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ScienceDirect
Physica A 378 (2007) 201–210
PHYSICA A
www.elsevier.com/locate/physa

An exactly solvable nonlinear model: Constructive effects of correlations between Gaussian noises

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Received 6 March 2006; received in revised form 16 October 2006
Available online 15 November 2006

Abstract

A system with two correlated Gaussian white noises is analysed. This system can describe both stochastic localization and long tails in the stationary distribution. Correlations between the noises can lead to a nonmonotonic behaviour of the variance as function of the intensity of one of the noises and to a stochastic resonance. A method for improving the transmission of external periodic signal by tuning parameters of the system discussed in this paper is proposed.

Keywords: Correlated Gaussian noises; Stochastic resonance

1. Introduction

Stochastic models with multiplicative, or parametrical, noise find numerous applications in a variety of branches of science and technology. Unfortunately, models for which analytical results are known are very scarce and any such a model deserves a thorough discussion. Recently, Denisov and Horsthemke [1] have discussed a model given by the equation

$$\dot{x} = -ax + |x|^{\alpha}\eta(t), \tag{1}$$

where $0 \leqslant \alpha \leqslant 1$, $\eta(t)$ is a Gaussian noise, possibly coloured, and have found that it can describe anomalous diffusion and stochastic localization. Denisov and Horsthemke have also discussed several physical systems in which models of type (1) can be useful; see references provided in their paper. Later Vitrenko [2] has generalized (1) to include two noise terms:

$$\dot{x} = -(a + \eta_1(t))x + |x|^{\alpha}\eta_2(t), \tag{2}$$

where $\eta_{1,2}$ are certain coloured and correlated Gaussian noises. This system has a very nice feature: for $0 < \alpha < 1$, it interpolates between a linear transmitter with multiplicative and additive noises ($\alpha = 0$) and a system that closely resembles a linear system with a purely multiplicative noise ($\alpha = 1$). These two linear

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0378-4371/$ - see front matter
doi:10.1016/j.physa.2006.10.056

systems are very well known in the literature (see e.g. Ref. [3] and references quoted therein). Vitrenko has formally linearized system (2) by means of a substitution that has been already used in Ref. [1]:

$$y = \frac{x}{|x|^{\alpha}}, \tag{3}$$

and solved the resulting equation for the trajectories. Converting back to the original variable proves to be rather tricky and that author has managed to do so only if the noises $\eta_{1,2}$ are correlated in a very specific (not to say peculiar) manner. It is now widely recognized that correlations between various noises can lead to many interesting effects. It is, however, possible that phenomena reported by Vitrenko result principally from the very specific form of correlations assumed by this author and are not generic to system (2). We find it interesting to see how the system behaves for the intermediate values of α when the correlation requirements are less restrictive than those discussed by Vitrenko.

Coloured noises introduce more complexity. However, if a dynamical effect is present in the white noise case, it also appears, perhaps in a distorted form, in the coloured case [4]. To simplify the discussion, we will assume that the noises are white. Finally, note that the expression $a + \eta_1(t)$ in Eq. (2) can be interpreted as a biased noise. The noise that multiplies $|x|^{\alpha}$ in Eq. (2) is not biased. To "symmetrize" the system, we include a bias in ξ_2 in our analysis. It is also convenient to have explicit expressions for noise amplitudes, or coupling constants between the noises and the dynamical variable. We thus recast Eq. (2) in the form

$$\dot{x} = -(a + p\xi_1(t))x + |x|^{\alpha}(b + q\xi_2(t)), \tag{4}$$

where $a > 0$, $0 \leqslant \alpha \leqslant 1$, $\xi_{1,2}$ are mutually correlated Gaussian white noises:

$$\langle \xi_i(t) \rangle = 0, \quad \langle \xi_i(t)\xi_i(t') \rangle = \delta(t - t'), \quad i = 1, 2, \tag{5a}$$

$$\langle \xi_1(t)\xi_2(t') \rangle = c\delta(t - t') \tag{5b}$$

and $c \in [-1, 1]$. If not otherwise specified, we interpret the noises in the sense of Ito. For the sake of terminology, we will call the noise $\xi_1(t)$ "multiplicative" and $\xi_2(t)$ "additive", even though this terminology is accurate only if $\alpha = 0$ (for $\alpha > 0$ both noises couple parametrically). Note that if a particle hits $x = 0$, it stays there forever if $\alpha > 0$. Accordingly, any fraction of the initial population that starts at $x = 0$ remains there and may be trivially excluded from the subsequent discussion of stationary distributions.

