SOFT THEOREMS FOR SIGMA MODELS WITH 2D-3D TARGET SPACE

M. Shifman

William I. Fine Theoretical Physics Institute, University of Minnesota

Karol Kampf, Jiri Novotny, Jaroslav Trnka +MS

64. Cracow School of Theoretical Physics, Zakopane, Tatra Mountains, Poland; , June 15-23, 2024// JUNE 18

Soft theorems: Connecting soft limits of scattering amplitudes and sigma models

History: Gell-Mann — Levy (1960) ⟷ Adler (1965)

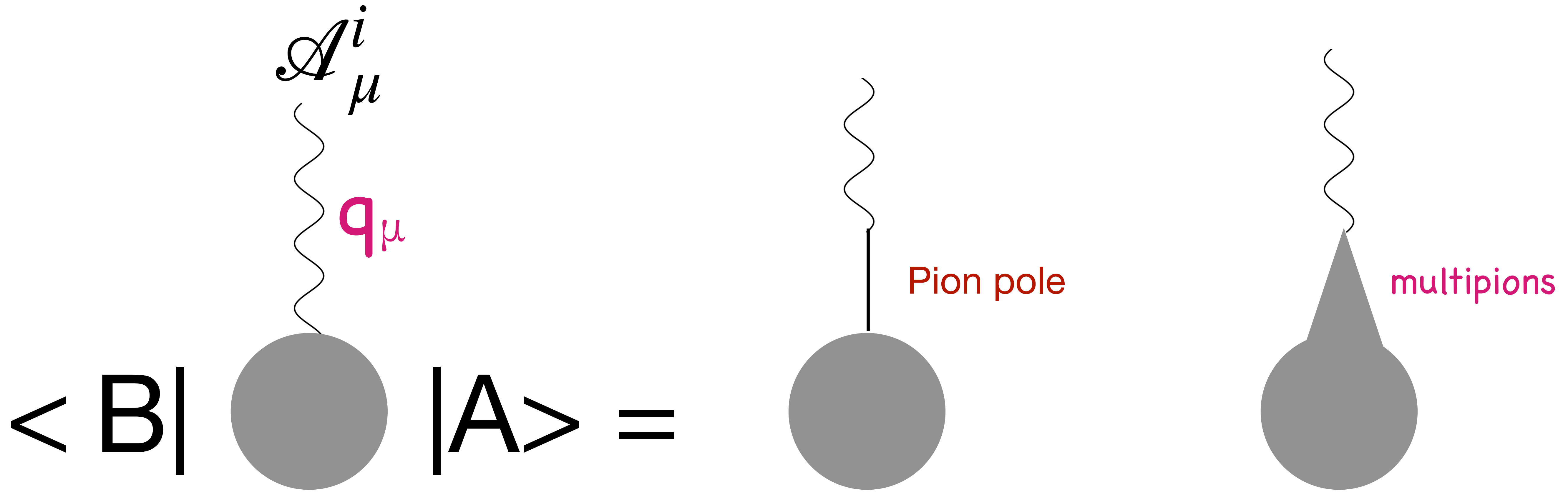
- The simplest case: spontaneous symmetry breaking G ⟶ H, sigma model for Goldstones has coset target space G/H

  Currents generating spontaneously broken symmetries are coupled to Goldstones, but are conserved

Example : (GML), SU(2) chiral symmetry in «QCD» ⟵ SU(2)×SU(2)/SU(2); Three axial currents are coupled to pions

and conserved as operators $\partial^\mu \mathscr{A}^i_\mu = 0$

Adler consistency condition // Adler zero



Non-singular in q

Absence of cubic vertices CRUCIAL!

M(q)

For pole

$$\langle \pi B | A \rangle \times (2\pi)^4 \delta^4(p_B + q - P_A) \times \frac{q_\mu}{q^2}$$

Multiply by q_μ and tend $q \to 0$

We then get $\langle \pi B | A \rangle = 0$ at $q \to 0$

Adler zero, hurrah !!!

Drastic changes the last decade or so

👉 1. How to establish soft theorems if your target space is NOT a coset? (I.e. massless scalar particles are not necessarily Goldstones!)

👉 2. Can one incorporate symmetries of sigma models directly into S-matrix?

