QCD critical point and the predictable randomness of relativistic fluids

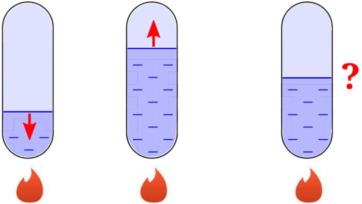
M. Stephanov

Outline

1. Introduction.
   - Critical point. History.
   - Critical point in QCD
   - Heavy-Ion Collisions
2. Fluctuations in equilibrium
   - Critical fluctuations
   - Intriguing data from RHIC BES I
3. Non-equilibrium (recent progress)
   - Hydrodynamics and fluctuations
   - Hydro+
   - Covariant relativistic formalism
   - Non-gaussian fluctuations
   - Freezeout of fluctuations

History

Cagniard de la Tour (1822): discovered continuos transition from liquid to vapour by heating water, alcohol, etc. in a sealed container.



What's in a name?

- Faraday (1844) – liquefying gases:

  "Cagniard de la Tour made an experiment some years ago which gave me occasion to want a new word."

  "...the beauty of the experiment & its general results has always in my eyes been so great that I have constantly regretted we had not a word wherewith we might talk & write freely about it."

  "Cagniard de la Tour point"

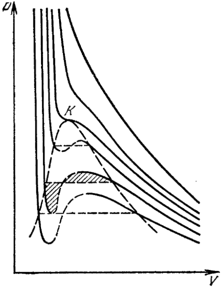
- Mendeleev (1860) – measured vanishing of liquid-vapour surface tension: "Absolute boiling temperature".

- Andrews (1869) – systematic studies of many substances established continuity of vapour-liquid phases as a universal phenomenon. Coined the name "critical point".

Theory

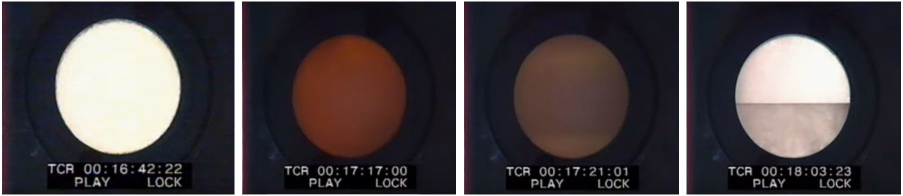
- van der Waals (1879) –
  "On the continuity of the gas and liquid state"
  (PhD thesis) – e.o.s. with a critical point.

  Law of corresponding states.



- Smoluchowski, Einstein (1908,1910) –
  explained critical opalescence as a *fluctuation* phenomenon.
- Landau – classical theory of critical phenomena.
- Fisher, Kadanoff, Wilson – fluctuation theory, scaling, RG (QFT).
- Universality extends also to ferromagnets (Curie point).

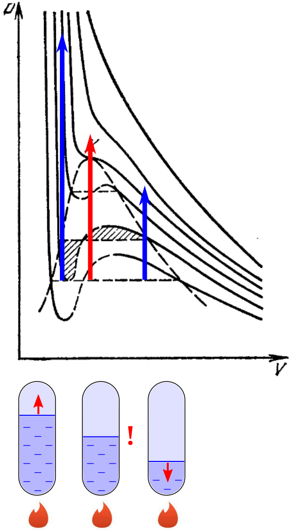
Critical opalescence



$T > T_c$ $T < T_c$

https://www.youtube.com/watch?v=cSliO89x7UU

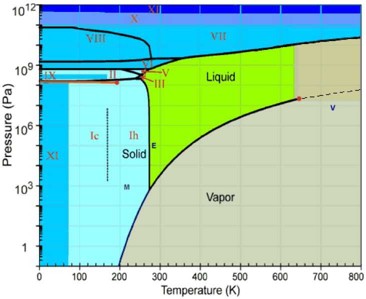
Cagniard de la Tour experiment as a density scan



Substance[13][14]	Critical temperature	Critical pressure (absolute)
Argon	−122.4 °C (150.8 K)	48.1 atm (4,870 kPa)
Ammonia[15]	132.4 °C (405.5 K)	111.3 atm (11,280 kPa)
Bromine	310.8 °C (584.0 K)	102 atm (10,300 kPa)
Caesium	1,664.85 °C (1,938.00 K)	94 atm (9,500 kPa)
Chlorine	143.8 °C (416.9 K)	76.0 atm (7,700 kPa)
Ethanol	241 °C (514 K)	62.18 atm (6,300 kPa)
Fluorine	−128.85 °C (144.30 K)	51.5 atm (5,220 kPa)
Helium	−267.96 °C (5.19 K)	2.24 atm (227 kPa)
Hydrogen	−239.95 °C (33.20 K)	12.8 atm (1,300 kPa)
Krypton	−63.8 °C (209.3 K)	54.3 atm (5,500 kPa)
CH_4 (methane)	−82.3 °C (190.8 K)	45.79 atm (4,640 kPa)
Neon	−228.75 °C (44.40 K)	27.2 atm (2,760 kPa)
Nitrogen	−146.9 °C (126.2 K)	33.5 atm (3,390 kPa)
Oxygen	−118.6 °C (154.6 K)	49.8 atm (5,050 kPa)
CO_2	31.04 °C (304.19 K)	72.8 atm (7,380 kPa)
N_2O	36.4 °C (309.5 K)	71.5 atm (7,240 kPa)
H_2SO_4	654 °C (927 K)	45.4 atm (4,600 kPa)
Xenon	16.6 °C (289.8 K)	57.6 atm (5,840 kPa)
Lithium	2,950 °C (3,220 K)	652 atm (66,100 kPa)
Mercury	1,476.9 °C (1,750.1 K)	1,720 atm (174,000 kPa)
Sulfur	1,040.85 °C (1,314.00 K)	207 atm (21,000 kPa)
Iron	8,227 °C (8,500 K)	
Gold	6,977 °C (7,250 K)	5,000 atm (510,000 kPa)
Water[2][16]	373.946 °C (647.096 K)	217.7 atm (22.06 MPa)

Critical point
– end of phase coexistence –
is a ubiquitous phenomenon

Water:



Is there one in QCD?

Physics of QCD

- QCD is an inherently *relativistic* theory of a fundamental force. Part of the Standard Model.
- Constituents of QCD – quarks and gluons – are (almost) massless. But hadrons (quantum excitations of QCD) are massive.

  $m_{\text{proton}} = E_{\text{QCD}}/c^2$

  This is the origin of almost all of the visible mass in the Universe.
- Color charges are “confined” and color forces are “hidden” within hadrons.
- High-energy collisions expose color degrees of freedom and high T environment “liberates” color forces (gluons) and color charges.

  The resulting new form of matter is Quark-Gluon Plasma.

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Q1: Can the two phases continuously transform into each other? *Yes.*

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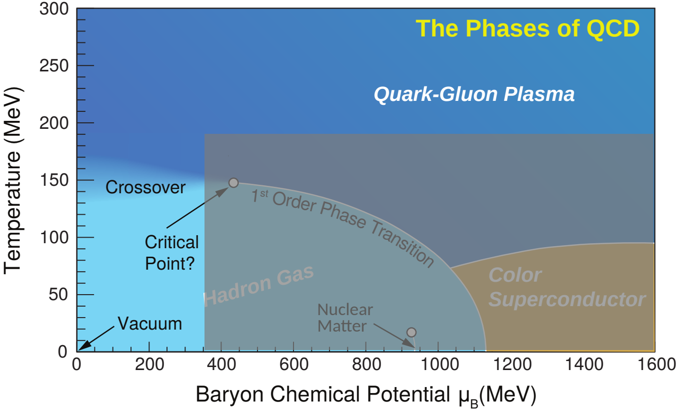
Lattice QCD at $\mu_B = 0$ – a crossover.



QCD in crossover region: no quasiparticles (not hadrons, not quarks/gluons). Strongly interacting matter (sQGP). More a liquid than a gas.

