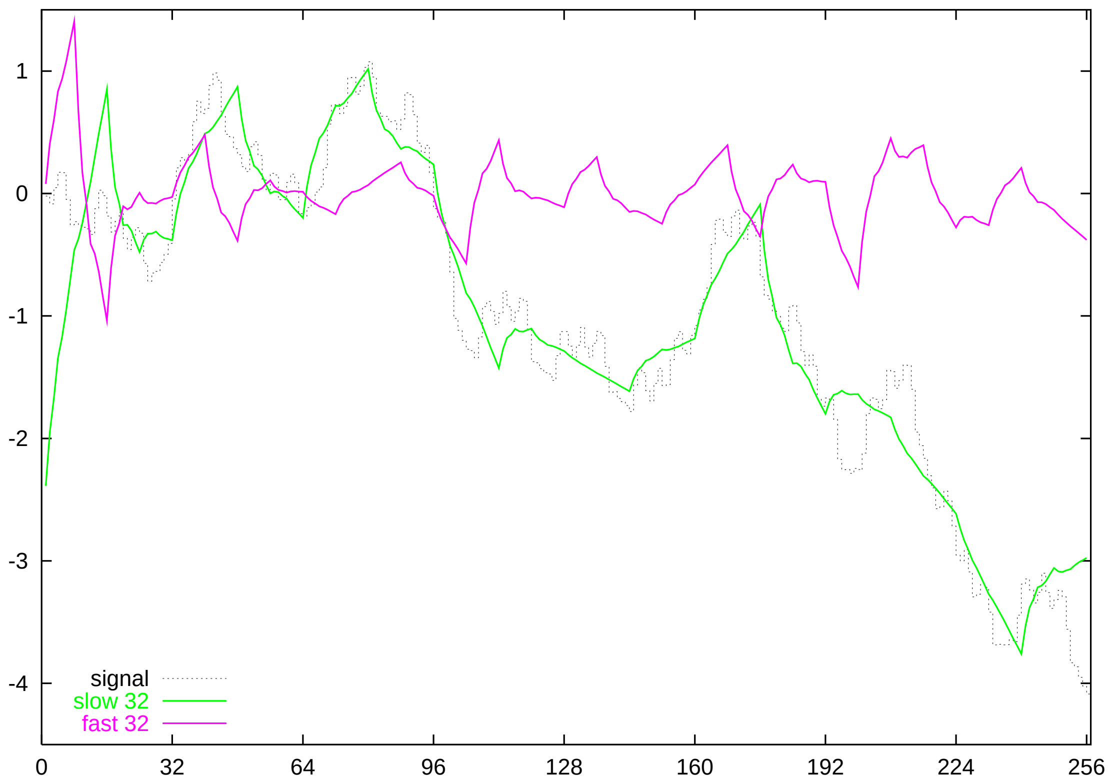

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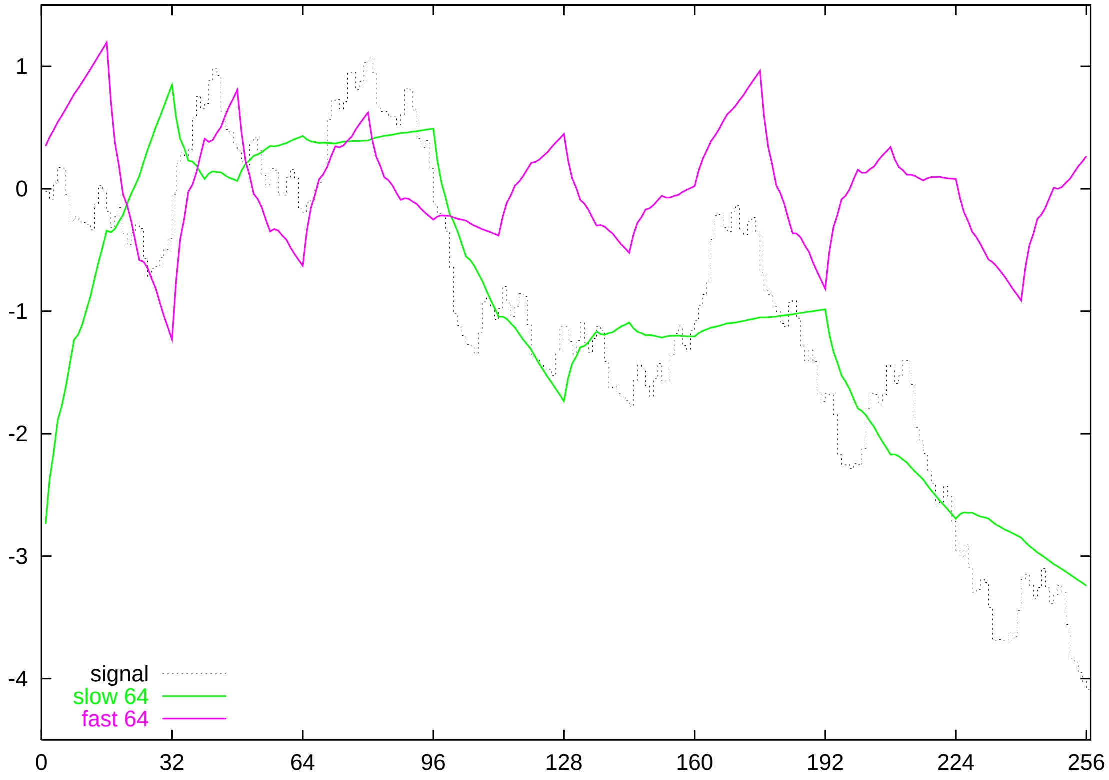
