Geometric quantum complexity as a probe of de Sitter horizon

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June 16, 2024

Problem Statement:

Question: Does quantum complexity capture information about the underlying cosmological spacetime?

OR

Is quantum complexity sensitive to the presence of cosmological horizons– de Sitter horizon in this case?

Yes.

based on S.Chowdhury, M. Bojowald and J. Mielczarek "*Upper bound on quantum complexity of time dependent oscillators*"- ongoing.

Contents

Introduction

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- Complexity quantifies the difficulty of performing a certain task (here task $\equiv$ constructing an unitary operator).
- Complexity $\equiv$ minimal number of elementary operations required to complete the task.
- Quantum circuit picture: *Minimum* number of universal gates $\{g_i\}$ required in the circuit that constructs the desired unitary U as a product of g_i's:

$$U = g_n g_{n-1} .... g_2 g_1 \mathbb{I} \tag{1}$$

- Requires finding the optimal circuit, which is a very challenging task.
- Geometrizing quantum complexity:
    - Idea proposed by Nielsen and his collaborators in [*Science 311 no. 5764,(2006), Quant.Inf.Comput. 6 (2006) 3, 213-262,Quant.Inf.Comput. 8 (2008)*].
    - The problem of determining complexity of a unitary operation is related to the problem of finding minimal length geodesics on the unitary manifold.
    - Optimal circuit $\equiv$ minimal geodesic on the unitary group manifold connecting $\mathbb{I}$ to U.
    - complexity $\equiv$ length of the minimal geodesic.

General recipe to geometrically compute complexity:

- Given a target unitary operator U_{target}, identify a set of fundamental operators $(\mathcal{O}_I)$ that form a closed commutator algebra and hence specify a Lie group.
- After identifying the $\mathcal{O}_I$'s, classify them as "*easy*" or "*hard*".
- To define the geometry, consider a metric (G_{IJ}) that accurately penalizes the directions along the hard operators such that moving in their direction is discouraged for geodesics in the Lie group.
- To determine the geodesics on the Lie groups equipped with G_{IJ}, solve the Euler-Arnold equation: [*V. Arnold, Ann. Inst. Fourier 16 (1966) 319*]

$$G_{IJ}\frac{dV^J(s)}{ds} = f_{IJ}^K V^J(s) G_{KL} V^L(s), \tag{2}$$

where f_{IJ}^K are the structure constants of the Lie algebra, defined by

$$[\mathcal{O}_I, \mathcal{O}_J] = i f_{IJ}^K \mathcal{O}_K. \tag{3}$$

- Given a solution $V^I(s)$, the trajectory in the group is given by

$$U(s) = \mathcal{P}\exp\left(-i\int_0^s ds'\ V^I(s')\mathcal{O}_I\right) \tag{4}$$

- The path ordered exponential is usually approached by using an iterative approach. The result gives as a *Dyson series*

$$U(s) = \mathbb{I} - i\int_0^s V^I(s')\mathcal{O}_I ds' + (-i)^2\int_0^s V^I(s')\mathcal{O}_I ds' \int_0^{s'} V^J(s'')\mathcal{O}_J ds'' + \cdots \tag{5}$$

- We will keep only the leading-order term in the Dyson series. Approximating the Dyson series implies deviations of the trajectory to the target unitary from the geodsic. The result is a distance greater than the geodesic length. Instead of getting the actual value we get an upper bound on the complexity.
- Finally we impose the boundary conditions:

$$U(s=0) = \mathbb{I} \;\text{ and }\; U(s=1) = U_{\text{target}} \tag{6}$$

to filter out the geodesics that realize the target unitary operator.
- The complexity of the target unitary operator is given by:

$$C[U_{\text{target}}] := \min_{\{V^I(s)\}} \int_0^1 ds\sqrt{G_{IJ}V^I(s)V^J(s)}, \tag{7}$$

where the minimization is over all solutions $\{V^I(s)\}$ of the Euler–Arnold equation.

Desired target unitary

Time dependent oscillator

- The Hamiltonian of an oscillator with time-dependent frequency can be written as:

$$H(t) = \frac{p^2}{2} + \frac{1}{2}\omega^2(t)q^2, \tag{8}$$

- The canonically conjugated variables q and p can be promoted to operators

$$q(t) = f(t)a_0 + f^*(t)a_0^\dagger, \qquad p(t) = g(t)a_0 + g^*(t)a_0^\dagger\,. \tag{9}$$

where a_0 and $a_0^\dagger$ are the annihilation and creation operators defined at some initial time t_0 and $g(t) = \dot{f}(t)$. The mode function $f(t)$ satisfies the following equation:

$$\ddot{f}(t) + \omega^2(t)f(t) = 0. \tag{10}$$

- The time evolution of the creation and the annihilation operator gives us the system's time evolution. The annihilation and the creation operator at any time t can be written as:

$$a(t) = \alpha^*(t)a_0 - \beta^*(t)a_0^\dagger \quad , \quad a^\dagger(t) = -\beta(t)a_0 + \alpha(t)a_0^\dagger, \tag{11}$$

where $\alpha, \beta \in \mathbb{C}$, are the so-called Bogoliubov coefficients.