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Q2: Is there phase coexistence, i.e., 1st order transition? *Likely.*

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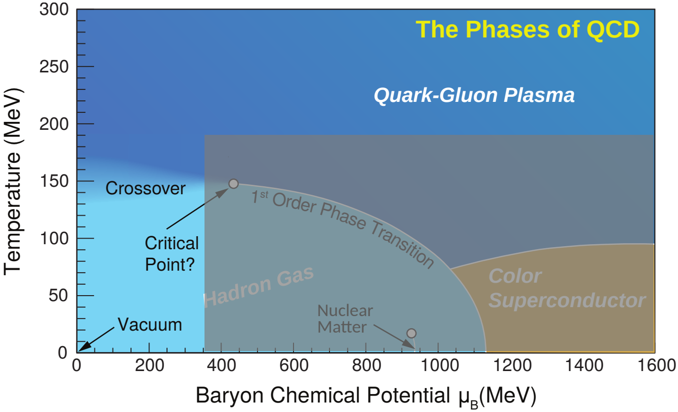

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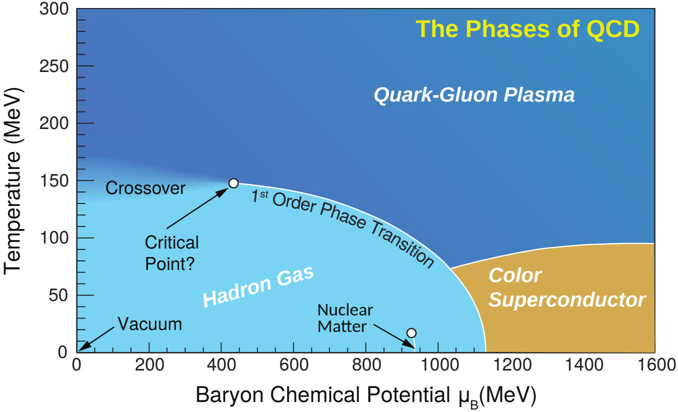


But 1st order transition (and thus C.P.) is ubiquitous in models of QCD: NJL, RM, Holography, Strong coupl. Lattice QCD, . . .

How can one discover the QCD critical point?

Essentially two approaches to discovering the QCD critical point. Each with its own challenges.

- Lattice simulations.

  The *sign problem* restricts reliable lattice calculations to $\mu_B = 0$.

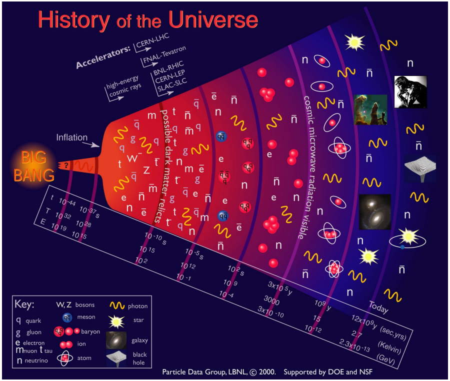
  Under different assumptions one can estimate the position of the critical point, assuming it exists, by extrapolation from $\mu = 0$.

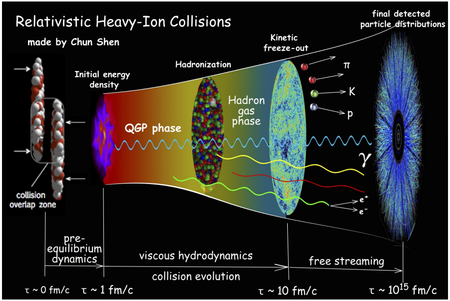
  Not reliable (yet).

  Recent results put the point at $\mu_B > 500$ MeV and $T < 130$ MeV.

- Heavy-ion collisions. *Non-equilibrium.*

Heavy-ion collisions vs the Big Bang





Similarity: expansion, accompanied by cooling, followed by freezeout.

CMB vs particles detected after h.i.c. – both thermally distributed.

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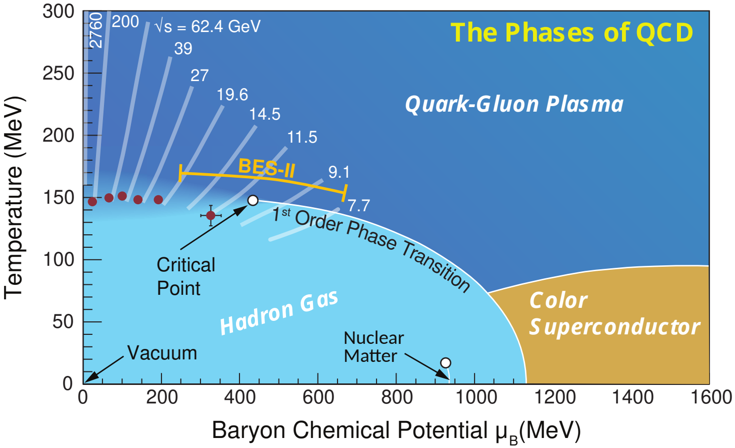
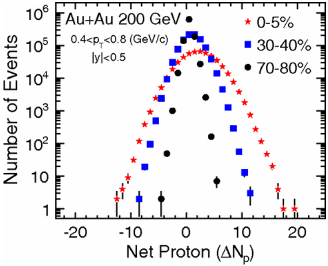
Scanning the QCD phase diagram

- Difference: tunable parameter μ_B via $\sqrt{s}$.



Fluctuations in heavy-ion collisions

- Another difference: single Event vs many repeated events (cosmic variance vs event-by-event fluctuations)
- Heavy-ion collision fireballs are large (thermodynamic), but not too large ($N \sim 10^2 - 10^4$ particles)
  - The fluctuations are small ($1/\sqrt{N}$), but measurable.
  - Close to Gaussian, but non-Gaussianity is also measurable.



- If there is a critical point fluctuation measures must be *non-monotonic* vs $\sqrt{s}$

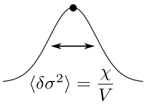
[PRL81(1998)4816]

Outline

Fluctuations and the critical point

- Equilibrium = maximum entropy.

$$P(\sigma) \sim e^{S(\sigma)} \qquad \textit{(Einstein 1910)}$$



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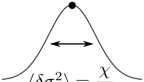
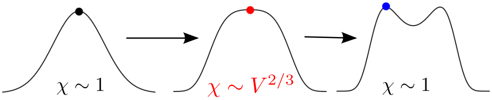


- At the critical point $P(\sigma)$ “flattens”. And $\chi \to \infty$ as $V \to \infty$.



CLT?

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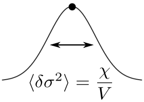

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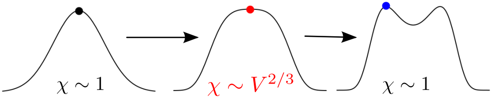

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$\delta\sigma$ is not an average of ∞ many *uncorrelated* contributions: $\xi \to \infty$

In fact, $\langle \delta\sigma^2 \rangle \sim \xi^2/V$.

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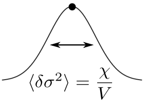

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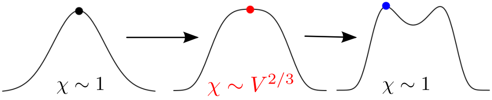

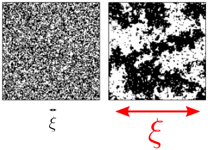

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In fact, $\langle\delta\sigma^2\rangle \sim \xi^2/V$.

Higher order cumulants

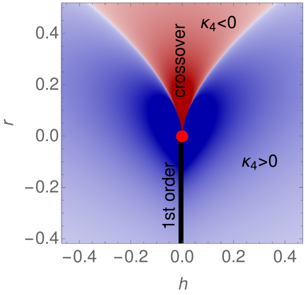
- $n > 2$ cumulants (shape of $P(\sigma)$) depend stronger on ξ.

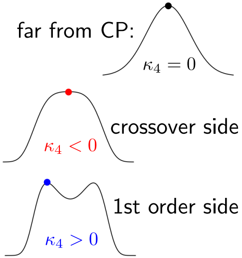
  E.g., $\langle \sigma^2 \rangle \sim \xi^2$ while $\kappa_4 = \langle \sigma^4 \rangle_c \sim \xi^7$ *[PRL102(2009)032301]*

- For $n > 2$, sign depends on which side of the CP we are.