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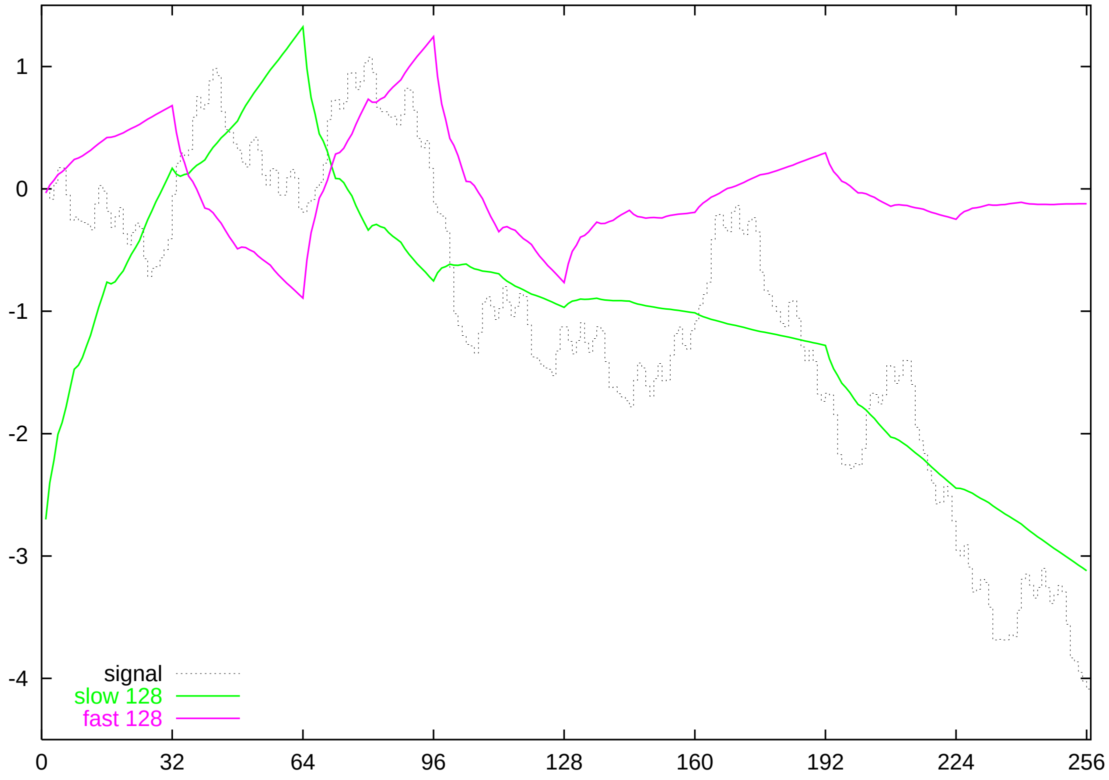

