Quantum Field Theory in Large N Wonderland: Three Lectures

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This wonderful set of notes will have a wonderful abstract wonderfully soon.

I. PREFACE

You may have heard that Quantum Field Theory is a well-developed, mature discipline.

That all the easy problems have been done long ago.

That there is nothing left to discover.

That is not true.

Welcome to QFT in large N wonderland!

II. INTRODUCTION

The aim of these lecture notes is to provide an accessible introduction to the physical applications of large N solution techniques for quantum field theory. They are aimed at advanced graduate students and early-career postdocs in theoretical physics, but they do contain new material that occasionally puzzles more senior researchers. The main guiding principle behind the lectures is that they offer techniques to obtain direct first-principle quantitative answers to physics problems of interest, with minimal specialized mathematical knowledge.

To keep the lecture notes readable, I have chosen to keep references at a minimum, with an emphasis on recent rather than older results.

That said, the use of large N techniques as opposed to perturbation theory has a long history in field theory, with many of the key results already obtained in the 1970s [1–3]. Unfortunately, subsequent research showed that large N techniques are not sufficient to solve specific non-abelian gauge theories of interest, such as QCD. As a consequence, the main theoretical tools for the study of QCD presently are perturbative (weak-coupling) expansions (e.g. [4–6]), lattice QCD (e.g.[7, 8]), as well as effective non-relativistic expansions (e.g. [9]).

In contradistinction to QCD, large N does play an important role in holographic conjectures of supersymmetric gauge theories, such as the conjectured dual of $\mathcal{N}=4$ Super-Yang-Mills theory in the large N limit to classical Einstein gravity in asymptotically AdS_5 spacetimes [10].

Coming full circle, the holographic conjectures for large N gauge theories did lead to conjectures for large N scalar theories, such as the conjectured dual of the O(N) model in 3 dimensions to higher-spin gravity in asymptotically AdS_4 spacetimes [11]. Unlike the case of gauge theories, where a proof of the gravity dual seems out of reach, the solvability of scalar field theories in the large N limit suggests the gravity dual theory can be derived, rather than conjectured [12, 13].

Despite the attractive feature of large N solvability, applications of large N techniques for scalar and fermionic theories has remained somewhat dormant since the 1970s. This provides opportunity for using large N techniques to solve problems of interest, such as calculation of transport coefficients [14–16], finite temperature correlators [17, 18], finite-density systems [19], as well as real-time evolution in quantum field theory [20].

Many problems which are intractable using standard perturbation theory surprisingly become ... [to be continued]

III. LECTURE 1: QUANTUM MECHANICS

Let's start with a simple test case where we can check our methods: quantum mechanics.

Quantum mechanics concerns itself with the spectrum of a Hamiltonian. For concreteness, let us consider the case of a one-dimensional system with Hamiltonian

$$\mathcal{H} = \frac{p^2}{2} + \lambda x^4\,, \tag{1}$$

where p, x are the momentum and position operator, respectively. The spectrum E_n of the Hamiltonian is defined through the time-independent Schrödinger equation,

$$\langle x|\mathcal{H}|n\rangle = \mathcal{H}\psi_n(x) = E_n\psi_n(x)\,, \tag{2}$$

where $\psi_n(x)$ are the wave-function eigenstates of $\mathcal{H}$.

What is the ground-state energy E_0 for the Hamiltonian (1) ?

It so happens that E_0 for the Hamiltonian (1) is not known analytically. I've chosen (1) deliberately partly because of this property, otherwise it would be too easy. However, note that *no* Hamiltonian with potential $V(x) \propto x^\alpha$ for $\alpha \in (2, \infty)$ has analytically known ground state energies, so the problem of finding E_0 is not contrived, but rather generic.

However, (1) shares certain important properties with Hamiltonian of the harmonic oscillator $V(x) \propto x^2$, in that it's spectrum for $\lambda > 0$ is real, discrete, and positive definite. It's just hard to calculate E_0.