  This dependence is also universal. *[PRL107(2011)052301]*

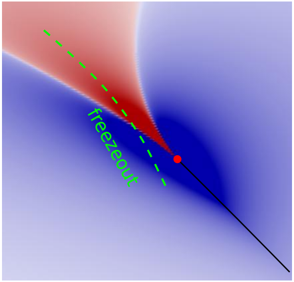
- Using Ising model:





Beam Energy Scan I: intriguing hints

κ_4 vs μ_B and T:



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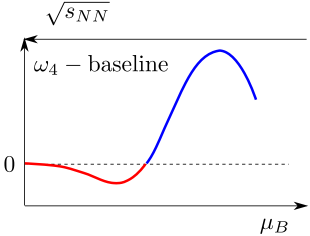

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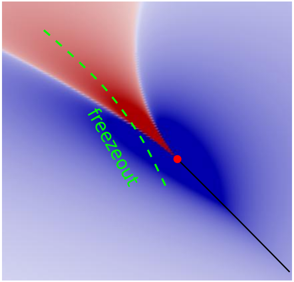

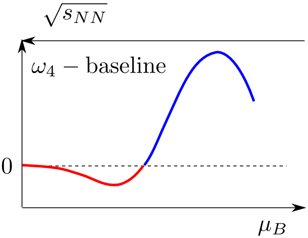

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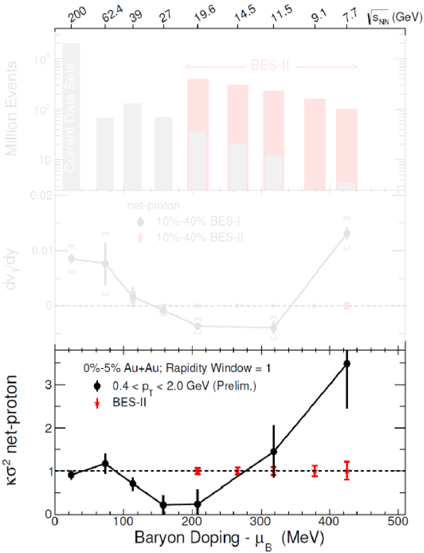

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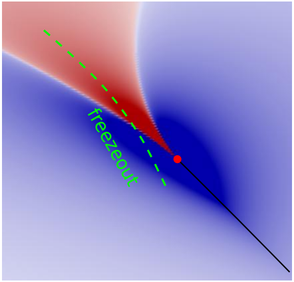

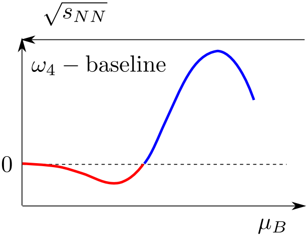

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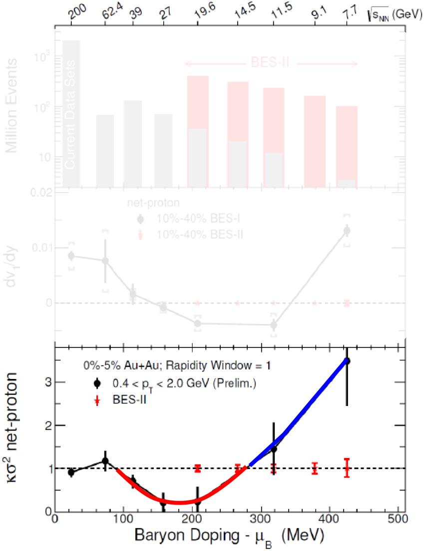

"intriguing hint" (2015 LRPNS)

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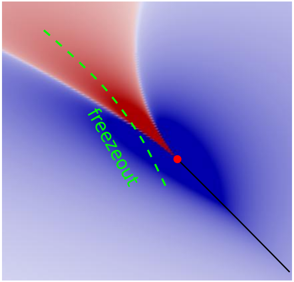

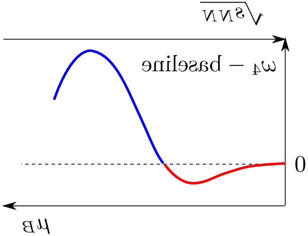

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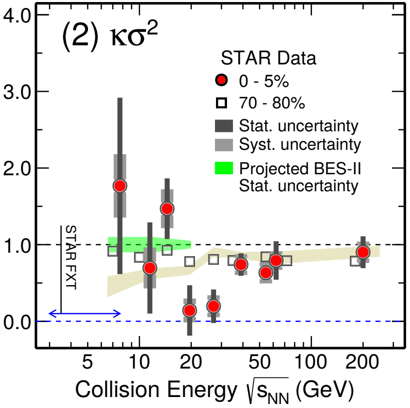

STAR 2001.02852

“non-monotonic with 3.1σ significance”

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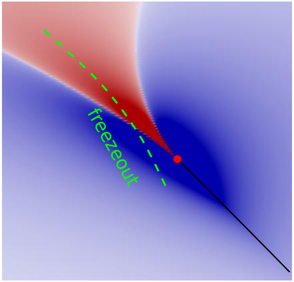

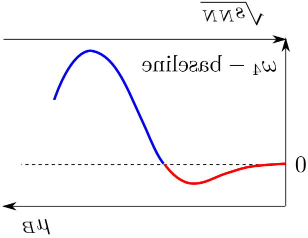

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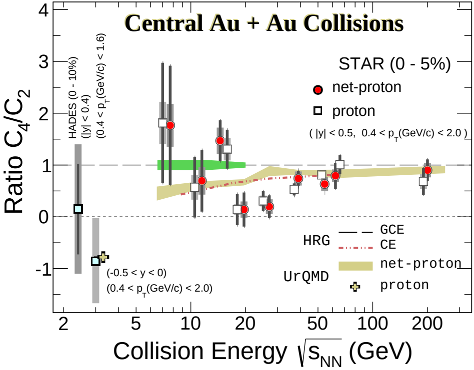

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Theory: beyond the equilibrium assumption

Predictions assume equilibrium, but in heavy-ion collisions

non-equilibrium physics is essential,
especially near the critical point.

Critical slowing down: certain slow degrees of freedom are further away from equilibrium. These degrees of freedom are directly related to fluctuations.

Challenge: develop hydrodynamics *with fluctuations* capable of describing *non-equilibrium* effects on fluctuation signatures of CP.

Outline

What is hydrodynamics and why randomness

Fluid left alone tends to equilibrium.

There are two time scales:

1) local thermodynamic equilibration – fast;

2) achieving homogenious conditions – slow.

Hydrodynamics describes the slower process: transport of conserved quantities.

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- Scale hierarchy in heavy-ion collisions:
  fireball size $\sim 10 - 15$ fm vs microscale $\sim 0.5 - 1$ fm
  – enough for hydrodynamics, but fluctuations are important.

Randomness in hydrodynamics

- Hydro variables obey conservation eqs ($\partial_\mu T^{\mu\nu} = 0$, $\partial_\mu J^\mu = 0$):

$$\partial_t \psi = -\nabla \cdot \text{Flux}[\psi];$$

where ψ is an *averaged* conserved density, e.g., T^{i0}, J^0, and Flux$[\psi]$ is the corresponding *averaged* flux, T^{ij}, J^i.

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- Operators T^{i0}, J^0 *coarse-grained* over "hydrodynamic cells" b, $\ell_{\text{wave}} \gg b \gg \ell_{\text{mic}}$, are stochastic variables and obey

$$\partial_t \breve{\psi} = -\nabla \cdot \Big(\text{Flux}[\breve{\psi}] + \text{Noise} \Big)$$

(Landau-Lifshits)

Randomness in hydrodynamics

Stochastic description

Random hydro variables: $\breve{\psi}$

$$\partial_t \breve{\psi} = -\nabla \cdot \Big(\text{Flux}[\breve{\psi}] + \text{Noise} \Big)$$

$+$ fewer variables and eqs.