👉 3. Some sigma models can be understood as consequence of a "generalized Adler zero" emerging in "exceptional" sigma models (whose leading interactions are uniquely fixed by a single coupling constant), namely, nonlinear sigma models, of the type CP(1), the Dirac-Born-Infeld (DBI) theory, and the so-called special Galileon.

- Cheung, K. Kampf, J. Novotny and J. Trnka (2015)

Generalized Adler zero: M(q) =O(q^σ),

$$\mathscr{L} = (\partial\phi)^2 \sum_m \lambda_{mn} \partial^m \phi^n$$

Indices: σ, m, $\rho = m/n$

Dirac-Born-Infeld action: $\mathscr{L} = -b^2\sqrt{-\det\left(\eta + \frac{F}{b}\right)} + b^2$

η is the Minkowski metric, F is the Faraday tensor

Galileon is a scalar field whose action is invariant under Galilean transformations, e.g.

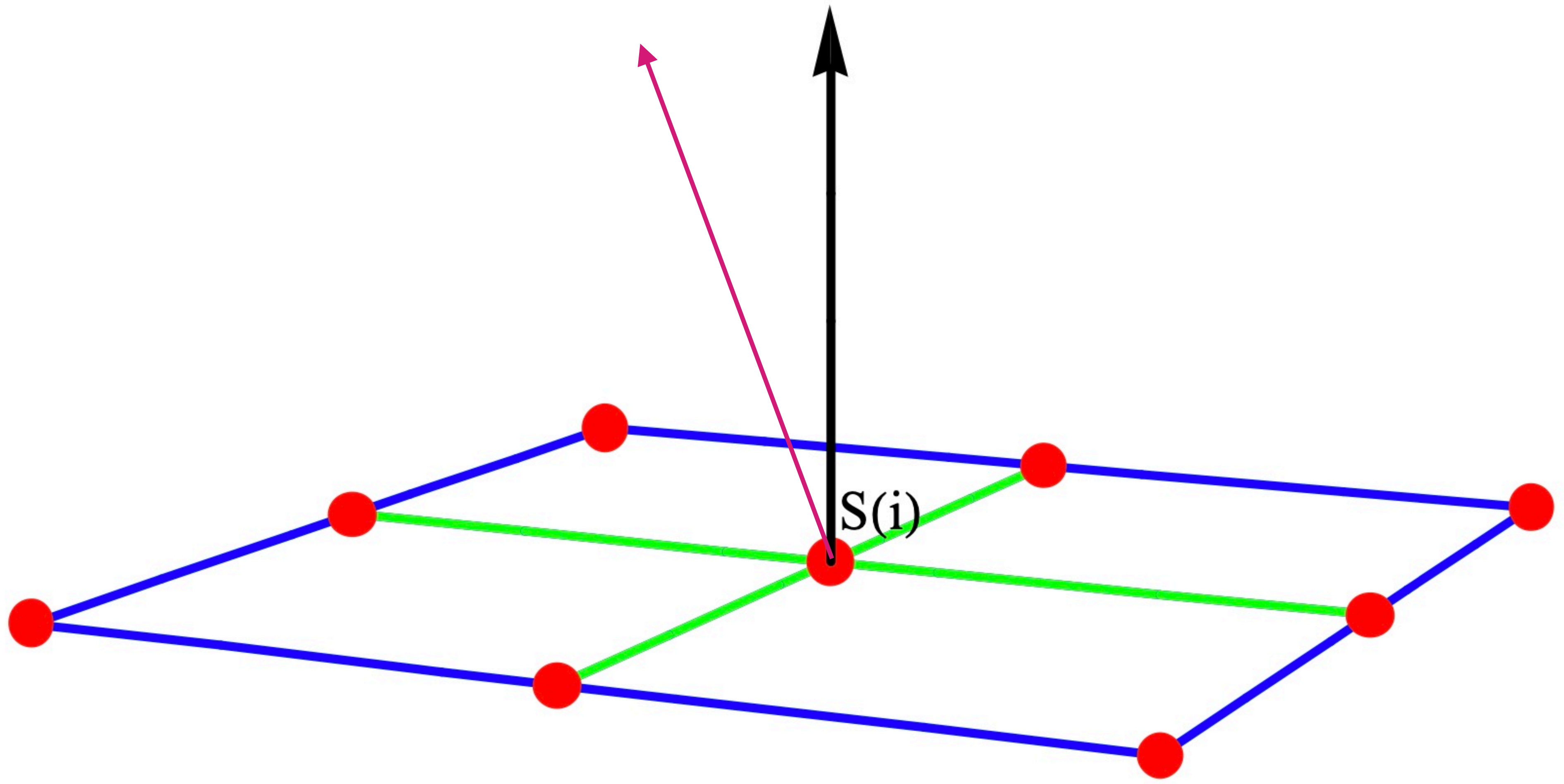
$$\mathrm{Gal}_3^1 \to \partial_\mu\pi\partial^\mu\pi(\partial^2)\pi - \partial_\mu\pi\partial^\mu\partial^\nu\pi\partial_\nu\pi$$

