Statistical Physics 1

1. Let $\{P_i\}_{i=1}^{N}$ be a probability distribution: $\forall i\colon\ P_i \geqslant 0$, $\sum\limits_i P_i = 1$. The Shannon-Gibbs entropy of a probability distribution is defined as

$$S = -k_B \sum_{i=1}^{N} P_i \ln P_i \,, \qquad k_B > 0 \,. \tag{1}$$

   Find the distribution $\{P_i\}$ that maximizes (1). What is the maximal value of the entropy?

2. Let P_i be the probability that a system is in the i-th state. There is a quantity E_i associated with each state. Find the probability distribution that maximizes the entropy (1) under the constraint that

$$\sum_{i=1}^{N} E_i P_i = E \,. \tag{2}$$

3. The Rényi entropy of a probability distribution is defined as

$$S_q = \frac{1}{1-q} \ln \left(\sum_{i=1}^{N} p_i^q \right) \,, \tag{3}$$

   where $q > 0$ and $q \neq 1$. Discuss the limiting cases $q \to 0$, $q \to 1$, $q \to \infty$. Calculate Rényi entropies of the distributions found in the previous two problems.

4. The Hamiltonian of a classic perfect gas has the form

$$H = \frac{1}{2m} \sum_{i=1}^{N} \vec{p}_i^{\,2} \,, \tag{4}$$

   where N is the number of particles, $\vec{p}_i$ is the momentum of the i-th particle amd m is the mass (uniform for all particles). Let the total energy of the gas belong to the interval $[E, E + \Delta E]$. Find the phase space volume of the system. Using this quantity, find the entropy, internal energy, temperature, presure and other thermodynamic properties of the gas.

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