$-$ stochastic

$-$ cutoff dependence (infinite noise)

Landau-Lifshits, Kapusta et al, Gale et al, Nahrgang et al, . . .

Deterministic description

$\psi \equiv \langle \breve{\psi} \rangle$, $G \equiv \langle \breve{\psi}\breve{\psi} \rangle$, etc.

$$\partial_t \psi = -\nabla \cdot \text{Flux}[\psi; G];$$

$$\partial_t G = -2\Gamma(G - \bar{G}[\psi]);$$

$-$ more variables and eqs.

$+$ deterministic

$+$ no cutoff dependence after renormalization

Andreev, Akamatsu et al, Yin et al, An et al, Martinez et al, . . .

Hydro+

Yin, MS, 1712.10305

- Near CP, for some d.o.f., $\tau_{\text{equilibration}} \gg \tau_{\text{micro}} \sim 1/T$. Hydrodynamics proper breaks down at $\tau_{\text{equilibration}} \sim \xi^3$.
- But it can be extended to shorter times by adding additional d.o.f.
- At the CP, the *slowest* (i.e., most out of equilibrium) new d.o.f. is the 2-pt function $\langle \delta m \delta m \rangle$ of the slowest hydro variable $m \equiv s/n$:

$$\phi_{\boldsymbol{Q}}(\boldsymbol{x}) = \int_{\Delta \boldsymbol{x}} \langle \delta m\,(\boldsymbol{x}_1)\ \delta m\,(\boldsymbol{x}_2) \rangle\ e^{i\boldsymbol{Q}\cdot\Delta\boldsymbol{x}}$$

  where $\boldsymbol{x} = (\boldsymbol{x}_1 + \boldsymbol{x}_2)/2$ and $\Delta\boldsymbol{x} = \boldsymbol{x}_1 - \boldsymbol{x}_2$.
- *In equilibrium* fluctuations are determined by thermodynamics: $\bar{\phi}_{\boldsymbol{Q}} = \dfrac{c_p}{n^2} f(\boldsymbol{Q}) \qquad (f(\boldsymbol{0}) = 1)$.

more

Relaxation of fluctuations towards equilibrium

- As usual, equilibration maximizes entropy $S = \sum_i p_i \log(1/p_i)$:

$$s_{(+)}(\epsilon, n, \phi_{\boldsymbol{Q}}) = s(\epsilon, n) + \underbrace{\frac{1}{2}\int_{\boldsymbol{Q}} \left(\log \frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}} - \frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}} + 1 \right)}_{\text{entropy relative to equilibrium}}$$

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Relaxation of fluctuations towards equilibrium

- As usual, equilibration maximizes entropy $S = \sum_i p_i \log(1/p_i)$:

$$s_{(+)}(\epsilon, n, \phi_{\boldsymbol{Q}}) = s(\epsilon, n) + \underbrace{\frac{1}{2}\int_{\boldsymbol{Q}} \left(\log \frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}} - \frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}} + 1 \right)}_{\text{entropy relative to equilibrium}}$$

more

- The equation for $\phi_{\boldsymbol{Q}}$ is a relaxation equation with rate

$$\Gamma(\boldsymbol{Q}) \approx 2DQ^2 \quad \text{for} \quad Q \ll \xi^{-1}, \quad D \sim 1/\xi.$$

- Impact of fluctuations on hydrodynamics:
  - "Renormalization" of bulk viscosity $\zeta \sim 1/\Gamma_\xi \sim \xi^3$.
  - (Non-analytic) frequency dependence of $\zeta(\omega)$ for $\omega \ll \Gamma_\xi$. "Long-time tails"

General covariant formalism

An, Basar, Yee, MS, 1902.09517, 1912.13456, 2212.14029

- Hydro+ is part of a more general theory for critical as well as non-critical fluctuations we would like to formulate.
- Expand stochastic hydro eqs. in $\{\delta m, \delta p, \delta u^\mu\} \sim \phi$

  $$\text{Flux}[\breve{\psi}] = \text{Flux}[\psi + \phi] = \text{Flux}[\psi] + \text{Flux'}[\psi]\phi + \tfrac{1}{2}\text{Flux''}[\psi]\phi\phi + \dots$$

  and then average,
  using equal-time correlator $G = \langle \phi\phi \rangle$ as a new variable

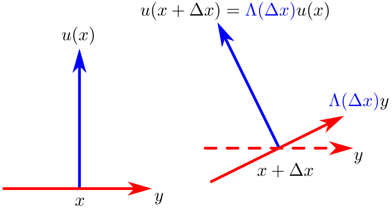
  - What is "equal-time" in *relativistic* hydro?
  - $\langle \phi(x)\phi(x) \rangle$ is singular (cutoff dependent). Renormalization?

Local equal time and *confluent* derivative

- $G = \langle \phi(t, \boldsymbol{x}_1)\phi(t, \boldsymbol{x}_2)\rangle$. In which frame?

  Natural choice is local rest frame, $u(x)$ at midpoint $x \equiv \frac{x_1+x_2}{2}$.

- Let $y \equiv x_1 - x_2$. How should we take $(\partial/\partial x)G(x; y)$ at "fixed y"?



not as $G(x+\Delta x; y) - G(x; y)$

But as $G(x + \Delta x; \Lambda(\Delta x)y) - G(x, y)$ – *confluent* derivative.

more

Renormalization

- Expansion of $\langle T^{\mu\nu} \rangle$ in fluctuations ϕ contains

more

$$\langle \phi(x)\phi(x) \rangle = G(x;0) = \int \frac{d^3q}{(2\pi)^3} W(x;q).$$

The integral is divergent (in equilibrium $G^{(0)}(x;y) \sim \delta^3(y)$).

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The integral is divergent (in equilibrium $G^{(0)}(x;y) \sim \delta^3(y)$).

- Such short-distance singularities can be absorbed into redefinion of EOS (i.e., pressure) and transport coefficients:

$$\begin{aligned}\langle T^{\mu\nu}(x)\rangle &= \epsilon u^\mu u^\nu + p(\epsilon,n)\Delta^{\mu\nu} + \Pi^{\mu\nu} + \Big\{G(x;0)\Big\} \\ &= \epsilon_R u_R^\mu u^\nu + p_R(\epsilon_R,n_R)\Delta_R^{\mu\nu} + \Pi_R^{\mu\nu} + \Big\{G_R(x;0)\Big\}.\end{aligned}$$

more

Phonon kinetics

An et al 1902.09517

- What is the meaning of $W(x;q)$ – Wigner transform of $G(x;y)$?
  - The 'longitudinal' components $W_L(x,\boldsymbol{q})$, corresponding to pressure and velocity fluctuations at $\delta(s/n) = 0$ (i.e., sound-sound), obey relativistic kinetic equation for phonons.

    $E = c_s(x)|\boldsymbol{q}|$ – dispersion relation in local rest frame.

    $W_L(x,\boldsymbol{q})$ corresponds to phase space phonon density (times E).
  - In a non-homogeneous fluid, the phonons experience gradient as well as inertial, Coriolis and Hubble forces. Due to acceleration, rotation and expansion of the fluid respectively.

more

non-Gaussian fluctuations are sensitive signatures of the critical point

Deterministic approach to non-Gaussian fluctuations

An et al 2009.10742, PRL

- *Infinite* hierarchy of coupled equations for cumulants $G_n^{\mathsf{c}} \equiv \langle \underbrace{\delta\breve{\psi} \ldots \delta\breve{\psi}}_{n} \rangle^{\mathsf{c}}$:

$$\partial_t \psi = -\nabla \cdot \mathsf{Flux}[\psi, G, G_3^{\mathsf{c}}, G_4^{\mathsf{c}}, \ldots];$$

$$\partial_t G = \mathsf{F}[\psi, G, G_3^{\mathsf{c}}, G_4^{\mathsf{c}}, \ldots];$$

$$\partial_t G_3^{\mathsf{c}} = \mathsf{F}_3[\psi, G, G_3^{\mathsf{c}}, G_4^{\mathsf{c}}, \ldots];$$

$$\vdots$$

Controlled perturbation theory

- *Small* fluctuations are *almost* Gaussian
- Introduce expansion parameter ε, so that $\delta\breve{\psi} \sim \sqrt{\varepsilon}$.