Since our goal is to learn something about quantum field theory rather than quantum mechanics, let's cast quantum mechanics into field theory language by using path integrals. A rigorous way to do this from first principles is to consider the canonical partition function

$$Z = \mathrm{Tr} e^{-\beta\mathcal{H}} = \sum_{n=0}^{\infty}\langle n|e^{-\beta\mathcal{H}}|n\rangle = \sum_{n=0}^{\infty} e^{-\beta E_n}\,, \tag{3}$$

where $\beta = \frac{1}{T}$ and T the temperature of the system. By inserting complete sets of states, one can turn the trace of the Boltzmann operator into a path integral (see the steps leading from (1.27) to (1.37) in the excellent open-access textbook [21])

$$Z = \int \mathcal{D}\phi e^{-S_E}\,, \quad S_E = \int_0^\beta \left[\frac{1}{2}\dot\phi^2(\tau) + \lambda\phi^4(\tau)\right]\,, \tag{4}$$

where S_E is the Euclidean action of the theory and the field $\phi(\tau)$ lives on the Euclidean circle $\tau \in [0, \beta]$ with periodic boundary conditions.

Unfortunately, trying to solve the path integral in (4) is just as hard (or maybe even harder) than trying to directly solve the eigenvalue problem (2). Some new idea is needed.

To develop this idea, let's do something counter-intuitive: instead of considering the quantum mechanics problem in one dimension (which was hard), how about quantum mechanics in higher dimensions? At first glance, if the problem was hard in one dimension, it seems unlikely that could make progress by trying to to solve it in two, three, etc. dimensions, but let's see.

Using the odd symbol N to denote the number of dimensions, the equivalent Hamiltonian to (1) is given by

$$\mathcal{H} = \frac{\vec{p}^2}{2} + \frac{\lambda}{N}\left(\vec{x}^2\right)^2 , \tag{5}$$

where $\vec{p} = (p_1, p_2, \ldots p_N)$ and $\vec{x} = (x_1, x_2, \ldots, x_N)$ are again the momentum and position operators for quantum mechanics in N dimensions. The appearance of N in the denominator of the coupling λ may appear arbitrary at first sight, but if one considers that $\vec{x}^2 = x_1^2 + x_2^2 + \ldots x_N^2$ are N contributions of the operator x^2 it becomes clear that $\frac{\lambda}{N}$ is the right normalization so that $\mathcal{H}$ scales appropriately with N. (Alternatively, or rather equivalently, think of λ as the appropriate 't Hooft coupling [1] for this theory).

The path integral for quantum mechanics in N dimensions follows the same steps as for one-dimensional quantum mechanics, except that there is a quantum field for every dimension, so we end up with

$$Z = \int \mathcal{D}\vec{\phi} e^{-S_E} , \quad S_E = \int_0^\beta \left[\frac{1}{2}\left(\partial_\tau \vec{\phi}\right)^2 + \frac{\lambda}{N}\left(\vec{\phi}^2\right)^2\right] , \tag{6}$$

and $\vec{\phi} = (\phi_1, \phi_2, \ldots, \phi_N)$.

Instead of a hard path integral over a single field ϕ as in (4), we now have a path integral over multiple fields $\vec{\phi}$ which are all coupled together. If anything, this seems much harder than our original hard problem, so it doesn't look like we've made any progress here.

Don't despair yet, I have a trick down my sleeve!

The trick is that I can solve an integral over a Dirac δ function:

$$\int d\sigma \delta(\sigma - f) = 1 , \tag{7}$$

for any real f. I can write a product of these integrals, and obtain a 'path-integral δ':

$$\prod_\tau \int d\sigma(\tau)\delta\left(\sigma(\tau)-f(\tau)\right)=\int \mathcal{D}\sigma\delta\left(\sigma-f\right)=1\,. \tag{8}$$

Since this is true for any function $f(\tau)$ on the Euclidean circle, I can take $f(\tau)=\vec{\phi}^2(\tau)$ and thus re-write the partition function (6) as

$$Z=\int \mathcal{D}\vec{\phi}\mathcal{D}\sigma\delta(\sigma-\vec{\phi}^2)e^{-S_E}\,,\quad S_E=\int_0^\beta d\tau\left[\frac{1}{2}\left(\partial_\tau\vec{\phi}\right)^2+\frac{\lambda}{N}\sigma^2\right]\,. \tag{9}$$