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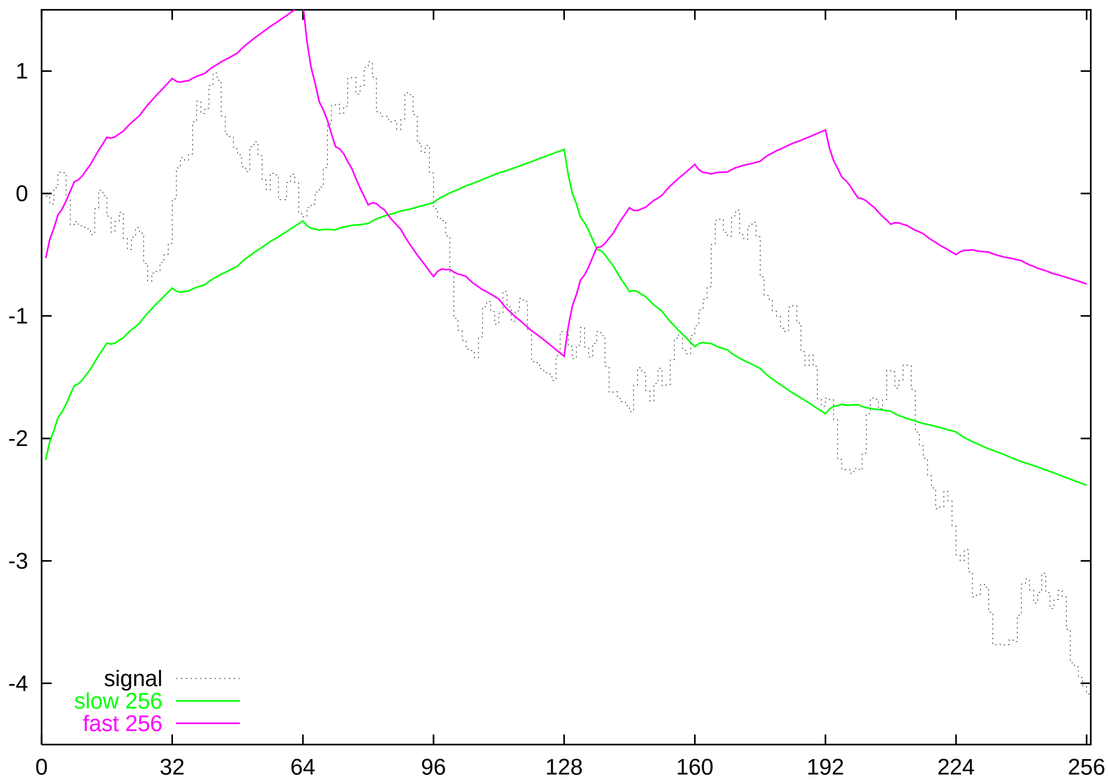

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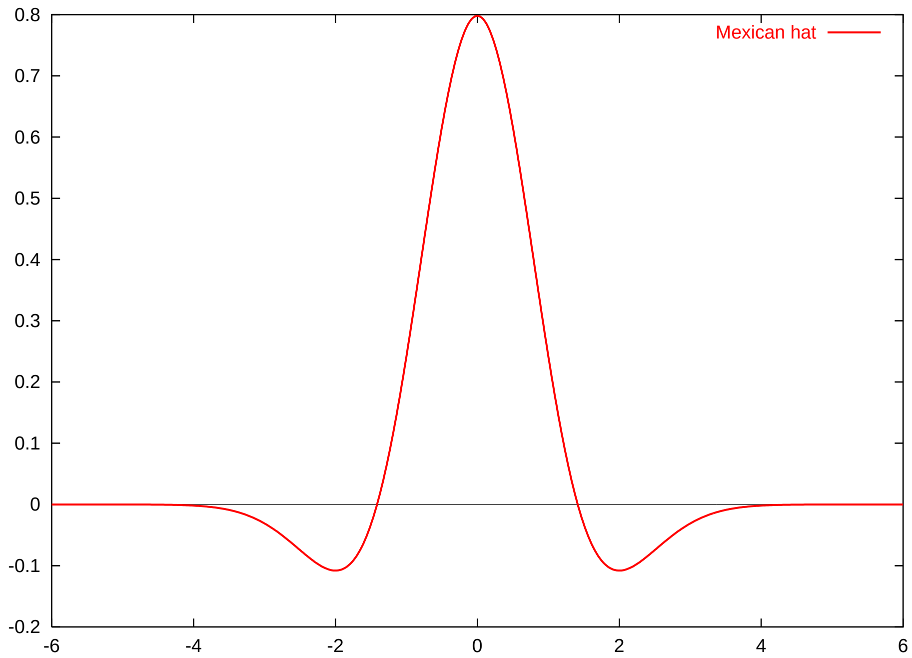
Other wavelet bases

There are infinitely many different wavelet bases. For example, Daubechies bases of higher order — the high pass filter kills local trends of orders higher than two.

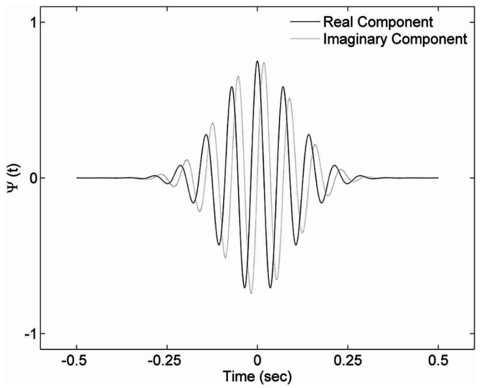
Haar wavelets are nor continuous. DAUB(4) wavelets are continuous, but not differentiable. Higher order Daubechies wavelets have non-differentiable higher derivatives. Wavelets that can be differentiated arbitrarily many times are considered very useful.

There are also bases that are not introduced through filter banks. For example, the “Mexican hat” is particularly popular.



$$\psi(x) = \sqrt{\frac{2}{\pi}} \left(1 - \frac{x^2}{2}\right) \exp\left(-\frac{x^2}{2}\right) \tag{9}$$

Morlet wavelet (Gabor wavelet)



$$\psi_\sigma(x) = c_\sigma \pi^{-\frac{1}{4}} e^{-\frac{x^2}{2}} \left(e^{i\sigma x} - e^{-\frac{\sigma^2}{2}} \right) \quad (10)$$

where $c_\sigma = \left(1 + e^{-\sigma^2} - 2e^{-\frac{3}{4}\sigma^2} \right)^{-\frac{1}{2}}$.