There is, in fact, one more reason for including $b \neq 0$ in our discussion. Much as substitution (3) linearizes system (4), another substitution, namely

$$z = \frac{|x|^{\alpha}}{x} \tag{6}$$

converts it to a noisy logistic equation

$$\dot{z} = (1 - \alpha)(a + p\xi_1(t))z - (1 - \alpha)(b + q\xi_2(t))z^2. \tag{7}$$

We have discussed this last system in Ref. [5,6] and found that $b \neq 0$ together with correlations between the noises can lead to a nonmonotonic behaviour of the variance $\langle z^2 \rangle - \langle z \rangle^2$ as a function of the intensity of the "additive" noise, q, and to a stochastic resonance [7] if the system is additionally stimulated by an external periodic signal. It would be naive to expect that these phenomena occur in system (4) in exactly the same manner as they do in (7). A nonlinear change of variables, especially in case of stochastic equations, can significantly alter the behaviour. We will see, however, that there are striking similarities between systems (7) and (4).

This paper is organized as follows: we construct the Fokker–Planck equation for system (4) in Section 2 and in Section 3 we present its stationary solutions. Then in Section 4 we discuss the constructive effects of the correlations between the noises; in particular, in Section 4.2 we give numerical evidence for the presence of the stochastic resonance. Conclusions are given in Section 5.

2. The Fokker–Planck equation

The problem of constructing a Fokker–Planck equation corresponding to a process driven by two correlated Gaussian white noises has been first discussed in Ref. [8], where the two noises have been decomposed into two independent processes. The same result has been later rederived in Ref. [9], where the authors have attempted to avoid an explicit decomposition of the noises but eventually resorted to a disguised form of the decomposition. The Fokker–Planck equation for correlated white noises has been also discussed in Refs. [10,11] and in several other papers; see, for example, Ref. [6] for a particularly simple rederivation.

A general Langevin equation

$$\dot{x} = h(x) + g_1(x)\xi_1(t) + g_2(x)\xi_2(t), \tag{8}$$

where $x(t)$ is a one-dimensional process and $\xi_{1,2}$ are as in Eq. (5), leads to the following Fokker–Planck equation in the Ito interpretation:

$$\frac{\partial P(x,t)}{\partial t} = -\frac{\partial}{\partial x} h(x)P(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2} B(x)P(x,t), \tag{9a}$$

where

$$B(x) = [g_1(x)]^2 + 2c\, g_1(x)g_2(x) + [g_2(x)]^2. \tag{9b}$$

In case of Eq. (4) we obtain

$$\frac{\partial P(x,t)}{\partial t} = \frac{\partial}{\partial x}(ax - b|x|^\alpha)P(x,t) + \frac{1}{2}\frac{\partial^2}{\partial x^2}(p^2x^2 - 2cpqx|x|^\alpha + q^2|x|^{2\alpha})P(x,t). \tag{10}$$

In the following we interpret $|x|^\alpha$ as $|x|^\alpha = (x^2)^{\alpha/2}$, where the square of x must be calculated prior to taking the fractional power. It is also apparent that the probability $P(x,t)$ does not depend on the absolute signs of the amplitudes p, q, but only on their relative sign. We assume that $\mathrm{sgn}(pq) = +1$. This comes at no loss of generality as Eq. (10) is invariant under a simultaneous change of signs of pq and c.

Finding stationary distributions corresponding to Eq. (10) is the main goal of this paper. This, in principle, could be handled by standard methods [12], but it would be very difficult due to the absolute value and the fractional powers. It is apparent that since the right-hand side of the corresponding stationary equation vanishes identically if $x = 0$, the term $\delta(x)$ should always be included in any stationary distribution. We now use substitution (3). After some algebra we eventually obtain

$$\begin{aligned}\frac{\partial P(y,t)}{\partial t} &= (1-\alpha)\frac{\partial}{\partial y}\left[ay - b + \frac{\alpha}{2y}(p^2y^2 - 2cpqy + q^2)\right]P(y,t) \\ &\quad + \frac{1}{2}(1-\alpha)^2\frac{\partial^2}{\partial y^2}(p^2y^2 - 2cpqy + q^2)P(y,t).\end{aligned} \tag{11}$$

The last term in the square brackets in Eq. (11) corresponds to the Ito interpretation [13]. This term is missing if the noises are interpreted according to Stratonovich.