Having a delta function inside a path integral is un-field theorist, so I use

$$\delta(x)=\int d\zeta e^{i\zeta x}\,, \tag{10}$$

to rewrite the path integral again as

$$Z=\int \mathcal{D}\vec{\phi}\mathcal{D}\sigma\mathcal{D}\zeta e^{-S_E}\,,\quad S_E=\int_0^\beta d\tau\left[\frac{1}{2}\left(\partial_\tau\vec{\phi}\right)^2+\frac{\lambda}{N}\sigma^2-i\zeta\left(\sigma-\vec{\phi}^2\right)\right]\,. \tag{11}$$

In this form, we have a path integral with two auxiliary fields σ,ζ, but since the action for σ is quadratic, we can integrate out σ explicitly:

$$Z=\int \mathcal{D}\vec{\phi}\mathcal{D}\zeta e^{-S_E}\,,\quad S_E=\int_0^\beta d\tau\left[\frac{1}{2}\left(\partial_\tau\vec{\phi}\right)^2+i\zeta\vec{\phi}^2+\frac{N\zeta^2}{4\lambda}\right]\,. \tag{12}$$

As a side remark, rewriting of the path integral for quartic potential using an auxiliary field is known as a Hubbard-Stratonovic transformation in the literature. When I started working on this, I didn't know about Hubbard-Stratonovic, so I came up with this version which works for other potentials of the form $V(x)\propto x^\alpha$ as well, not just $\alpha=4$ [22]. Apparently, sometimes ignorance is an advantage when working on a new subject.

The partition function (12) is quadratic in the field $\vec{\phi}$, so we can formally integrate out those fields as well, giving

$$Z=\int \mathcal{D}\zeta e^{-S_E}\,,\quad S_E=\frac{N}{2}\mathrm{Tr}\ln\left[-\partial_\tau^2+2i\zeta\right]+\int_0^\beta d\tau\frac{N\zeta^2}{4\lambda}\,. \tag{13}$$

So far, everything has been exact.

Splitting the auxiliary field ζ into zero-mode and fluctuations

$$\zeta(\tau)=\zeta_0+\zeta'(\tau)\,, \tag{14}$$

we have

$$S_E=\frac{N}{2}\mathrm{Tr}\ln\left[-\partial_\tau^2+2i\zeta_0\right]+\frac{N\beta\zeta_0^2}{4\lambda}+\mathcal{O}(\zeta'^2)\,. \tag{15}$$

The path integral over the fluctuations ζ' cannot be calculated analytically in closed form. However, since it is a single field, the integral over the fluctuations cannot give a contribution of order $e^{\mathcal{O}(N)}$ to the path integral. So in the limit of large N, the (complicated) contribution from the fluctuations are sub-dominant.

The calculation simplifies in the large N limit!

For $N \gg 1$, we thus have

$$\lim_{N\gg 1} Z = \int d\zeta_0 e^{-S_{R0}}\,, \quad S_{R0} = \frac{N}{2}\mathrm{Tr}\ln\left[-\partial_\tau^2 + 2i\zeta_0\right] + \frac{N\beta\zeta_0^2}{4\lambda}\,. \tag{16}$$

Instead of a path integral, the large N partition function is given in terms of a single integral, but the expression in the action still needs some work.

Since ζ_0 is τ-independent, it effective acts as a mass term, and we can calculate the trace of the logarithm of the operator as

$$\mathrm{Tr}\ln\left[-\partial_\tau^2 + 2i\zeta_0\right] = \sum_n \langle n|\ln\left[-\partial_\tau^2 + 2i\zeta_0\right]|n\rangle = \sum_n \ln\left[\omega_n^2 + 2i\zeta_0\right]\,, \tag{17}$$

when using $\langle\tau|n\rangle = e^{i\omega_n\tau}$ with $\omega_n = 2\pi nT$ the bosonic Matsubara frequencies. The "thermal" sum can be calculated using methods from thermal quantum field theory [21], or by straightforward comparison to the partition function of the harmonic oscillator. Let's do the latter: for the harmonic oscillator, the partition function is

$$Z_{HO} = \int \mathcal{D}\phi e^{-\frac{1}{2}\int_0^\beta\left[\dot\phi^2 + m^2\phi^2\right]} = e^{-\frac{1}{2}\mathrm{Tr}\ln\left[-\partial_\tau^2 + m^2\right]}\,, \tag{18}$$

because it is a Gaussian integral. But we know the spectrum of the harmonic oscillator is $E_n = m\left(n + \frac{1}{2}\right)$, so we can calculate the harmonic oscillator partition function as

$$Z_{HO} = \sum_{n=0}^\infty e^{-\beta E_n} = \frac{1}{2\sinh\left(\frac{m\beta}{2}\right)}\,. \tag{19}$$