Three point Haar wavelets

$$\underbrace{\begin{bmatrix} a_1 & a_2 & a_3 & & & & & & \\ b_1 & b_2 & b_3 & & & & & & \\ c_1 & c_2 & c_3 & & & & & & \\ & & & a_1 & a_2 & a_3 & & & \\ & & & b_1 & b_2 & b_3 & & & \\ & & & c_1 & c_2 & c_3 & & & \\ & & & & & & \ddots & & \\ & & & & & & a_1 & a_2 & a_3 \\ & & & & & & b_1 & b_2 & b_3 \\ & & & & & & c_1 & c_2 & c_3 \end{bmatrix}}_{\mathcal{W}_N,\ N=3^p} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ \vdots \\ x_{N-2} \\ x_{N-1} \\ x_N \end{bmatrix} = \begin{bmatrix} s_1 \\ d_1 \\ d_2 \\ s_2 \\ d_3 \\ d_4 \\ \vdots \\ s_{N/3} \\ d_{2N/3-1} \\ d_{2N/3} \end{bmatrix} \rightarrow \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_{N/3} \\ d_1 \\ d_2 \\ \vdots \\ d_{2N/3} \end{bmatrix} \tag{11}$$

Then

$$\mathcal{W}_{N/3} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_{N/3} \end{bmatrix} = \dots \tag{12}$$

etc

Construction of three-point Haar wavelets

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

orthogonal
b: high-pass
c: high-pass

$$\begin{aligned}
a_1^2 + a_2^2 + a_3^2 &= 1 \\
b_1^2 + b_2^2 + b_3^2 &= 1 \\
c_1^2 + c_2^2 + c_3^2 &= 1 \\
a_1 b_1 + a_2 b_2 + a_3 b_3 &= 0 \\
a_1 c_1 + a_2 c_2 + a_3 c_3 &= 0 \\
b_1 c_1 + b_2 c_2 + b_3 c_3 &= 0 \\
b_1 + b_2 + b_3 &= 0 \\
c_1 + c_2 + c_3 &= 0
\end{aligned}$$

9 parameters, 8 equations — one-parameter family of solutions.

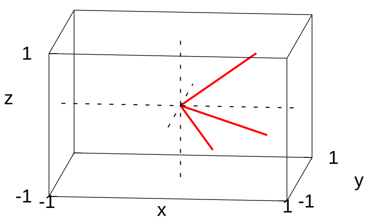
A Haar wavelet: $a_1 = a_2 = a_3 = \dfrac{1}{\sqrt{3}}$

A single non-trivial function: $c_1 = -b_3, c_2 = -b_2, c_3 = -b_1$

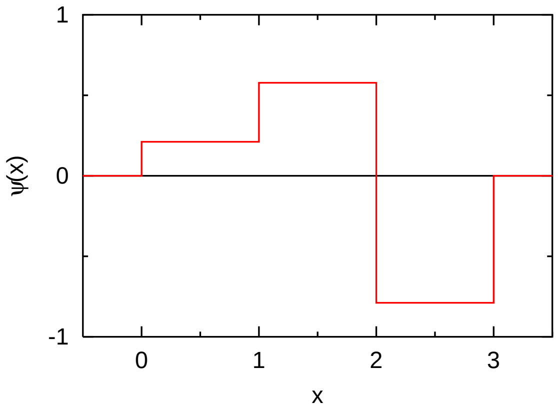
$$
\begin{aligned}
b_1^2 + b_2^2 + b_3^2 &= 1 \\
b_2^2 + 2b_1 b_3 &= 0 \\
b_1 + b_2 + b_3 &= 0
\end{aligned}
$$

The only non-trivial solution:

$$\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1 & 1 \\ \frac{1}{2}\left(\sqrt{3}-1\right) & 1 & -\frac{1}{2}\left(\sqrt{3}+1\right) \\ \frac{1}{2}\left(\sqrt{3}+1\right) & -1 & -\frac{1}{2}\left(\sqrt{3}-1\right) \end{bmatrix}$$



$$\psi(x) = \begin{cases} \frac{1}{2\sqrt{3}}\left(\sqrt{3}-1\right) & 0 \leqslant x \leqslant 1 \\ \frac{1}{\sqrt{3}} & 1 < x \leqslant 2 \\ -\frac{1}{2\sqrt{3}}\left(\sqrt{3}+1\right) & 2 < x \leqslant 3 \\ 0 & \text{otherwise} \end{cases}$$