The stationary distribution solves an equation that is fairly easy to integrate:

$$\left[\left(a + \left(1 - \frac{1}{2}\alpha\right)p^2\right)y - b - cpq + \frac{\alpha q^2}{2y}\right]P_{\mathrm{st}} + \frac{1}{2}(1-\alpha)(p^2y^2 - 2cpqy + q^2)\frac{\mathrm{d}P_{\mathrm{st}}}{\mathrm{d}y} = 0. \tag{12}$$

3. Stationary distributions

Before proceeding to the general case, let us discuss the case where there is only one noise present.

3.1. No "multiplicative" noise

If there is no "multiplicative" noise, $p = 0$, and no bias in the "additive" noise, $b = 0$, our problem reduces to that discussed in Ref. [1]. In this case, Eq. (11) corresponds to the following Langevin equation:

$$\dot{y} = -(1-\alpha)\left(ay + \frac{\alpha q^2}{2y}\right) + (1-\alpha)^2 q \xi_2(t), \tag{13}$$

which, in turn, corresponds to an overdamped motion in a potential

$$V_{\rm eff}(y) = \frac{1}{2}(1-\alpha)ay^2 + \frac{1}{2}(1-\alpha)\alpha q^2 \ln|y|. \tag{14}$$

The effective potential (14) has an infinite noise-created well at $y = 0$ which traps Brownian particles; this well is missing if the noises are interpreted according to Stratonovich. Curiously, in another context we have observed a similar phenomenon, where noise interpreted according to Ito created an insurmountable barrier restricting particles to one half of the real axis [14]. A similar barrier is observed in the noisy logistic system (7), cf. Ref. [6]. Loosely speaking, the change of variables (6) converts an infinite barrier into an infinite well.

The stationary equation now takes the form

$$\left(ay + \frac{\alpha q^2}{2y}\right)P_{\rm st} + \frac{1}{2}(1-\alpha)q^2\frac{{\rm d}P_{\rm st}}{{\rm d}y} = 0 \tag{15}$$

and is solved by

$$P_{\rm st}(y) = \frac{N}{|y|^{\alpha/(1-\alpha)}}\exp\left(-\frac{ay^2}{(1-\alpha)q^2}\right), \tag{16}$$

where N is a normalization constant. If we want to transform back to the original variable, we must remember that $P_{\rm st}(y)$ is a probability distribution and, therefore, the Jacobian of substitution (3) must be included in the transformation. Finally,

$$P_{\rm st}(x) = \frac{N}{|x|^{2\alpha}}\exp\left(-\frac{a\left(x^2\right)^{1-\alpha}}{(1-\alpha)q^2}\right). \tag{17}$$

Distribution (17) is normalizable for $a>0$ and $0\leqslant\alpha<\frac{1}{2}$. For $\alpha = 0$ it reduces to a standard Gaussian distribution, and for $0<\alpha<\frac{1}{2}$ it mildly diverges at $x = 0$. If α increases towards $\frac{1}{2}$, the divergence becomes more pronounced. At the same time, though, tails of the distribution get heavier which is characteristic for anomalous diffusion. In other words, if $\alpha = 0$, the stationary distribution is nonsingular, but the "less additive" the system becomes as α increases, the more pronounced the singularity is and the tails of the distribution get flatter.

The presence of a bias, $b\neq 0$, introduces some asymmetry in the exponential term, but the overall behaviour remains much the same:

$$P_{\rm st}(x) = \frac{N}{|x|^{2\alpha}}\exp\left(\frac{2bx|x|^{-\alpha} - a\left(x^2\right)^{1-\alpha}}{(1-\alpha)q^2}\right). \tag{18}$$

If the noises are interpreted according to Stratonovich, we obtain

$$P_{\rm st}^{\rm Strat}(x) = \frac{N'}{|x|^{\alpha}}\exp\left(\frac{2bx|x|^{-\alpha} - a(x^2)^{1-\alpha}}{(1-\alpha)q^2}\right) \tag{19}$$

and the distribution is normalizable for $0\leqslant\alpha<1$.