  Then $G^{\mathsf{c}}_n \equiv \varepsilon^{n-1}$ and to leading order in ε:

$$\partial_t \psi = -\nabla \cdot \mathsf{Flux}[\psi] + \mathcal{O}(\varepsilon);$$

$$\partial_t G = -2\Gamma(G - \bar{G}[\psi]) + \mathcal{O}(\varepsilon^2);$$

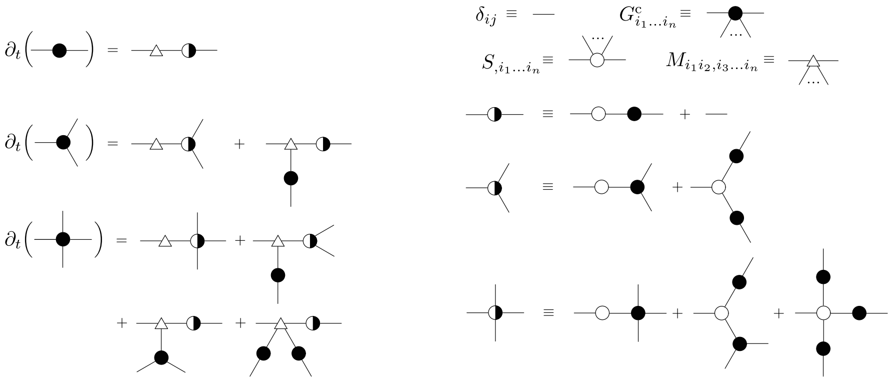
$$\vdots$$

$$\partial_t G^{\mathsf{c}}_n = -n\Gamma(G^{\mathsf{c}}_n - \bar{G}^{\mathsf{c}}_n[\psi, G, \ldots, G^{\mathsf{c}}_{n-1}]) + \mathcal{O}(\varepsilon^n);$$

  To leading order, the equations are iterative and “linear”.
- In hydrodynamics the small parameter is $(q/\Lambda)^3$, i.e., fluctuation wavelength $1/q \gg$ size of hydro cell $1/\Lambda$ (UV cutoff).

Diagrammatic representation

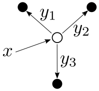
Systematically expand in ε and truncate at leading order:



- Leading order in ε $\Leftrightarrow$ tree diagrams.
- In higher-orders, loops describe feedback of fluctuations (e.g., long-time tails).

Generalizing Wigner transform

- Definition:



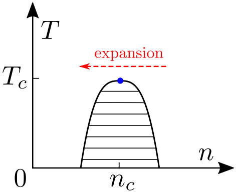
$$W_n(\boldsymbol{x};\boldsymbol{q}_1,\ldots,\boldsymbol{q}_n) \equiv \int d\boldsymbol{y}_1^3 \ldots \int d\boldsymbol{y}_n^3\, G_n\left(\boldsymbol{x}+\boldsymbol{y}_1,\ldots,\boldsymbol{x}+\boldsymbol{y}_n\right)$$
$$\delta^{(3)}\left(\frac{\boldsymbol{y}_1+\ldots+\boldsymbol{y}_n}{n}\right) e^{-i(\boldsymbol{q}_1\cdot\boldsymbol{y}_1+\ldots+\boldsymbol{q}_n\cdot\boldsymbol{y}_n)};$$

$$G_n\left(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_n\right) = \int \frac{d\boldsymbol{q}_1^3}{(2\pi)^3} \ldots \int \frac{d\boldsymbol{q}_n^3}{(2\pi)^3}\, W_n(\boldsymbol{x},\boldsymbol{q}_1,\ldots,\boldsymbol{q}_n)$$
$$\delta^{(3)}\left(\frac{\boldsymbol{q}_1+\ldots+\boldsymbol{q}_n}{2\pi}\right) e^{i(\boldsymbol{q}_1\cdot\boldsymbol{x}_1+\ldots+\boldsymbol{q}_n\cdot\boldsymbol{x}_n)}\,.$$

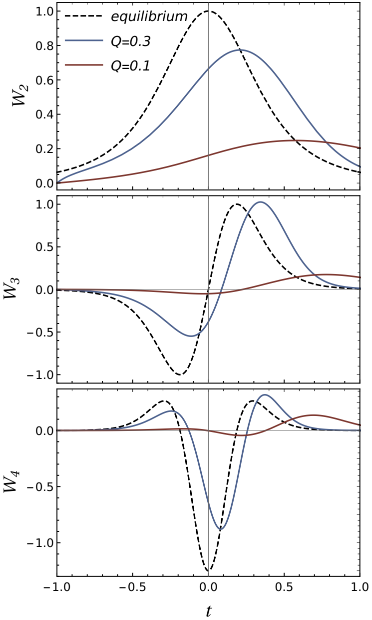
  - Properties similar to the usual ($n=2$) Wigner transform.
  - Takes advantage of the scale separation:
    long-scale x-dependence and short-scale $\boldsymbol{y}_n$-dependence.

Example: expansion through a critical region

An et al 2009.10742, PRL



- Two main features:
  - Lag, "memory".
  - Smaller Q – slower evolution. Conservation laws.
- Critical point signatures depend on the scale of fluctuations probed.



Experiments measure particles, not hydro variables

Freezing out (critical) hydrodynamic fluctuations

- Cooper-Frye deals with with 1-particle observables.
  We need 2-particle (and n-particle) *correlations*.
- Critical contribution to fluctuations of $f(x,p)$: *1104.1627, PRL*

$$\delta f = \frac{\partial f}{\partial \sigma}\delta\sigma, \quad \text{via} \quad \delta m = g\delta\sigma.$$

$$\langle \delta\sigma\delta\sigma \rangle \sim \text{ F.T. } \phi_{\boldsymbol{Q}}$$

- Critical contribution to observables

$$\langle \delta N^2 \rangle = \int_{\boldsymbol{p}_1}\int_{\boldsymbol{p}_2} \langle \delta f_1 \delta f_2 \rangle \sim \int_{\boldsymbol{p}_1}\int_{\boldsymbol{p}_2} \frac{\partial f_1}{\partial \sigma}\frac{\partial f_2}{\partial \sigma} \text{ F.T. } \underbrace{\phi_{\boldsymbol{Q}}}_{\text{Hydro+}}$$

EOS, transport coeffs. $\longrightarrow$ Hydro+ $\longrightarrow$ Observables

An example of implementation: *Pradeep et al, 2204.00639, PRD*

more

How do the *non-gaussian* fluctuations freeze out?

Maximum entropy freezeout of fluctuations

Pradeep, MS, 2211.09142, PRL

- Freezeout: translation of correlators of hydrodynamic fluctuations (n-point functions) $H_n = \langle \delta\epsilon \ldots \delta\epsilon \rangle$ to particle correlators $G_n = \langle \delta f \ldots \delta f \rangle$. more
- Conservation laws relate momentum space integrals of G_n to H_n, but there are ∞ many possibilities/solutions for G_n matching these constraints. Because f and G_n are functions of $\boldsymbol{p}$'s in addition to $\boldsymbol{x}$'s.

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- Conservation laws relate momentum space integrals of G_n to H_n, but there are ∞ many possibilities/solutions for G_n matching these constraints. Because f and G_n are functions of $\boldsymbol{p}$'s in addition to $\boldsymbol{x}$'s.
- There is a special solution which maximizes the entropy!
  - for $n = 1$ equivalent to Cooper-Frye
  - for critical fluctuations equivalent to the σ field coupling
  - but applies much more generally
- Work in progress – implement in a hydro model and estimate *nonequilibrium* expectations for multiplicity cumulants in BES

(Karthein, Pradeep, MS, Rajagopal, Yin)

Some open questions in fluctuation hydrodynamics

- Fluctuation feedback – loops, renormalization, etc.
- Relation to path-integral (Schwinger-Keldysh) formulation of hydrodynamics.
- First-order phase transition and fluctuations
- . . .