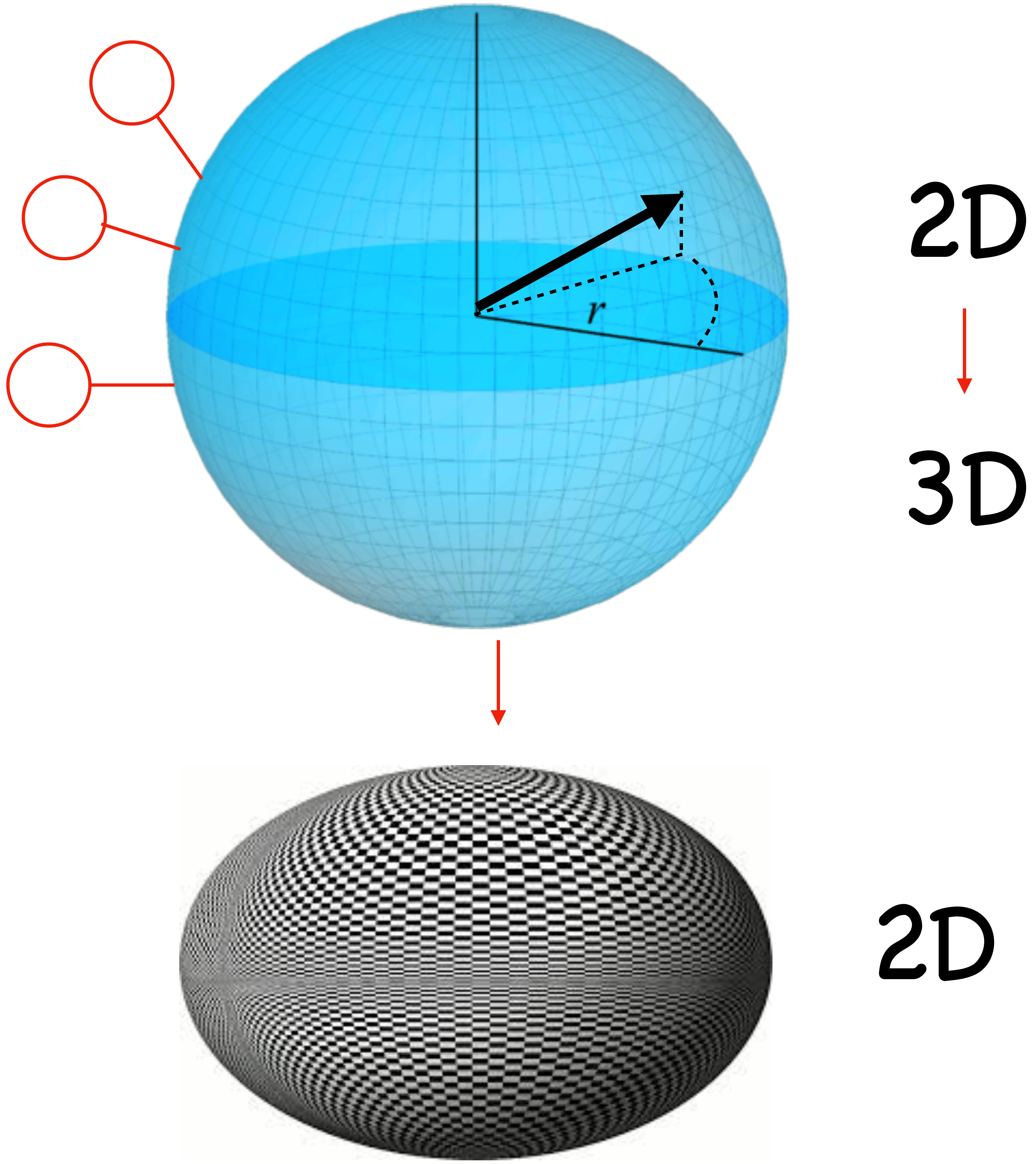
Variety of Soft Recursion Relations: What particular sigma models will be considered?