3.2. No "additive" noise

If there is no "additive" noise, $q = 0$, and no bias, $b = 0$, dynamics (4) reduces to that of a linear transmitter with a multiplicative noise and no other external forcing [3]. The only normalizable stationary solution is $P_{\rm st}(x) = \delta(x)$, corresponding to all particles eventually collapsing to their common resting point. If $b\neq 0$, there is no stationary solution as some particles go to the resting point, but some can escape to infinity.

3.3. The general case

If the noises are *not* maximally correlated, $|c| \neq 1$, the general solution reads

$$P_{\rm st}(x) = \frac{N \exp[2(bp - caq)/(1-\alpha)\sqrt{1-c^2}p^2 q \arctan((px|x|^{-\alpha} - cq)/\sqrt{1-c^2}\,q)]}{|x|^{\mu}[q^2 - 2cpqx|x|^{-\alpha} + p^2(x^2)^{1-\alpha}]^{(1+a/(1-\alpha)p^2)}}, \tag{20}$$

where N is again a normalization constant and μ depends on the interpretation of the noises: $\mu = 2\alpha$ in the Ito, and $\mu = \alpha$ in the Stratonovich interpretation, cf. Eqs. (17) and (19) above.

If the noises are interpreted according to Ito, $\mu = 2\alpha$, principal properties of distribution (20), despite its complicated form, are easy to find. Because the inverse tangent function, $\arctan(\cdot)$, is limited, the exponential term is also limited and convergence properties of (20) depend solely on its denominator. One can easily see that this distribution is normalizable for all $0 \leqslant \alpha < \frac{1}{2}$. The distribution has rather heavy tails. It has a convergent first moment if $a > \frac{1}{2}p^2$. The second moment is convergent if a stronger condition, $a > p^2$, is satisfied. Note that these are the same conditions that need to be satisfied for the existence of the moments of both the linear [3] and the noisy logistic [6] systems.

In general, distribution (20) is not symmetric. Apart from the central singularity, it has an additional peak whose location depends on the sign of c: if $c > 0$, the peak is located to the right of $x = 0$, and if $c < 0$, it is located to the left. If $|c| \lesssim 1$, the height of this peak can be very large. The asymmetry is physically introduced by the interplay of the "exponential" and "multiplicative" noises; on a formal level, the asymmetry results from the exponential term and can be removed if

$$bp - caq = 0. \tag{21}$$

It is important to understand the origin of this phenomenon. The asymmetric broadening results from the bias—the force acting in one direction is, on the average, larger than the force acting in the opposite one. In system (4), the parameter $b \neq 0$ acts as one source of the bias; it has been introduced for this specific purpose. It is also known that correlations between two noises can effectively introduce another bias, see e.g. Refs. [3,8,10]. If condition (21) is met, the two sources of bias nullify each other. To see this, let us represent the two correlated Gaussian white noises $\xi_{1,2}$ as linear combinations of two *independent* GWNs $\psi_{1,2}$:

$$\xi_1(t) = \psi_1(t), \tag{22a}$$

$$\xi_2(t) = c\psi_1(t) + \sqrt{1-c^2}\psi_2(t). \tag{22b}$$

With condition (21) satisfied, the Langevin equation (4) now takes the form

$$\dot{x} = -(a + p\psi_1(t))\left(x - \frac{b}{a}|x|^{\alpha}\right) + \sqrt{1-c^2}q|x|^{\alpha}\psi_2(t). \tag{23}$$

The system now behaves as if it were driven by two uncorrelated white noises, one of which is unbiased. As a result, the bias-induced asymmetric broadening of the stationary distribution disappears.

In the Stratonowich interpretation, $\mu = \alpha$ in Eq. (20), the distribution is normalizable for $0 \leqslant \alpha < 1$ and the first and second moments exist if $a > \frac{1}{2}\alpha p^2$ and $a > \frac{1}{2}(1+\alpha)p^2$, respectively. Curiously, much of our discussion on the asymmetries introduced by the bias and the correlations remains the same: condition (21) for the mutual cancellation of the two sources of bias does not depend on the noise interpretation chosen.