Summary

- Is there a critical point between QGP and hadron gas phases?

  Heavy-Ion collision experiments may answer.

  The quest for the QCD critical point challenges us to creatively apply existing concepts and develop new ideas.

- Non-monotonic behavior of fluctuation measures (especially non-Gaussian) – universal signatures of a critical point.

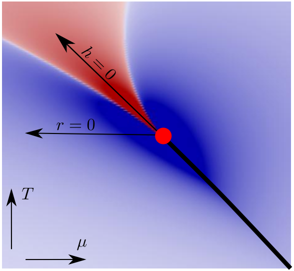
- In H.I.C., the signatures of criticality are subject to non-equilibrium effects. The interplay of fluctuations and non-equilibrium dynamics opens interesting questions.

More

QCD and observables near CP

κ_4 vs μ_B and T:

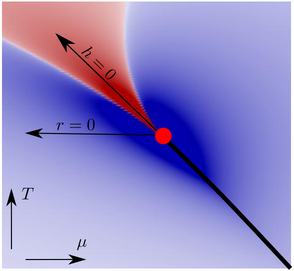
back



- $(r, h) \to (\mu - \mu_{\rm CP}, T - T_{\rm CP})$

κ_4 vs μ_B and T:

back



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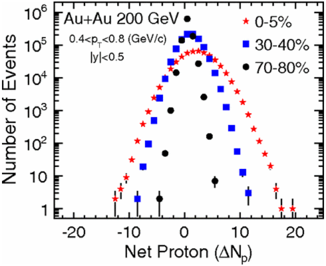


- Experiments do not measure σ. Fluctuations of σ are "imprinted" on hadron multiplicity fluctuations:

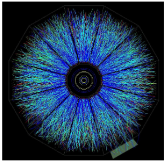
$$\kappa_n(N) = N + \mathcal{O}(\kappa_n(\sigma))$$

MS, 1104.1627

Pradeep et al 2109.13188

Heavy-Ion Collisions. Thermalization. Freezeout.

STAR



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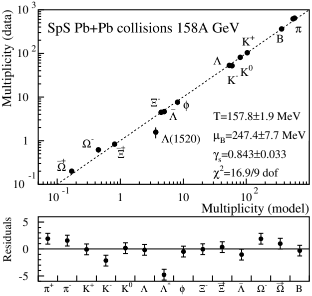

(Becattini et al)

“Little Bang”

- The final state looks *thermal*.
- Similar to CMB.

back

Heavy-Ion Collisions. Thermalization. Freezeout.

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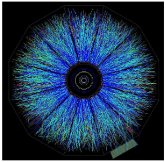

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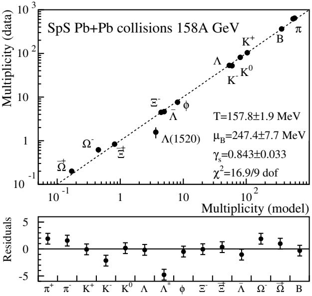

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(Becattini et al)

back

Assumption for the next part of this talk

H.I.C. are sufficiently close to equilibrium that we can study thermodynamics at freezeout T and μ_B — as a first approximation.

What are the additional slow modes?

- An *equilibrium* thermodynamic state is completely characterized by average values $\bar{\epsilon}$, $\bar{n}$, $\ldots$.

  Fluctuations of ϵ, n are given by eos: $P \sim \exp(S_{\text{eq}}(\epsilon, n))$.

back

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  back

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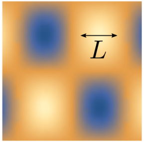
- Hydrodynamics describes *partial-equilibrium states*, i.e., equilibrium is only local, because equilibration time $\sim L^2$.

  Fluctuations in such states are not necessarily in equilibrium.



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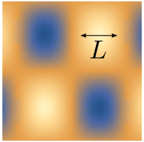


- Thus measures of fluctuations, e.g., 2pt functions, are *additional* variables needed to characterize a partial-equilibrium state.

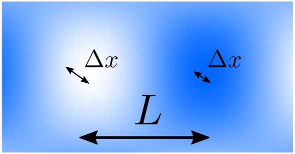
New variables in Hydro+

- At the CP the new variable is 2-pt function $\langle \delta m \delta m \rangle$:

$$\phi_{\boldsymbol{Q}}(\boldsymbol{x}) = \int_{\Delta \boldsymbol{x}} \left\langle \delta m \left(\boldsymbol{x} + \frac{\Delta \boldsymbol{x}}{2} \right) \delta m \left(\boldsymbol{x} - \frac{\Delta \boldsymbol{x}}{2} \right) \right\rangle e^{i \boldsymbol{Q} \cdot \Delta \boldsymbol{x}}$$

where $m \equiv n/s$ ("baryon asymmetry") – the slowest mode.

- Wigner transformed because dependence on $\boldsymbol{x}$ ($\sim L$) is much slower than on $\Delta \boldsymbol{x}$ ($\sim \xi$, $\sqrt{L}$).



back

Relaxation of fluctuations towards equilibrium

- As usual, relaxation toward equilibrium, or maximum of entropy:

$$s_{(+)}(\epsilon, n, \phi_{\boldsymbol{Q}}) = s(\epsilon, n) + \frac{1}{2}\int_{\boldsymbol{Q}} \left(1 - \frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}} + \log\frac{\phi_{\boldsymbol{Q}}}{\bar{\phi}_{\boldsymbol{Q}}}\right)$$

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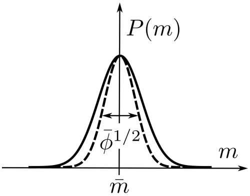
- Entropy = log # of states, which depends on the width of $P(m_{\boldsymbol{Q}})$, i.e., $\phi_{\boldsymbol{Q}}$:
  - Wider distribution – more microstates – more entropy: $\log(\phi/\bar{\phi})^{1/2}$;

    vs

  - Penalty for larger deviations from peak entropy (at $\delta m = 0$): $-(1/2)\phi/\bar{\phi}$.

  Maximimum of $s_{(+)}$ is achieved at $\phi = \bar{\phi}$.



--- equilibrium (variance $\bar{\phi}$)

—- actual (variance ϕ)

back

Hydro+ mode kinetics

- The equation for $\phi_{\boldsymbol{Q}}$ is a relaxation equation:

$$(u \cdot \partial)\phi_{\boldsymbol{Q}} = -\gamma_\pi(\boldsymbol{Q})\pi_{\boldsymbol{Q}}, \quad \pi_{\boldsymbol{Q}} = -\left(\frac{\partial s_{(+)}}{\partial \phi_{\boldsymbol{Q}}}\right)_{\epsilon,n}$$

  $\gamma_\pi(\boldsymbol{Q})$ is known from mode-coupling calculation in ‘model H’ (Kawasaki). It is universal.

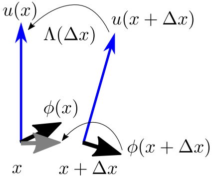
- Characteristic rate: at $Q \sim \xi^{-1}$, $\gamma_\pi(Q) \sim \xi^{-3}$.

  Vanishes at CP, leading to breakdown of hydrodynamics.

Confluent derivative, connection and correlator

Take out dependence of *components* of ϕ due to change of $u(x)$:

$$\Delta x \cdot \bar{\nabla}\phi = \Lambda(\Delta x)\phi(x + \Delta x) - \phi(x)$$



Confluent two-point correlator:

$$\bar{G}(x, y) = \Lambda(x_1 - x)\,\langle\phi(x_1)\,\phi(x_2)\rangle\,\Lambda(x_2 - x)^T$$

(boost to $u(x)$ – rest frame at midpoint)

$$\bar{\nabla}_\mu \bar{G}_{AB} = \partial_\mu \bar{G}_{AB} - \bar{\omega}^C_{\mu A}\bar{G}_{CB} - \bar{\omega}^C_{\mu B}\bar{G}_{AC} - \mathring{\omega}^b_{\mu a}\, y^a \frac{\partial}{\partial y^b}\bar{G}_{AB}\,.$$

Connection $\bar{\omega}$ corresponds to the boost Λ.

back

Connection $\mathring{\omega}$ makes sure derivative is independent of the choice of basis triad $\boldsymbol{e}_a(x)$ needed to express $\boldsymbol{y} \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2$ in local rest frame.