Model 1: U(1) fibration of CP(1)~O(3): 3D target space

Model 2: U(1) Lie-algebraic generalization of CP(1)~O(3): 2D target space

$\mathbf{S}_i \mathbf{S}_i = 1$ unit (iso)vector





Nearest-neighbors isotropic interaction

$$\mathscr{H} \to \mathscr{L}_{\text{Euclid}} = \frac{1}{2g^2}\,\partial_\mu \mathbf{S} \partial_\mu \mathbf{S} \to G\,\partial_\mu \phi^\dagger \partial_\mu \phi = \frac{1}{2g^2}\,\frac{1}{(1+\phi^\dagger\phi)^2}\,\partial_\mu \phi^\dagger \partial_\mu \phi$$

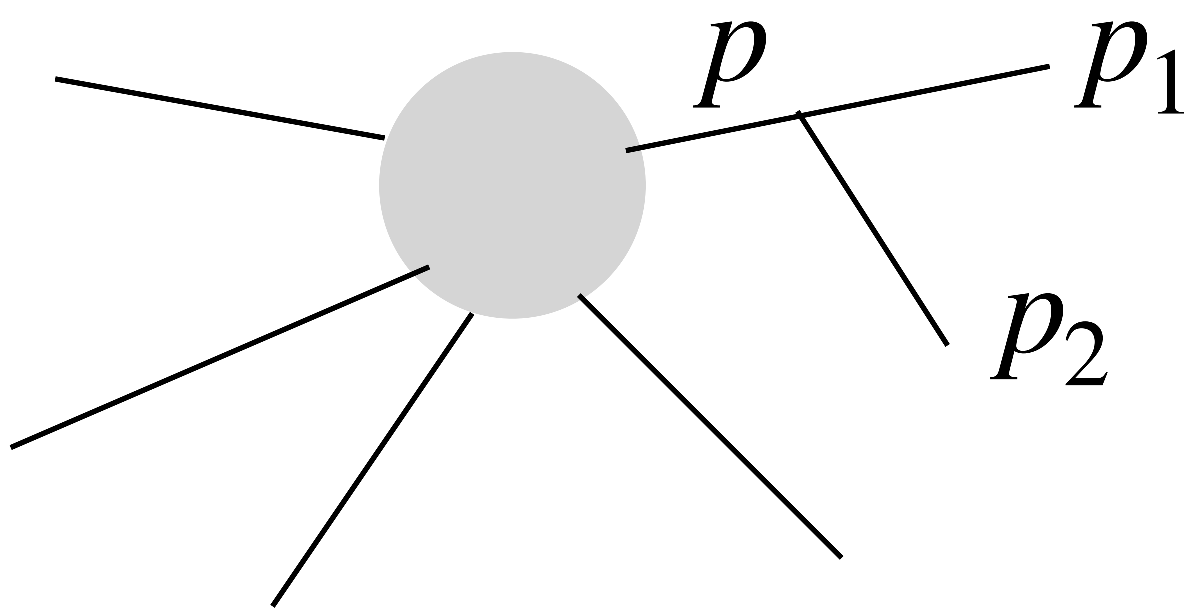
Adler Zeroes: if T= coset G/H, e.g. CP(1) or PCM. (Modulo some nuances)

Spontaneously broken currents are crucial!

If not ALL massless particles are Goldstone (T$\neq$ coset), then

Adler zeros $\rightarrow$ Recursions

Typically,



$p = p_1 + p_2 \rightarrow$ pole singularity

To avoid Adler zero and arrive at recursions, generically, both even- and odd-particle amplitudes $\neq$ 0. Some other necessary conditions apply, e.g. linear terms in nonlinearly realized sim transformations

Cheung review + Cheung, K. Kampf, J. Novotny and J. Trnka (2015)

Noncollinear magnetic phenomena in correlated electron systems in cont. limit are described by a sigma model on target space with a geometry that interpolates between 2D sphere S_2 and 3D sphere S_3

Gell-Mann—Levy representation for three pions

$$\mathscr{H} = \frac{1}{2g^2}\int dx \left\{ \left[\sum_{a=1,2,3} J^a_\mu J^a_\mu \right] - \kappa J^3_\mu J^3_\mu \right\},$$