Mother three-point Haar wavelet

The basis:

$$\psi_{jkl}(x) = (-1)^l 3^{-k/2} \psi\left((-1)^l 3^k (x-j)\right) \tag{13}$$

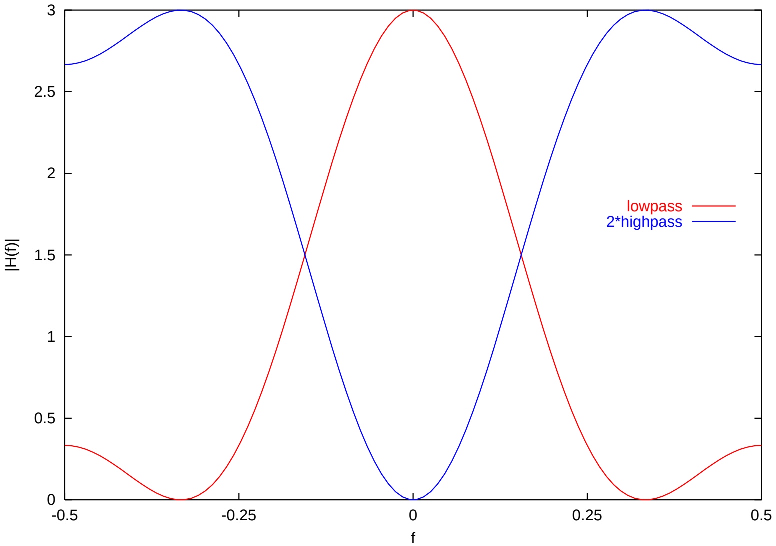
plus a constant scaling function.

The transfer functions:

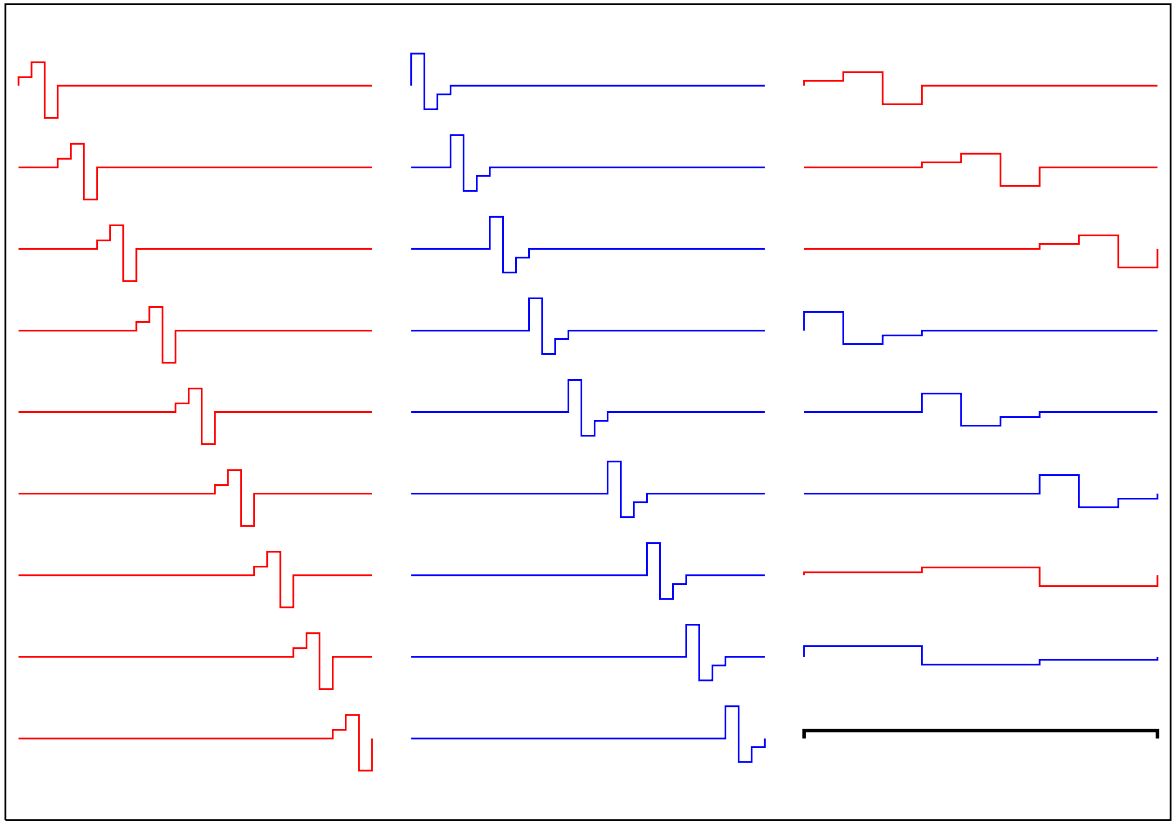
Low pass: $H(f) = \frac{1}{\sqrt{3}}(1 + 2\cos 2\pi f)$

First high pass: $H(f) = \frac{1}{\sqrt{3}}(1 - 2\cos 2\pi f) - \frac{i}{\sqrt{3}}\sin 2\pi f$