Comparing the last two equation leads to

$$\frac{1}{2}\mathrm{Tr}\ln\left[-\partial_\tau^2 + m^2\right] = \ln\left[2\sinh\left(\frac{m\beta}{2}\right)\right] \tag{20}$$

As a consequence, we get for (16)

$$\lim_{N\gg 1} Z = \int d\zeta_0 e^{-N\ln\left[2\sinh\left(\frac{\sqrt{2i\zeta_0}\beta}{2}\right)\right] - \frac{N\beta\zeta_0^2}{4\lambda}}\,. \tag{21}$$

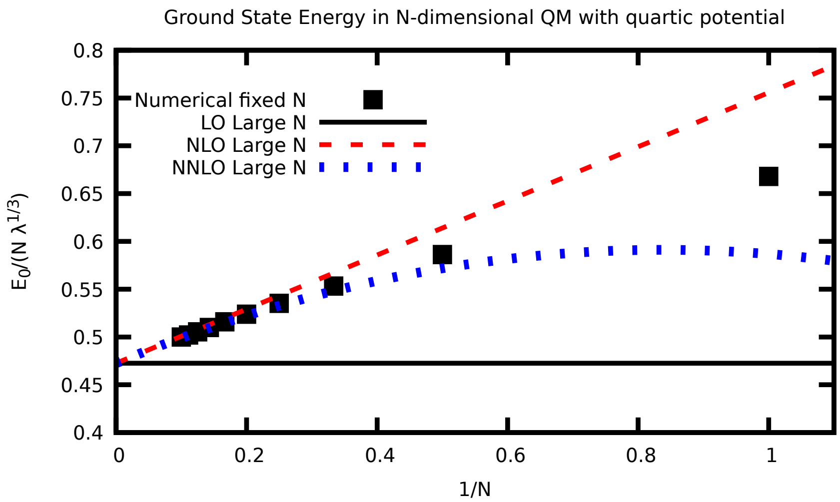

FIG. 1. Ground state energy $\frac{E_0}{\lambda^{\frac{1}{3}}N}$ as a function of components N. Shown are numerical results from table I for $n = 31$, and the analytic results in the large N limit: LO from (25), NLO from (30, and NNLO is left as an exercise.

This is the expression for the partition function of quantum mechanics in $N \gg 1$ dimensions at finite temperature. If we care about the ground state energy, we want to consider the low temperature limit $\beta \to \infty$. In this limit, the result simplifies to

$$\lim_{\beta\gg 1}\lim_{N\gg 1} Z = \int d\zeta_0 e^{-\frac{N\beta\sqrt{2i\zeta_0}}{2}-\frac{N\beta\zeta_0^2}{4\lambda}}\,. \tag{22}$$

For large N, the exponential is typically very small, except for the regions of the integral where the action is at a minimum. This is formally encoded in the saddle point method, so that integrals such as (22) can be evaluated exactly in closed form at large N. We find

$$\lim_{\beta\gg 1}\lim_{N\gg 1} Z = e^{-\beta E(\zeta^*)}\,, \tag{23}$$

where $\zeta_0 = \zeta^*$ is the solution to the saddle point condition

$$\frac{dE(\zeta^*)}{d\zeta^*} = 0 = \frac{N}{2\sqrt{2i\zeta^*}} + \frac{N\zeta^*}{2\lambda} \longrightarrow i\zeta^* = \frac{(2\lambda)^{\frac{2}{3}}}{2}\,. \tag{24}$$