$U(x) \in \mathrm{SU}(2)$

$0 \leq \kappa \leq 1$

$$J_\mu = -\,iU^\dagger \partial_\mu U \equiv \sum_a 2\, J^a_\mu T^a\,, \quad J^a_\mu = \mathrm{Tr}\left(J_\mu T^a\right), \quad U^\dagger U = 1$$

If $\kappa = 0$, then $U(1)$ Principal Chiral Model GML

Target = 3D sphere
Coset SU(2)×SU(2)/SU(2)

If $\kappa = 1$, then CP^1 model

Target = 2D sphere
Coset SU(2)/U(1) and sl(2)

If $0 < \kappa < 1$, then $U(1)$ fibtration over CP^1

Adler Zeroes → Soft Recursion Relations

[SU(N)/(SU(N-1) × U(1))] × U(1) ~ SU(N)/SU(N-1) ⟶ dim= 2N-1

For generic k cubic vertices are present

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_I \partial^\mu \phi_I + \frac{1}{2} K_{IJK} \partial_\mu \phi_I \partial^\mu \phi_J \phi_K + \mathcal{O}(\phi^4)$$

$$N^J_\mu = \sum_{I=1}^{N} F^J_I \partial_\mu \phi_I + \sum_{L,K=1}^{N} \mathcal{K}^J_{LK} \phi_K \partial_\mu \phi_L + \mathcal{O}(\phi^3)$$

★ $$\lim_{p\to 0} \sum_{I=1}^{N} F^J_I A_n(\alpha+\phi_I(p), \beta) = \sum_{I \in \alpha \cup \beta} \sum_{K=1}^{N} \mathcal{C}^J_{IK} A^{K,I}_{n-1}(\alpha, \beta) .$$

In the soft limit $p \to 0$ n-leg amplitude through a sum of (n-1) amplitudes ✌️

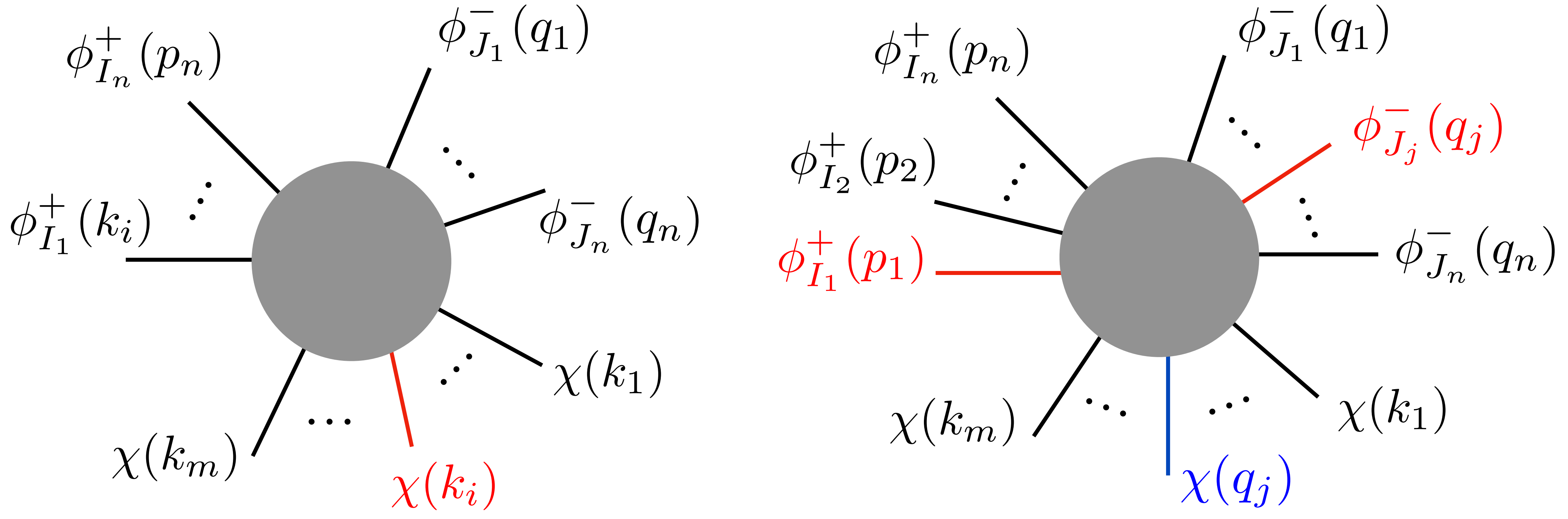
[SU(2)/U(1)]×U(1)