We then define the Wigner transform $W_{AB}(x; q)$ of $\bar{G}_{AB}(x; y)$.

Renormalization

Expansion of $\langle T^{\mu\nu}\rangle$ contains $\langle\phi(x)\phi(x)\rangle = G(x;0) = \int \frac{d^3q}{(2\pi)^3}\, W(x;q)$.

This integral is divergent (equilibrium $G^{(0)}(x;y) \sim \delta^3(y)$).

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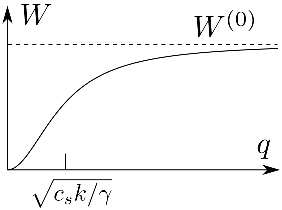

$$W(x,q) \sim \underbrace{W^{(0)}}_{Tw} + \underbrace{W^{(1)}}_{\partial u/q^2} + \widetilde{W}$$

(∼"OPE" or gradient expansion)

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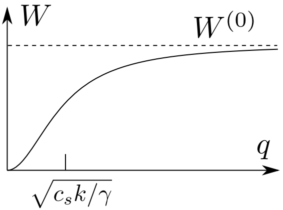

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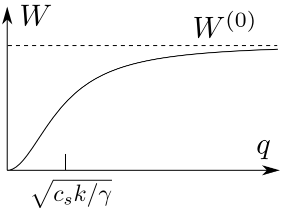

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$$\langle T^{\mu\nu}(x)\rangle = \epsilon u^\mu u^\nu + p(\epsilon,n)\Delta^{\mu\nu} + \Pi^{\mu\nu} + \Big\{G(x,0)\Big\}$$
$$= \epsilon_R u_R^\mu u^\nu + p_R(\epsilon_R,n_R)\Delta_R^{\mu\nu} + \Pi_R^{\mu\nu} + \Big\{\tilde{G}(x,0)\Big\}\ .$$

Renormalized e.o.s. and transport coefficients

- Fluctuation corrections to kinetic coefficients are positive. back
- Corrections to pressure and bulk viscosity vanish for conformal e.o.s.

$$p_R(\epsilon_R, n_R) \quad = \quad p(\epsilon_R, n_R) + \frac{T\Lambda^3}{6\pi^2}\left((1 - c_s^2 - 2\dot{T} + \dot{c}_s) + \frac{1}{2}(1 - \dot{c}_p)\right),$$

$$\eta_R \quad = \quad \eta + \frac{T\Lambda}{30\pi^2}\left(\frac{1}{\gamma_L} + \frac{7}{2\gamma_\eta}\right),$$

$$\zeta_R \quad = \quad \zeta + \frac{T\Lambda}{18\pi^2}\left(\frac{1}{\gamma_L}(1 - 3\dot{T} + 3\dot{c}_s)^2 + \frac{2}{\gamma_\eta}\left(1 - \frac{3}{2}(\dot{T} + c_s^2)\right)^2 + \frac{9}{4\gamma_\lambda}(1 - \dot{c}_p)^2\right),$$

$$\lambda_R \quad = \quad \lambda + \frac{T^2 n^2 \Lambda}{3\pi^2 w^2}\left(\frac{c_p T}{(\gamma_\eta + \gamma_\lambda) w} + \frac{c_s^2}{2\gamma_L}\right).$$

$$\gamma_\eta \equiv \frac{\eta}{w}, \quad \gamma_\zeta \equiv \frac{\zeta}{w}, \quad \gamma_\lambda \equiv \frac{\kappa}{c_p} = D, \quad \dot{X} \equiv \left(\frac{\partial \log X}{\partial \log s}\right)_m.$$

Phonon kinetic equation

- The components of $W(x, \boldsymbol{q})$, corresponding to p and u^μ fluctuations at $\delta(s/n) = 0$, obey ($N_L \equiv W_L/(wc_s|\boldsymbol{q}|)$)

$$\underbrace{\left[(u+v)\cdot\bar{\nabla} + f\cdot\frac{\partial}{\partial q}\right] N_L}_{\mathcal{L}[N_L]\text{ – Liouville op.}} = -\gamma_L \boldsymbol{q}^2 \left(N_L - \underbrace{\frac{T}{c_s|\boldsymbol{q}|}}_{N_L^{(0)}} \right)$$

- Kinetic eq. for phonons with $E = c_s|\boldsymbol{q}|$, $\boldsymbol{v} = \partial E/\partial \boldsymbol{q}$

$$f_\mu = \underbrace{-E(a_\mu + 2v^\nu \omega_{\nu\mu})}_{\text{inertial + Coriolis}} \underbrace{-q^\nu \partial_{\perp\mu} u_\nu}_{\text{“Hubble”}} - \bar{\nabla}_{\perp\mu} E\,.$$

- N_L – phonon distribution function, relaxes to Bose-distribution.

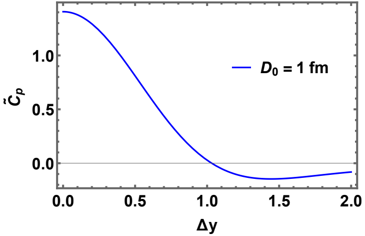
Note: Lots of algebra with many *miraculous* cancellations.

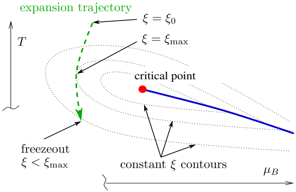
back

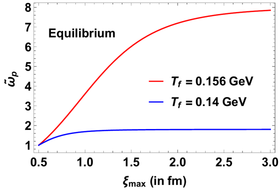
Freeze out in Hydro+: model calculation and lessons

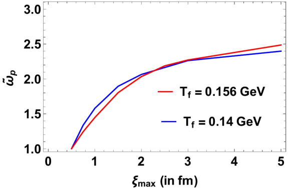
Pradeep et al, 2204.00639, PRD

- Effect of conservation laws:
  - particle (anti)correlations
  - suppression relative to equilibrium critical expectations









($\xi_{\max}$ – how close fireball gets to CP; $T_{\rm f}$ – how long it evolves after passing CP.)

Signal less sensitive to $T_{\rm freezeout}$ due to noneq. effects.

back

Earlier work, problems and questions

- "Fluctuating Cooper-Frye:" *Kapusta-Muller-MS 2011*

$$\delta f_A = \left(\delta\alpha\frac{\partial}{\partial\alpha} + \delta\beta\frac{\partial}{\partial\beta} + \delta u\frac{\partial}{\partial u}\right) f_A(\alpha,\beta,u)$$

Then, multiplicity fluctuation correlator:

$$\langle\delta f_A \delta f_B\rangle = \underbrace{\langle\delta\alpha\delta\alpha\rangle}_{\text{from hydro}} \left(\frac{\partial}{\partial\alpha} f_A\right)\left(\frac{\partial}{\partial\alpha} f_B\right) + \ldots \qquad (*)$$

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- Problem:
  consider ideal gas, no correlations, i.e. $\langle f_A f_B\rangle = \delta_{AB} f_A$

  but there are fluctuations of $\delta\alpha$, $\delta\beta$, etc. even in ideal gas $\Rightarrow$ equation $(*)$ produces incorrect result: spurious correlations.