3.4. Maximally correlated noises

Distribution (20) does not have a universal limit $|c| \to 1$. Instead, if $c = \pm 1$, we need to solve Eq. (12) directly and then convert back to the original variable. We obtain a candidate solution

$$P_{\rm trial}(x) \sim \frac{\exp(2(bp \mp aq)/(1-\alpha)p^2(q \mp px|x|^{-\alpha}))}{|x|^{\mu}|q \mp px|x|^{-\alpha}|^{2(1+a/(1-\alpha)p^2)}}, \tag{24}$$

where μ is as in Eq. (20) and the $\mp$ sign is the opposite of the sign of the correlation coefficient, $c = \pm 1$. However, the right-hand side of (24) is not normalizable. If $q \mp px|x|^{-\alpha} = 0$, the exponential in (24) hits its essential singularity. This singularity is eliminated if a special case of condition (21), namely

$$bp \mp aq = 0, \tag{25}$$

holds. In this case, either $p = q = 0$ and the system becomes fully deterministic, or the Langevin dynamics (4) takes a particularly simple form

$$\dot{x} = -(a + p\psi_1(t))\left(x \mp \frac{q}{p}|x|^\alpha\right) = -(a + p\psi_1(t))\left(x - \frac{b}{a}|x|^\alpha\right), \tag{26}$$

cf. Eq. (22) above, and the stationary Fokker–Planck equation (12) in the Ito interpretation factorizes to

$$\frac{py \mp q}{2py}\left[[(2a + (2-\alpha)p^2)y \mp \alpha pq]P_{\rm st}(y) + (1-\alpha)py(py \mp q)\frac{\mathrm{d}P_{\rm st}(y)}{\mathrm{d}y}\right] = 0. \tag{27}$$

The regular part of this equation, the one in the square brackets, again does not lead to a normalizable solution, but a singular distribution $\delta(py \mp q)$ solves Eq. (27). After transforming back to the original variable, this singular distribution corresponds to $\delta(x \mp (q/p)^{1/(1-\alpha)})$, which in turn corresponds to one of the stationary points of Eq. (26). As it can be easily verified, this stationary point is stable if the multiplicative noise is sufficiently weak, $a > p^2$. The other stationary point of Eq. (26), $x = 0$, is not even deterministically stable. We, therefore, conclude that if the noises are maximally correlated and condition (25) holds, the stationary distribution in the Ito interpretaion reads

$$P_{\rm st}(x) = \delta(x \mp (q/p)^{1/(1-\alpha)}), \tag{28}$$

provided the noise is sufficiently weak. Otherwise, and in particular in the Stratonovich interpretation, a stationary distribution does not exist. There is a striking similarity between systems (4) and (7), where a similar situation occurs [6]: if a condition analogous to (25) is satisfied and the noises are maximally correlated, the noisy logistic system has a δ-like stationary distribution. If the noises are maximally correlated but the counterpart of condition (25) does not hold, a stationary distribution does not form in the noisy logistic system, either.

4. Constructive effects of correlations

From now on, we interpret the noises only in the sense of Ito.

As we have seen, a delicate interplay between the correlations and the bias can significantly alter the shape of the stationary distribution. We may expect that this can lead to various unexpected properties of system (4).

4.1. Nonmonotonic behaviour of the variance

Recall that depending on the parameters, the stationary distribution of the system discussed in this paper can be nearly limited to very narrow peaks; with a different set of parameters, these peaks can be asymmetrically broadened. The second central moment of a probability distribution, $\langle x^2\rangle - \langle x\rangle^2$, if convergent, is perhaps one of its simplest and most easily comprehended characteristics. It is interesting to see how the second moment of distribution (20) behaves as a function of the "additive" noise strength. Because of the complicated analytical structure of this distribution, we have not been able to evaluate the integrals $\int_{-\infty}^{\infty} x\, P_{\rm st}(x)\, \mathrm{d}x$, $\int_{-\infty}^{\infty} x^2\, P_{\rm st}(x)\, \mathrm{d}x$ analytically. We have done so numerically instead. Example results are presented in Fig. 1; parameters chosen correspond to a convergent second moment. As we can see, a minimum of the variance as a function of the "additive" noise strength is clearly visible. This minimum is fairly deep if the correlations are large and becomes very shallow as the correlations decrease. Note that if $b < 0$, the minimum appears for negative values of the correlation coefficient (not plotted).