Plugging this saddle point back into the partition function, we get

$$E(\zeta^*) = \frac{3(2\lambda)^{\frac{1}{3}}N}{8} \simeq \lambda^{\frac{1}{3}}N \times 0.47247\ldots \tag{25}$$

Comparison between (23) and (3) shows that this is the ground state energy for quantum mechanics in $N \gg 1$ dimensions interacting via quartic potential. It is exact in the large

N limit, and is smoothly connected to the ground state energy for finite, but large N, cf. Fig. 1. But even if we boldly extrapolate this result to $N=1$, we find that it only differs from the numerically calculated ground state energy of the one-dimensional quartic anharmonic oscillator

$$E_0 = \lambda^{\frac{1}{3}} \times 0.66799\ldots \tag{26}$$

by only about 30 percent (see Tab.I in the appendix and Ref. [23]).

We can add a $\frac{1}{N}$ improvement to the large N result of the ground state energy without too much trouble. Expanding S_E in the exact partition function (13) to second order in fluctuations around the saddle: $\zeta = \zeta_0 + \zeta'(\tau)$ and performing a Fourier-transform

$$\zeta'(\tau) = \int \frac{dk}{2\pi} e^{ik\tau} \zeta'(k)\,. \tag{27}$$

In the zero temperature limit, we obtain

$$\lim_{\beta\gg 1} \lim_{N\gg 1} Z = e^{-\frac{3(2\lambda)^{\frac{1}{3}}}{8} N\beta} \int \mathcal{D}\zeta' e^{-\int \frac{dk}{2\pi} \frac{N|\zeta'(k)|^2}{4\lambda} - 2N \int \frac{dk}{2\pi} |\zeta'(k)|^2 \Pi(k)}\,, \tag{28}$$

with

$$\Pi(k) = \frac{1}{2} \int \frac{dp}{2\pi} \frac{1}{(p^2 + (2\lambda)^{\frac{1}{3}})((p+k)^2 + (2\lambda)^{\frac{1}{3}})} = \frac{1}{2} \frac{1}{(2\lambda)^{\frac{1}{3}}(k^2 + 4(2\lambda)^{\frac{2}{3}})}\,. \tag{29}$$

Performing the path integral over ζ' leads to the large N ground state energy given by

$$E_0 = \frac{3(2\lambda)^{\frac{1}{3}}}{8} N + \frac{1}{2} \int \frac{dk}{2\pi} \ln\left(1 + \frac{2(2\lambda)^{\frac{2}{3}}}{k^2 + 4(2\lambda)^{\frac{2}{3}}}\right) = (2\lambda)^{\frac{1}{3}} \left(\frac{3}{8} N + \frac{\sqrt{6}-2}{2}\right) + \mathcal{O}(N^{-1})\,. \tag{30}$$

Calculating the NNLO large N correction is possible with similar techniques, and obtaining the result (31) is left as an exercise (see below).

Extrapolating the NLO ground state energy for N=1 and comparing to the numerically calculated result for the N=1 theory (26), one finds that the NLO result is off by only about 13 percent. Agreement with quantum mechanics in higher dimensions is better, as can be seen in Fig. 1. Clearly, large N expansion techniques work quantitatively well in capturing the ground state energy for quantum mechanics at fixed and not too small N.

As a final note, let me point out that the fact that the NNLO correction does not improve on the disagreement for N=1, but helps with larger N, is in agreement with the expectation that the large N series expansion is asymptotic, just like the perturbative series expansion.

Guide to further reading

Considering N-component field theory in dimension less than four is an interesting application of the above techniques. Here are a few suggestions for further reading

- The scalar O(N) model in 2+1 dimensions was studied at finite temperature in [17]. In this case, the field theory is super-renormalizable, and the large N expansion allows solution of this field theory for all values of the coupling. In particular, this includes a calculation of the exact large N shear viscosity coefficient [16].

- Theories with fermions, as well as certain supersymmetric theories in 2+1 dimensions can also be solved with the same technique, see [24, 25].

- Three dimensional QED with many flavors of electrons does not suffer from the problems encountered in four dimensions and can also be solved with similar techniques. The thermodynamics of large N_f QED_3 was worked out in Ref. [26], and the curious 'fractional photon' in the strong coupling limit was pointed out in Ref. [27]. While it is possible to calculate transport coefficients in the strong coupling and large N_f limit of QED_3/QCD_3 along the lines of Ref. [14, 15], no such results currently exist in the literature.