- The Bogoliubov coefficients can be expressed in terms of the mode function as:

$$\alpha = -i(\tilde{f}g^* - f^*\tilde{g}), \quad \beta = i(\tilde{f}g - f\tilde{g}), \tag{12}$$

where f and $\tilde{f}$ are the mode functions in two different regimes.

- The Bogoliubov coefficients also satisfy the normalization condition:

$$|\alpha|^2 - |\beta|^2 = 1, \tag{13}$$

which allows the Bogoliubov coefficients to be parametrized hyperbolically as:

$$\alpha(t) = e^{-i\theta(t)}\cosh(r(t)), \qquad \beta(t) = e^{-i(\phi(t)-\theta(t))}\sinh(r(t)). \tag{14}$$

- Using this parametrization, (11) can be written as:

$$a(t) = e^{i\theta(t)}\cosh(r(t))a_0 - e^{i(\phi(t)-\theta(t)}\sinh(r(t))a_0^\dagger, \tag{15}$$

$$a^\dagger(t) = e^{-i\theta(t)}\cosh(r(t))a_0 - e^{-i(\phi(t)-\theta(t))}\sinh(r(t))a_0^\dagger. \tag{16}$$

Desired target unitary

- The above equation can be simply represented as a unitary transformation,

$$a(t) = U^\dagger(t) a(t_0) U(t) \tag{17}$$

- In order for the above transformation to hold, $U(t)$ needs to be of the following form

$$U = S(r(t), \phi(t)) R(\theta(t)), \tag{18}$$

  where $S(r, \phi)$ and $R(\theta)$ are popularly known as the squeezing and the rotation operator and are expressed as

$$S(\xi(t)) = \exp\left(\frac{1}{2}(\xi^*(t) a^2 - \xi(t) a^{\dagger 2})\right), \quad R(\theta(t)) = \exp\left(i\theta(t)\frac{a^\dagger a + a a^\dagger}{2}\right). \tag{19}$$

  where $\xi(t) = r(t) e^{i\phi(t)}$.

- Therefore, the target unitary operator in this case is the product of the unitary operators S and R:

$$U_{\text{target}} = S(r(t), \phi(t)) R(\theta(t)). \tag{20}$$

Geometric complexity of the target unitary

Complexity of the target unitary

- We need a set of Hermitian operators ($\mathcal{O}_I$) that can be used to build U_{target} and is closed with respect to taking commutators. For our U_{target},

$$\mathcal{O}_1 = \frac{a^2 + a^{\dagger 2}}{4}, \quad \mathcal{O}_2 = \frac{i(a^2 - a^{\dagger 2})}{4}, \quad \mathcal{O}_3 = \frac{aa^\dagger + a^\dagger a}{4}, \tag{21}$$

  satisfy the commutation relations

$$[\mathcal{O}_1, \mathcal{O}_2] = -i\mathcal{O}_3, \quad [\mathcal{O}_1, \mathcal{O}_3] = -i\mathcal{O}_2, \quad [\mathcal{O}_2, \mathcal{O}_3] = i\mathcal{O}_1\,, \tag{22}$$

  forming the $\mathfrak{su}(1,1)$ Lie algebra.

- In terms of these $\mathcal{O}_I$, the target unitary operator in terms of the generators can be written as:

$$U_{\text{target}} = \exp\bigg(-2ir(t)(\sin(\phi(t))\mathcal{O}_1 + \cos(\phi(t))\mathcal{O}_2)\bigg)\exp(2i\theta(t)\mathcal{O}_3). \tag{23}$$

- With the choice $G_{IJ} = \delta_{IJ}$, the Euler–Arnold equations can be written for individual components of the tangent vector as

$$\frac{dV^1}{ds} = -2V^2V^3, \quad \frac{dV^2}{ds} = 2V^1V^3, \quad \frac{dV^3}{ds} = 0. \tag{24}$$

- The general solutions to Eq. (24) are

$$V^1(s) = v_1 \cos(2v_3 s) - v_2 \sin(2v_3 s), \quad (25)$$
$$V^2(s) = v_1 \sin(2v_3 s) + v_2 \cos(2v_3 s), \quad (26)$$
$$V^3(s) = v_3 \quad (27)$$

  with integration constants v_i, $i = 1, 2, 3$, determined by the condition that the target unitary is reached in the group manifold at $s = 1$.
- The complexity of the target unitary operator $C[U_{\text{target}}] = \sqrt{v_1^2 + v_2^2 + v_3^2}$
- The final step involves deriving the v_I's from the boundary condition. $U(s = 1) = U_{\text{target}} = \exp(-2ir(t)(\sin(\phi(t))\mathcal{O}_1 + \cos(\phi(t))\mathcal{O}_2) \exp(2i\theta(t)\mathcal{O}_3)$.
- Apply BCH formula to express the product as a single exponential.