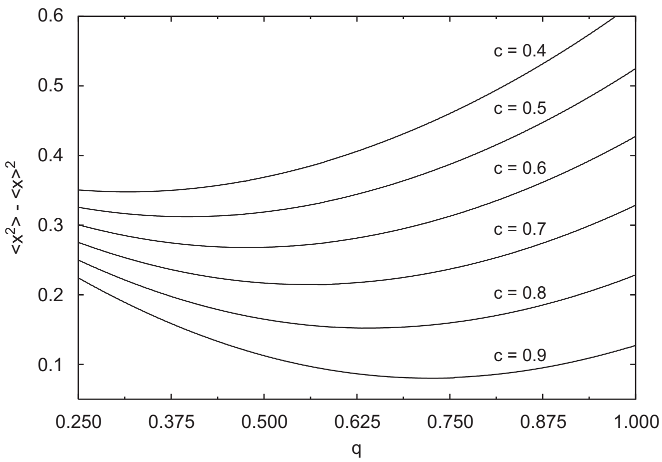

Fig. 1. Nonmonotonic behaviour of the variance of distribution (20) as a function of the "additive" noise strength, q, for various correlations between the noises, c. Other parameters are $\alpha = \frac{1}{8}$, $a = 1.25$, $b = 1.0$, $p = 1.0$.

4.2. Stochastic resonance

Now suppose that the system discussed in this paper is additionally stimulated by an external, periodic signal. The Langevin equation takes the form

$$\dot{x} = -(a + p\xi_1(t))x + |x|^{\alpha}(b + q\xi_2(t) + A\cos(\Omega t + \phi)) \tag{29}$$

where the noises are as in (5). Because we do not know exact solutions of a time-dependent Fokker–Planck equation corresponding to Eq. (29), we have solved Eq. (29) numerically with the Euler–Maryuama algorithm, consistent with the Ito interpretation, and a timestep equal to 2^{-12}. To generate the correlated noises $\xi_{1,2}$, we have first generated two independent Gaussian white noises $\psi_{1,2}$; we have used the Marsaglia algorithm [15] for that purpose and the famous Mersenne Twister [16] has been used as the underlying uniform generator. Then the correlated noises are created as linear combinations of the two uncorrelated ones, see Eq. (22) above. Example trajectories of system (29) and associated power spectra, averaged over 128 realizations of the noise and on the initial phase of the signal, ϕ, are presented in Fig. 2. The shape of the trajectories and the power spectra strongly depend on the parameters of the system, and on the correlation coefficient, c, and the strength of the "additive" noise, q, in particular. In general, the higher the correlations, the more ordered the trajectories are. It is worth noting that higher harmonics of the driving frequency can be visible in the power spectra, indicating a nonlinear nature of the coupling between the signal and the dynamical variable.

To quantify these observations, we will use the signal-to-noise ratio (SNR) as a measure of the stochastic resonance:

$$\mathrm{SNR} = 10\log_{10}\frac{S_{\mathrm{signal}}}{S_{\mathrm{noise}}(f = \Omega/2\pi)}, \tag{30}$$

where S_{signal} is the height of the peak in the power spectrum at the driving frequency, and $S_{\mathrm{noise}}(f)$ is the frequency-dependent noise-induced background. Several other measures of the stochastic resonance have been proposed [17], but we choose the SNR as the simplest, oldest and most commonly used one. Selected results, averaged on both realizations of the noises and the initial phase, are presented in Fig. 3. For high values of the correlation coefficient, a clear maximum in the SNR is visible. This shows that there is an optimal level of the "additive" noise that maximizes the ratio of power transmitted through coherent oscillations induced by the driving signal to that transmitted by the irregular ones, or that there is a stochastic resonance in system (29). For correlations only slightly larger than zero, the resonance is very small and it disappears for $c \leqslant 0$. Note that this happens if the asymmetry parameter, b, is greater than zero. For $b < 0$ the stochastic resonance occurs for negative correlations and reaches its largest magnitude at $c = -1$. In the symmetric case, $b = 0$,