Source of the problem and a solution

- Pairs of correlated particles erroneously include "pairs" made of the same particle counted twice.
- A solution *Li-Springer-MS '13, Plumberg-Kapusta '20*
  for charge fluctuations – subtract the contribution of ideal gas to $\langle\delta n\delta n\rangle$ in hydrodynamics and apply equation $(*)$ only to the remainder:

$$\langle\delta n\delta n\rangle \equiv \langle\delta n\delta n\rangle_{\rm ideal} + \Delta\langle\delta n\delta n\rangle$$

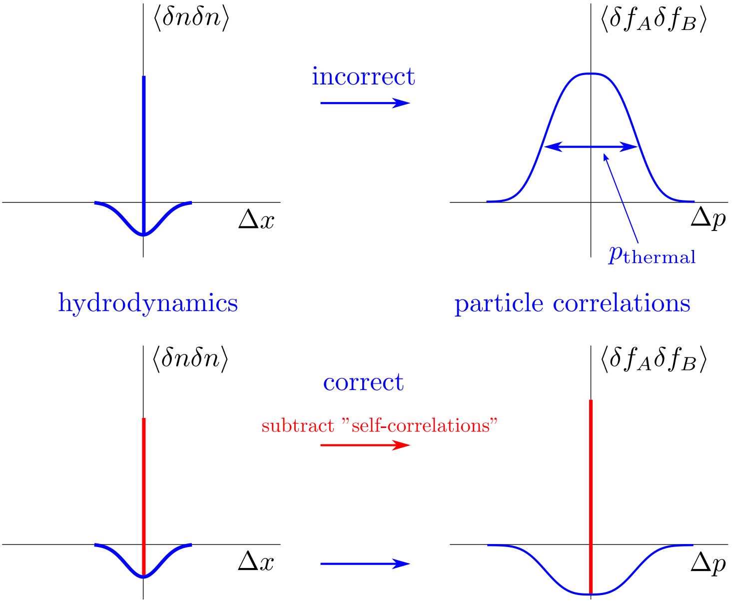
$$\langle\delta f_A\delta f_B\rangle = \delta_{AB}f_A + \underbrace{\Delta\langle\delta n\delta n\rangle\left(\frac{\partial}{\partial n}f_A\right)\left(\frac{\partial}{\partial n}f_B\right)}_{\text{balance function}}$$

- Similarly, for critical contribution to fluctuations, $\langle\delta\sigma\delta\sigma\rangle_{\rm critical} \sim \xi^2$ translates into deviation from baseline:

$$\langle\delta f_A\delta f_B\rangle = \underbrace{\delta_{AB}f_A}_{\text{baseline}} + \underbrace{\mathcal{O}(\xi^2)}_{\text{critical contribution}}$$

MS-Rajagopal-Shuryak 1999

Thermal smearing and “self-correlations”



Open questions

How to deal with

- Temperature, velocity fluctuations?
- Non-critical fluctuations?
- Non-gaussian fluctuations?

Maximum entropy freezeout: *Pradeep-MS 2211.09142*

Revisit one-point/single-particle observables

- Locally matching conserved quantities before/after freezeout: $n(x) = \sum_A q_A f_A(x)$ and $\epsilon(x) u^\mu(x) = \sum_A p_A^\mu f_A(x)$.

  Problem: these equations for f_A have infinitely many solutions.

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  Problem: these equations for f_A have infinitely many solutions.
- Which solution maximizes Bolzmann entropy?
  $S_0 = -\sum_A f_A \log f_A$
  Answer: $f_A = e^{\alpha_A q_A + \beta u \cdot p_A}$ — Cooper-Frye.
  - Matching also dissipative viscous stress and diffusive current gives $f_A = e^{\alpha_A q_A + \beta u \cdot p_A} + \underbrace{\Delta f_A}_{\text{non-equilibrium correction}}$. *(Everett-Chattopadhyay-Heinz 2021)*

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- Fluctuations? *Pradeep-MS 2022*

Maximum entropy freezeout of fluctuations

- We want to match fluctuations of $\{n, \epsilon, u^\mu\} \equiv \Psi_a$,
  to fluctuations of f_A so that $\Psi_a = \sum_A P_a^A f_A$ event-by-event

  i.e., $G_{AB} \equiv \langle \delta f_A \delta f_B \rangle$ must obey $(P_a^A = \{q_A, p_A, \dots\})$

$$\underbrace{\langle \delta\Psi_a \delta\Psi_b \rangle}_{H_{ab}} = \sum_{AB} P_a^A P_b^B \underbrace{\langle \delta f_A \delta f_B \rangle}_{G_{AB}}$$

  Again, for G_{AB}, there are infinitely many solutions.

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  Again, for G_{AB}, there are infinitely many solutions.

- Entropy? Is a functional of fluctuations, i.e., of G_{AB}, G_{ABC}, etc. E.g,

$$S_2 = S_0 + \underbrace{\frac{1}{2}\mathrm{Tr}\left[\log GC + GC + 1\right]}_{\text{relative entropy, } G = -C^{-1} \equiv \bar{G}}$$, where $C_{AB} = \frac{\delta^2 S}{\delta f_A \delta f_B}$.

Maximum entropy solution

- Relative entropy is maximized (subject to constraints) by

  $$G^{-1}_{AB} = \bar{G}^{-1}_{AB} + (H^{-1} - \bar{H}^{-1})_{ab} P^a_A P^b_B$$

  - Also for non-gaussian correlators *(Pradeep-MS 2022)*.
  - Note: when $H = \bar{H} \to G_{AB} = \bar{G}_{AB} = f_A \delta_{AB}$.

- Linearizing in $\Delta H \equiv H - \bar{H}$ we obtain the desired generalization of earlier results:

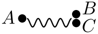
  $$G = \underbrace{\bar{G}}_{\text{baseline}} + \underbrace{(\bar{H}^{-1} P \bar{G})^T \Delta H (\bar{H}^{-1} P \bar{G})}_{\text{correlations}}$$

Non-gaussian correlators ($n \geq 3$ particles)

Linearied equations are simple and intuitive:

$$G_{AB} = \bar{G}_{AB} + \Delta G_{AB}, \quad H_{ab} = \bar{H}_{ab} + \Delta H_{ab},$$

$$G_{ABC} = \Big[\underbrace{\bar{G}_{ABC}} + \underbrace{3\Delta G_{AD}\delta_{DBC}} + \underbrace{\widehat{\Delta} G_{ABC}}_{\text{irreducible correlation}} \Big]_{\overline{ABC}} \leftarrow \text{permutation average}$$



$$H_{abc} = \left[\bar{H}_{abc} + 3\Delta H_{ad}\delta_{dbc} + \widehat{\Delta} H_{abc} \right]_{\overline{abc}}$$

Maximum entropy method gives:

$$\Delta G_{AB} = \Delta H_{ab} (\bar{H}^{-1} P \bar{G})^a_A (\bar{H}^{-1} P \bar{G})^b_B$$

$$\hat{\Delta} G_{ABC} = \hat{\Delta} H_{abc} (\bar{H}^{-1} P \bar{G})^a_A (\bar{H}^{-1} P \bar{G})^b_B (\bar{H}^{-1} P \bar{G})^c_C$$

Critical fluctuations

- The contribution of critical fluctuations matches the simple model often used in the literature *(MS 2011)*:

$$\delta f_A^{\rm critical} = \delta\sigma \left(\frac{\partial}{\partial\sigma} f_A\right)$$

where critical field σ couples to mass so that $\delta m_A = g_A \delta\sigma$.

Thus $$\langle \delta f_A \delta f_B \rangle = \underbrace{\delta_{AB} f_A}_{\text{Poisson baseline}} + \underbrace{\langle \delta\sigma\delta\sigma\rangle \left(\frac{\partial}{\partial\sigma} f_A\right)\left(\frac{\partial}{\partial\sigma} f_B\right)}_{\text{critical contribution} \sim g_A g_B}$$

- Now, within maximum entropy approach, we can determine the couplings g_A of the critical mode from the equation of state.

Concluding summary (ME freezeout)

- Maximum entropy approach for single-particle observables $=$ traditional Cooper-Frye freezeout.
- Maximum entropy approach solves the problem of freezing out of hydrodynamic fluctuations.
- The method is very general and works for gaussian and non-gaussian, for critical and non-critical fluctuations.
- Agrees with existing methods where such are available.
- Allows determination of critical field coupling parameters crucial to predicting the magnitude of CP signatures in terms of the EOS parameters.

back