SU(2) Example (Soft Recursion Relations)

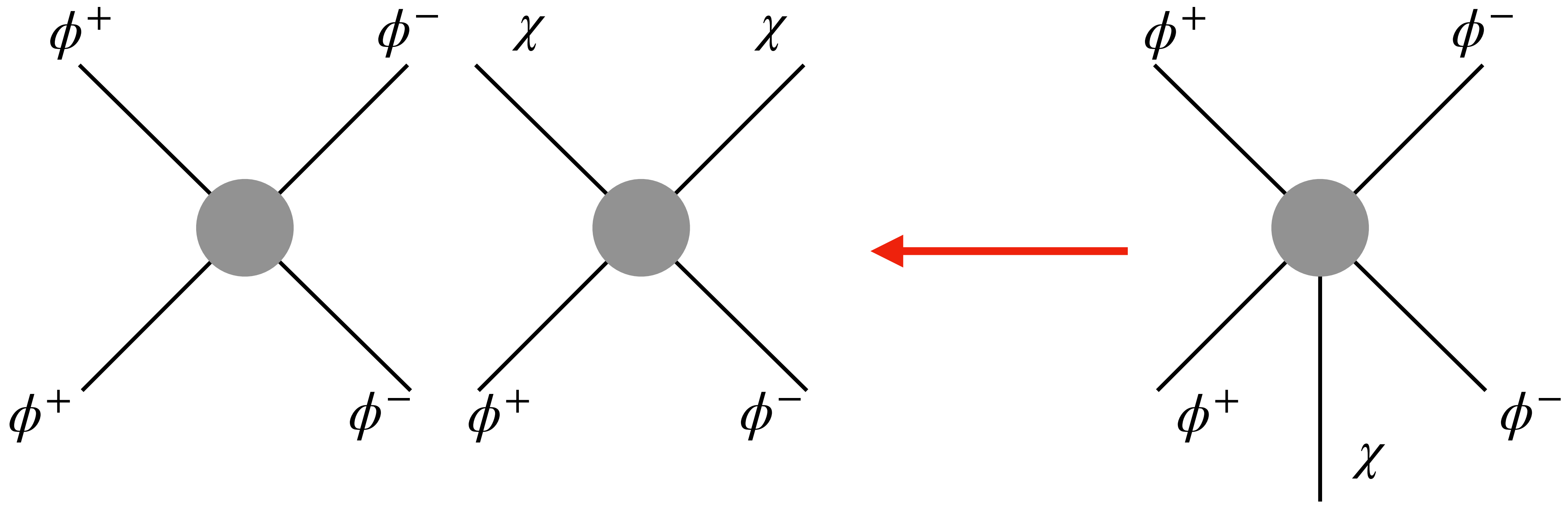
3 massless ϕ^+, ϕ^-, χ

Verify 5-leg ampl vs. sum of 4-leg ampl

GENERAL [SU(N)/(SU(N-1)×U(1))] × U(1)



[SU(2)/U(1)]×U(1)



$$A_4(\phi_1^+, \phi_2^+, \phi_3^-, \phi_4^-) = \frac{1}{4F^4}(3F_0^2 - 8F^2)s_{12}\,,$$

$$A_4(\phi_1^+, \phi_2^-, \chi_3, \chi_4) = \frac{F_0^2}{4F^4}s_{12}\,,$$

$$A_5(\phi_1^+, \phi_2^+, \phi_3^-, \phi_4^-, \chi_5) = \frac{\mathrm{i}F_0}{F^6}\left(F^2 - \frac{F_0^2}{2}\right)(s_{12} - s_{34}).$$