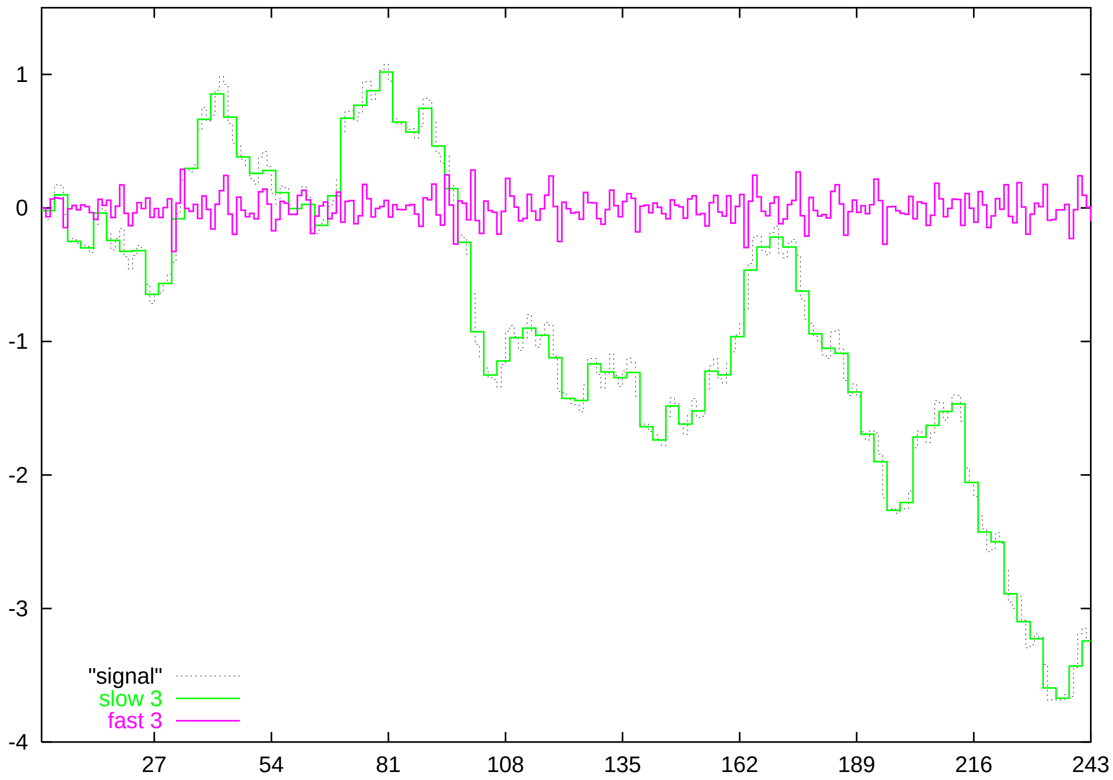
Second high pass: $H(f) = -\frac{1}{\sqrt{3}}(1 - 2\cos 2\pi f) - \frac{i}{\sqrt{3}}\sin 2\pi f$