$$e^X e^Y = e^Z \quad (28)$$

  where, $Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + \cdots$

- We will focus on $Z = X + Y$ and $Z = X + Y + \frac{1}{2}[X, Y]$.
- Warning: Neglecting the nested commutator terms in the BCH formula changes the target unitary operator.

$$U_{\text{target}} = e^{X} e^{Y}, \quad U^{(1)}_{\text{target}} \approx e^{X+Y}, \quad U^{(2)}_{\text{target}} \approx e^{X+Y+\frac{1}{2}[X,Y]} \tag{29}$$

- An approximation in the BCH formula places the final operator closer to the identity than desired and therefore under-estimates the distance. It implies that the curves we consider do not reach the exact target unitary we are interested in.
- Interpret the result as approximate upper bound.
- The boundary condition $U(s=1) = U^{(1)}_{\text{target}}$ gives

$$v_3 = -2\theta(t), \tag{30}$$

$$v_1 = -4\theta(t) r(t) \csc(2\theta(t)) \sin(2\theta(t) - \phi(t)), \tag{31}$$

$$v_2 = 4\theta(t) r(t) \csc(2\theta(t)) \cos(2\theta(t) - \phi(t)). \tag{32}$$

- The upper bound on the complexity of $U^{(1)}_{\text{target}}$ is therefore given by

$$C[U^{(1)}_{\text{target}}] \lesssim 2\sqrt{\theta(t)^2 (1 + 4\, r(t)^2 \csc^2(2\theta(t)))}. \tag{33}$$

- More generally, the upper bound can be written in terms of the Bogoliubov coefficients by realizing that the parameters r, θ, and ϕ can be parameterized by

$$r = \mathrm{arcsinh}|\beta|, \quad \theta = -\arg(\alpha), \quad \phi = -\arg(\alpha\beta). \tag{34}$$

  This allows us to rewrite the upper bound (33) as

$$C[U^{(1)}_{\mathrm{target}}] \lesssim 2\sqrt{\arg(\alpha(t))^2(1 + 4\mathrm{arcsinh}^2|\beta(t)|\csc^2(2\arg(\alpha(t))))}. \tag{35}$$

- For, $U(s=1) = U^{(2)}_{\mathrm{target}}$, the complexity upper bound becomes

$$C[U^{(2)}_{\mathrm{target}}] \lesssim 2\sqrt{\theta(t)^2(1 + 4r(t)^2(1+\theta(t)^2)\csc^2(2\theta(t)))} \tag{36}$$

Scalar field on de Sitter background

- Each mode of a free quantum field on a non-static background behaves like a harmonic oscillator with time-dependent frequency.
- For the *massless* case the frequency function is given by:

$$\omega_{dS}^2(\tau) = k^2 - \frac{2}{\tau^2}, \qquad \tau \to \text{conformal time..} \tag{37}$$

- The Bogoliubov coefficients can be obtained using $\alpha = -i(\tilde{f}g^* - f^*\tilde{g})$, and $\beta = i(\tilde{f}g - f\tilde{g})$, in which f will be the Minkowski mode function and the $\tilde{f}$ will be the de Sitter one, *i.e.*:

$$f(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}, \qquad \tilde{f}(\tau) = \frac{e^{-ik\tau}}{\sqrt{2k}}\left(1 - \frac{i}{k\tau}\right). \tag{38}$$

- The Bogoliubov coefficients are:

$$\alpha(\tau) = 1 - \frac{1}{2k^2\tau^2} - \frac{i}{k\tau}, \qquad \beta(\tau) = \frac{e^{-2ik\tau}}{2k^2\tau^2}, \tag{39}$$

- Complexity:

$$C[U^{(1)}_{\text{target}}] \sim 0 \qquad (\text{sub Horizon modes})$$

$$\sim \ln a \qquad (\text{super Horizon modes}) \quad \left(a = \text{scale factor} = -\frac{1}{H\tau}\right)$$

Summary

Summary

- Quantum complexity of a unitary operator $\equiv$ length of the minimal geodesic in the unitary group manifold formed by the fundamental operators required to construct U.
- Approximating the Dyson series gives us an upper bound instead of the actual value of complexity.
- When the mode is inside the horizon, the value of complexity is significantly low.
- Complexity increases as the logarithm of the scale factor after the mode exits the horizon.
- An indication that geometric complexity might be used to capture information about the underlying cosmological spacetime- presence of cosmological horizons to be precise.

Thank you!