The soft theorem ★ for $p_1 \to 0$ predicts,

$$\begin{aligned}\lim_{p_1\to 0} A_5 &= \frac{\mathrm{i}F_0}{2F^2}A_4(\phi_5^+, \phi_2^+, \phi_3^-, \phi_4^-) \\ &\quad - \frac{\mathrm{i}F_0}{2F^2}\left[A_4(\phi_2^+, \phi_3^-, \chi_4, \chi_5) + A_4(\phi_2^+, \phi_4^-, \chi_3, \chi_5)\right] \\ &= -\frac{\mathrm{i}F_0}{F^6}\left(F^2 - \frac{F_0^2}{2}\right)s_{34}\,,\end{aligned}$$

$$F_0 = \frac{1}{g}\sqrt{1-\kappa}\,, \quad F = \frac{1}{\sqrt{2}g}\,, \quad s_{ij} = (p_i + p_j)^2$$

Praszałowicz-Fest

Michal was instrumental in many developments in the Skyrmion theory which were mentioned in Larry McLerran presentation

A few remarks after Larry McLerran: today's advances in Skyrmions and related theories

From page 6:

$$\mathscr{H} = \frac{1}{2g^2}\int dx \left\{ \left[\sum_{a=1,2,3} J^a_\mu J^a_\mu \right] - \kappa J^3_\mu J^3_\mu \right\},$$

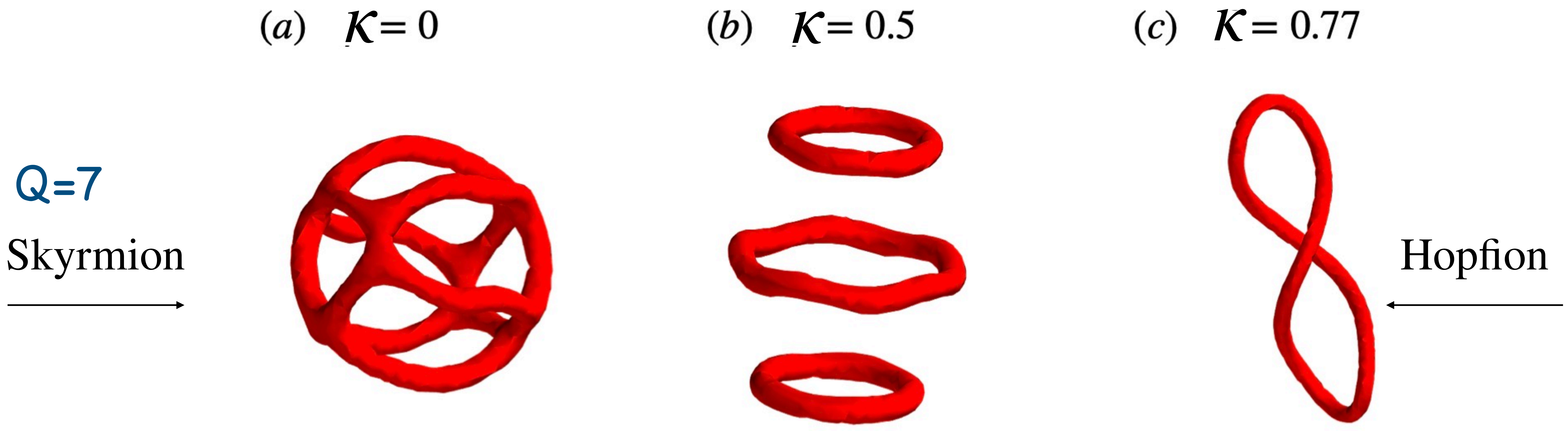
$$\kappa = 1 \to CP^1; \quad \kappa = 0 \to PCM \to \text{Skyrme model}$$

Faddeev-Niemi Hopfions

U(1) fibration over CP^{N-1}

Limit $\kappa \sim 1$ reduces to a frustrated magnetic system earlier considered by Sutcliffe as a host to Hopfions;

Limit $\kappa \sim 0$ is similar to the 3D Skyrme model.



(a) Position curve of a Q=7 Skyrmion at $\kappa = 0$; (b) For positive relatively small κ this settles to a three loop configuration; The three-loop solution may be tracked for increasing κ, but it develops an instability at κ = 0.77 and settles to a distinct buckled loop configuration

Carlos Naya, Daniel Schubring, Mikhail Shifman, Zhentao Wang, 2022

Conclusions

Dynamics of Goldstone massless particles for target spaces of SU(N)/SU(N-1) i.e. U(1) fibrations of CP^{N-1}:

Special Recursions. Lagrangian can be uniquely fixed if we start from these recursions.

Skymion and Hopfion topological numbers are related; There is a continuous path in κ leading from one to another