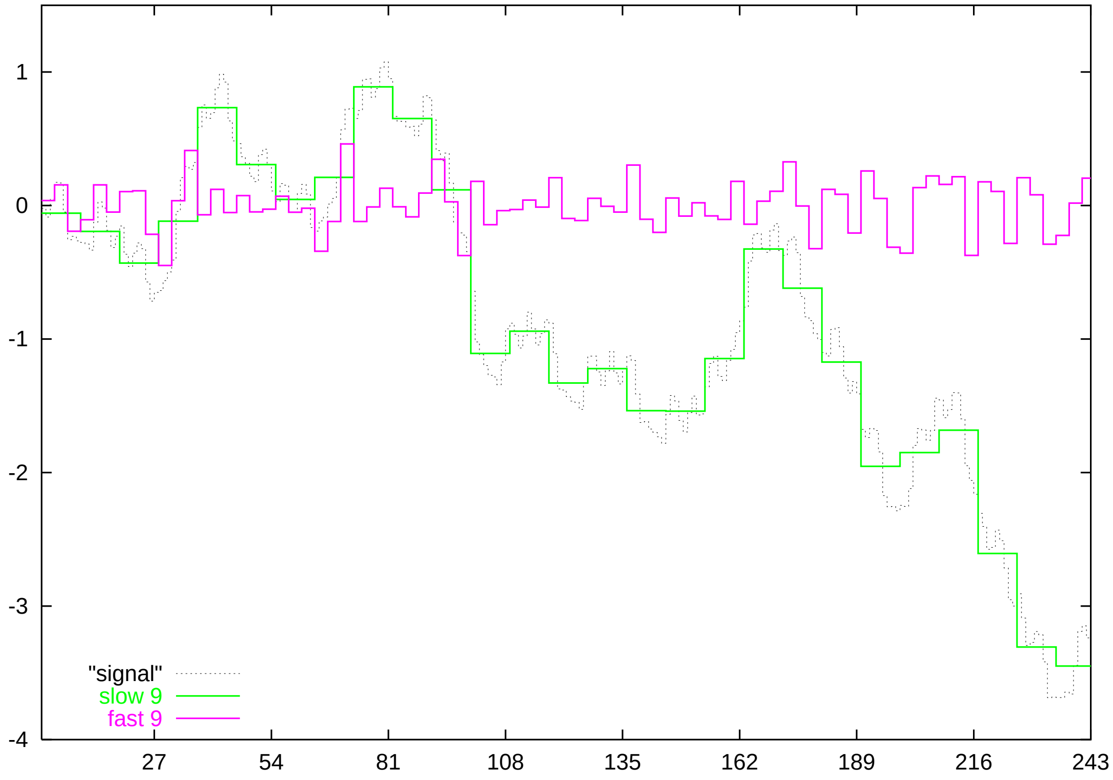


The basis

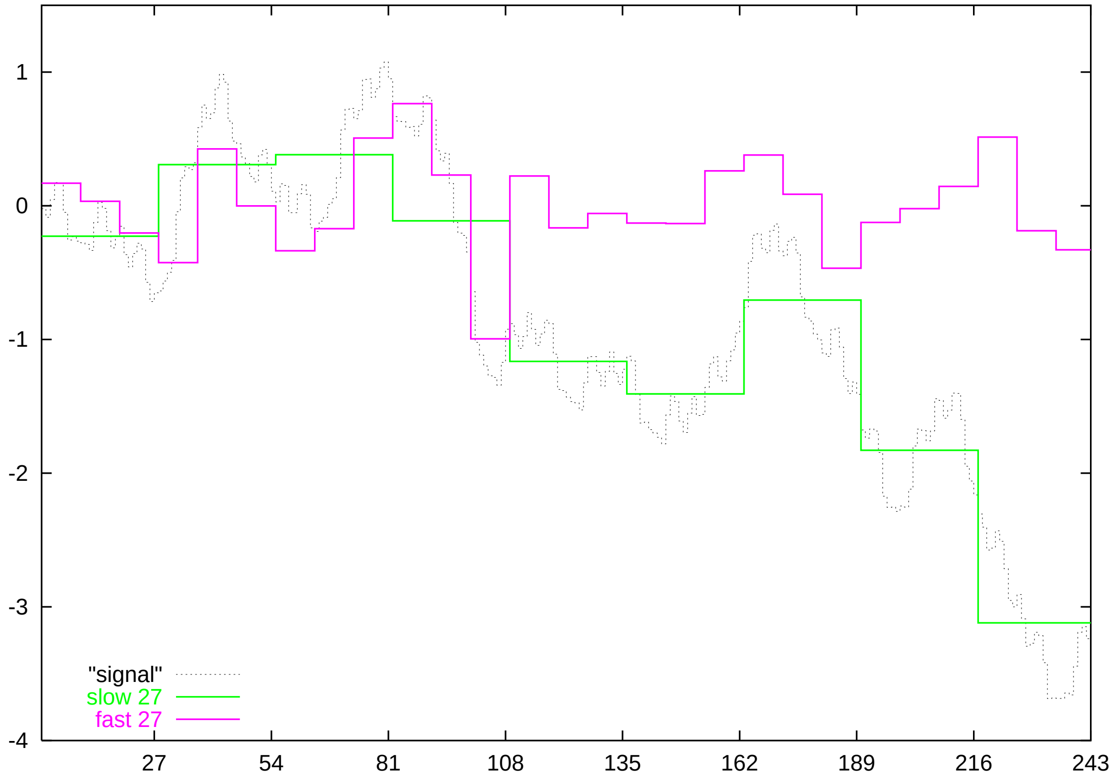


Multiresolution analysis with three point Haar wavelets





1
0
-1
-2
-3
-4
27
54
81
108
135
162
189
216
243
"signal"
slow 9
fast 9



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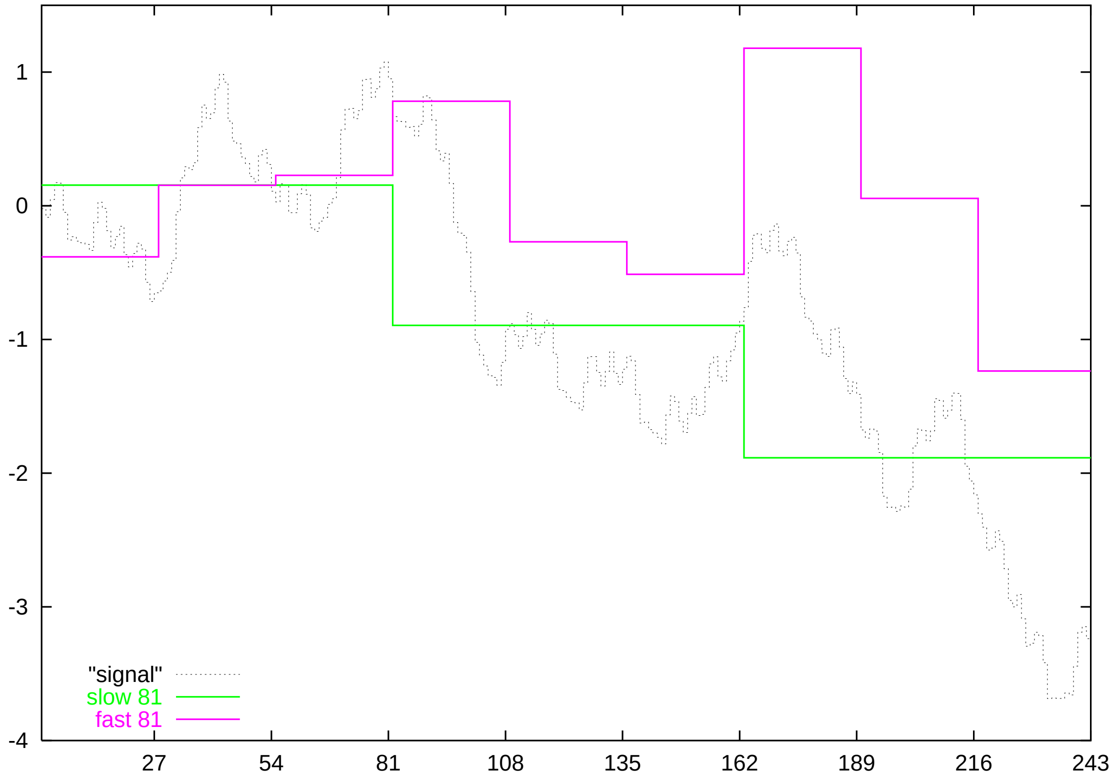