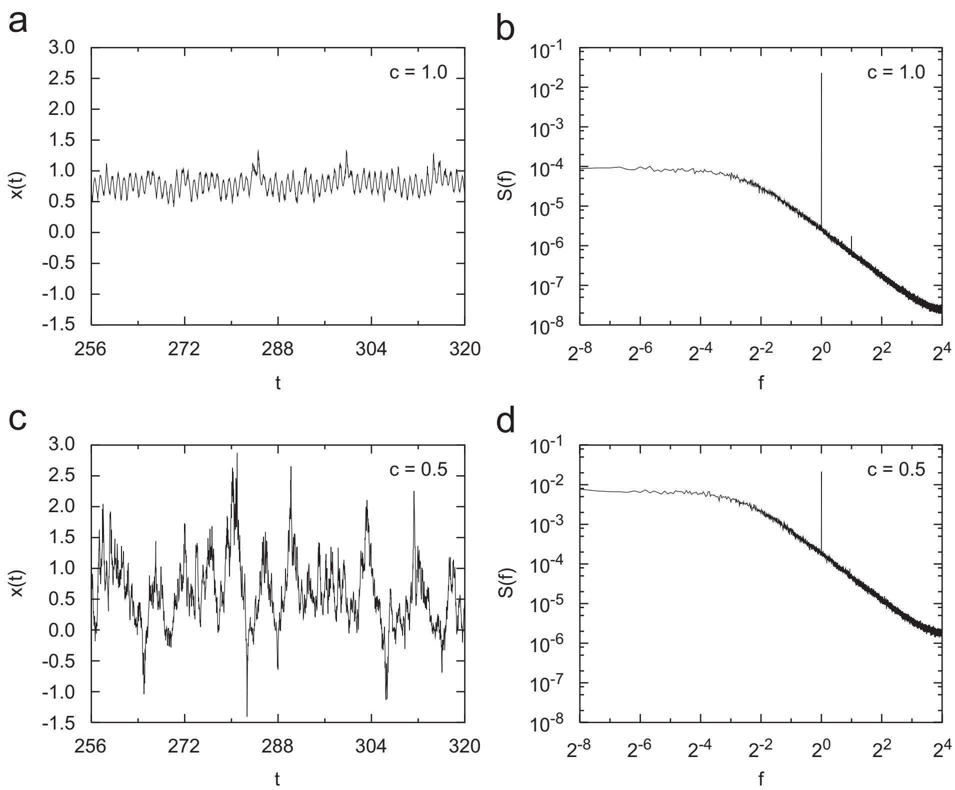

Fig. 2. Panel (a): a fragment of a typical realization of process (29) with $c = 1$. Panel (b): the corresponding power spectrum averaged over 128 realizations. Panels (c), (d): same as (a), (b) above, but with $c = 0.5$. Other parameters, common for all panels: $\alpha = \frac{1}{8}$, $a = 1.25$, $b = 1.0$, $p = 1.0$, $q = 0.8$, $A = 1$ and $\Omega = 2\pi$.

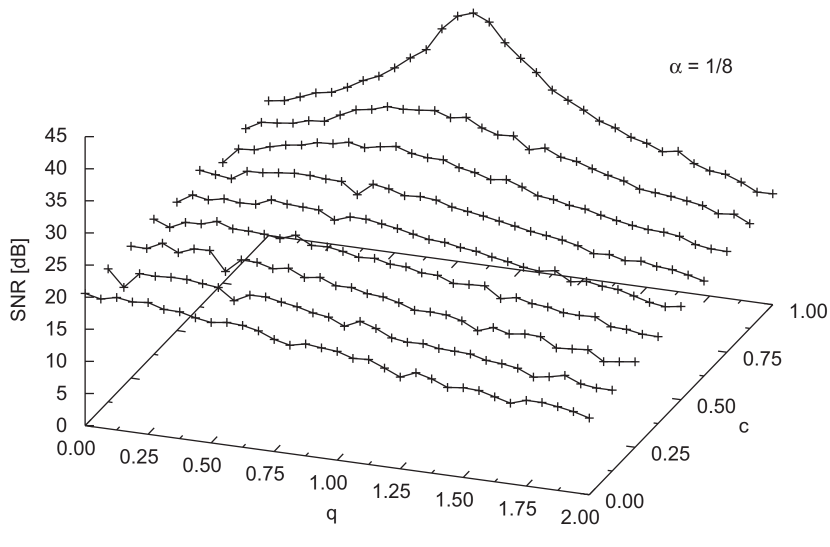

Fig. 3. Signal-to-noise ratio for system (29). Parameters are $\alpha = \frac{1}{8}$, $a = 1.25$, $b = 1.0$, $p = 1.0$, $A = 1$ and $\Omega = 2\pi$. Curves presented correspond, back to front, to the following values of the correlation coefficient: $c = 1.000$, 0.875, 0.750, 0.625, 0.500, 0.375, 0.250, 0.125 and 0.000.