135
162
189
216
243
"signal"
slow 27
fast 27



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-3
-4
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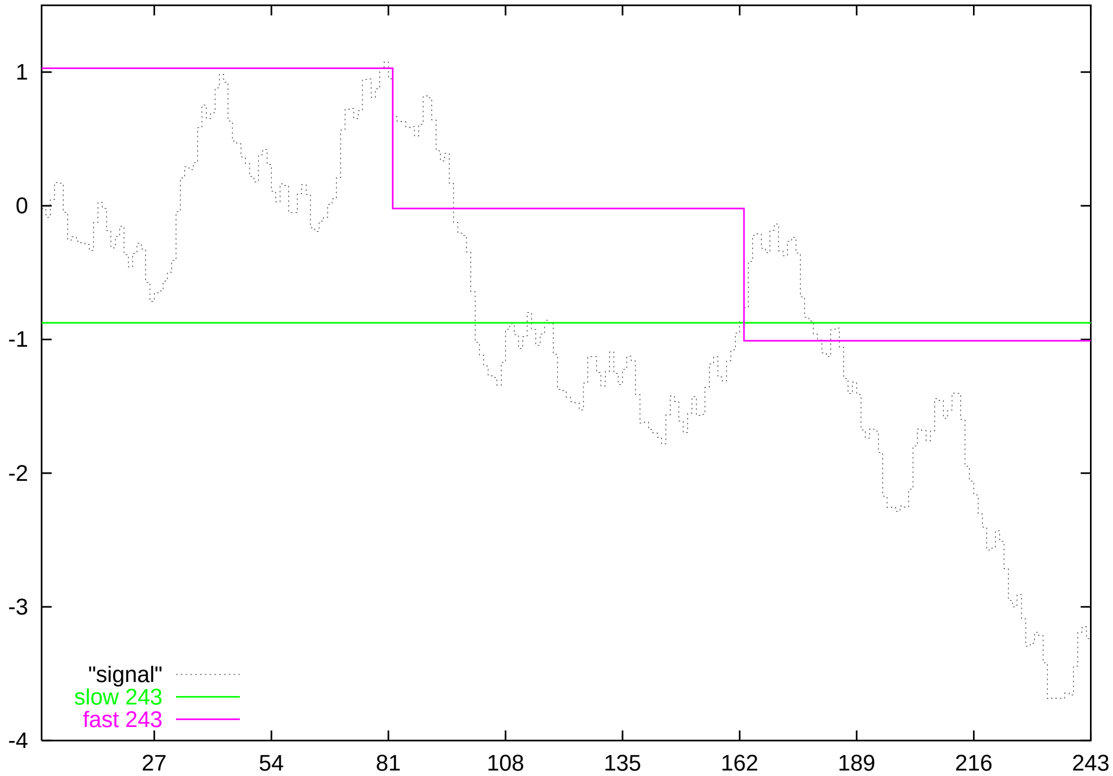
108
135
162
189
216
243
"signal"
slow 81
fast 81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"signal"
slow 243
fast 243