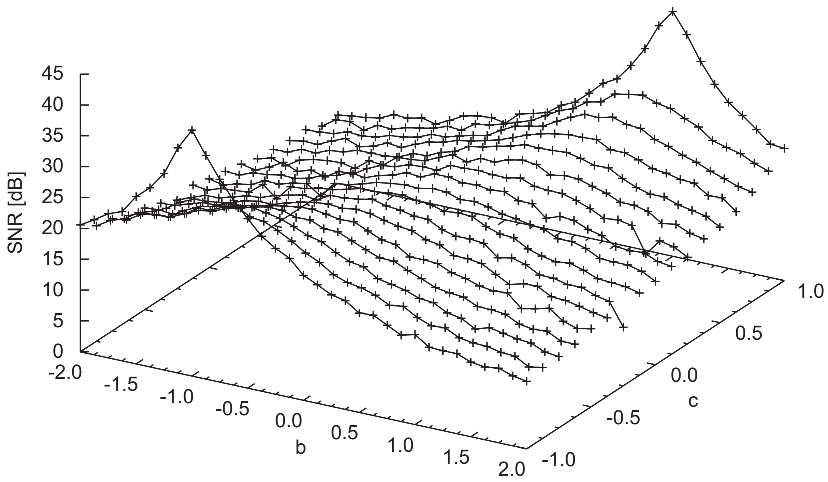

Fig. 4. Signal-to-noise ratio for system (29) as a function of the bias parameter, b, and the correlation coefficient, c. Other parameters are $\alpha = \frac{1}{8}$, $a = 1.25$, $p = 1.0$, $q = 0.8$, $A = 1$ and $\Omega = 2\pi$.

there is no stochastic resonance. Again, these features of the stochastic resonance resemble very much those of the noisy logistic system (7) discussed in Ref. [6].

4.3. Response to a change of deterministic parameters

We have shown in the two previous subsections that the system discussed here can be optimized by choosing an appropriate level of the "additive" noise. In practice, however, controlling the amplitude of the noise or correlations between the two sources of the noise can be very difficult. Tuning the deterministic part of the system may be much easier to achieve, and as our discussion of the asymmetric broadening of the distribution in Section 3.3 has shown, by changing the bias parameter, b, we can optimize the system even if the noise amplitudes and the correlation coefficient are not known.

To test for that, we have again numerically simulated the externally stimulated system (29) by the same means that have been used in Section 4.2 above. This time amplitudes of the two noises have been kept constant and the bias parameter has been varied. Selected results are presented in Fig. 4. As we can see, changing the bias does optimize the system. Clear maxima in the SNR are visible. These maxima are most pronounced if correlations are large, $|c| \lesssim 1$, but they are present also for $|c| \simeq 0$, even though the overall shape of the curves is much flatter. For the uncorrelated case, $c = 0$, the weak maximum coincides with $b = 0$ which is to be expected due to symmetry of the system. To put it in a slightly different way, we can see that the uncorrelated system transmits an external signal badly. Any correlations between the noises potentially improve the transmission properties. The system can be optimized to reach its full potential by appropriately adjusting its deterministic parameters.

5. Conclusions

In this paper we have discussed a nonlinear system with two correlated sources of Gaussian white noises. A closely related system has been discussed previously by Vitrenko in Ref. [2]. We have been mainly interested in what happens when the restrictions on correlations between the noises imposed by that author are lifted and, additionally, when the "additive" noise becomes biased. We have shown that this system can display both stochastic localization and heavy tails in its stationary distribution which is characteristic for anomalous diffusion. This agrees with previously published results [1,2]. It is worth noting, though, that authors of that references obtained their results under the assumption that the noises were coloured, we have shown that the same happens for white noises as well.

Next, we have shown that correlations present in the system discussed here can lead to interesting constructive effects of the noise: to a nonmonotonic behaviour of the variance of the stationary distribution and to a stochastic resonance. Finally, we have shown that the system can be optimized to an external periodic signal not only by varying amplitudes of the noises, but also by tuning the deterministic parameters of the system when the noise amplitudes and the correlation coefficient between the noises remain, in principle, unknown.

Surprisingly, system (4) discussed here is related to the noisy logistic system (7) that we have discussed previously [6]. As we have shown, many, but not all, properties of these two systems are strikingly similar.

Acknowledgements

I am most grateful to an anonymous referee for their thorough criticism of a previous version of this paper. After this research had been completed I learnt that authors of Ref. [18] provided another generalization of systems discussed in Refs. [1,2].

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