- The O(N) model in 2+1 dimensions was conjectured to have a gravity dual in the strong coupling limit, cf. Ref. [11]. There are encouraging works on reconstructing the bulk geometry from the boundary field theory in Refs. [12, 13].

- Higher dimensional O(N) models are not thought to be perturbatively renormalizable. However, O(N) models in odd dimensions (in particular in five dimensions) may be non-perturbatively renormalizable [28]. This has led to recent studies of O(N) models in odd dimensions, e.g. in Refs. [29–31].

Homework Problems Lecture 1

1. Calculate E_0 in one-dimensional quantum mechanics with Hamiltonian (1) using perturbation theory $\lambda \ll 1$. Compare your result to the numerically obtained result (26) and discuss.

2. Calculate E_0 in N-dimensional quantum mechanics with Hamiltonian (5) to order NNLO (including terms of order N^{-1} in E_0) in a large N expansion. Show that

$$E_0^{\rm NNLO} \simeq -0.1689 N^{-1} \lambda^{\frac{1}{3}}\,. \tag{31}$$

3. Instead of quantum mechanics, now consider quantum field theory in 2+1 dimensions with Euclidean action

$$S_E = \int_0^\beta d\tau \int d^2x \left[\frac{1}{2}\partial_\mu \vec{\phi}\cdot\partial_\mu\vec{\phi} + \frac{\lambda}{N}\left(\vec{\phi}^2\right)^2\right]\,. \tag{32}$$

Using the same techniques as for quantum mechanics, find the expression for the LO large N partition function Z at finite temperature equivalent to (21). Defining the entropy density as $s = \frac{d}{dT}\frac{\ln Z}{\beta V}$, evaluate it at infinite coupling $s_\infty \equiv \lim_{\lambda\to\infty} s$. Show that

$$\frac{s_\infty}{s_{\rm free}} = \frac{4}{5}\,, \tag{33}$$

where $s_{\rm free}$ is the thermal entropy density of N free bosons in 2+1 dimensions.

4. Consider again quantum field theory in 2+1 dimensions with Euclidean action (32). In Fourier space, the propagator for the scalar field ϕ at zero temperature can be parametrized as $G(k) = (k^2)^{-1+\frac{\eta}{2}}$ with η the critical exponent. Calculate the first non-vanishing term of the critical exponent in a large N expansion and show that in the strong coupling limit $\lambda \to \infty$

$$\eta = \frac{8}{3N\pi^2} + \mathcal{O}\left(N^{-2}\right)\,. \tag{34}$$

LECTURE 2: NON-RELATIVISTIC NEUTRONS

Consider the QCD phase diagram, sketched in Fig. 2. Most regions of this phase diagram are hard to access using first-principles QCD calculations, and this is especially true for the region of low temperature and finite baryon density relevant for neutron stars.

I only know of one exception to this statement: effective field theory (EFT).

EFTs are bona-fide field theories that are constructed out of the known symmetries, relevant degrees of freedoms, and a derivative expansion. Some well-known EFTs are chiral effective theory [32] and relativistic fluid dynamics [33].

EFTs have distinct advantages: they correspond to controlled, improvable first-principles calculations, and are often possible in regions where other approaches fail.

The main disadvantage to EFTs is that they invariably contain a finite number of free parameters that need to be fixed by other means, e.g. from experiment.

In the following, I will consider a particular EFT for QCD at low temperature and finite baryon density relevant for neutron stars: pionless EFT, denotes as $\not\pi$ EFT [34].

To build $\not\pi$ EFT, consider the energy scales relevant for low-temperature QCD: the nucleon masses $M \sim 940$ MeV, the pion masses $m_\pi \sim 135$ MeV and the deuteron binding energy $B \sim 2.2$ MeV. If we aim at a theory that only captures the deuteron, we need to include the nucleons, but can neglect excitations with energies much less than the pion mass. Hence we are driven to consider a theory of non-relativistic nucleons with kinetic energy $E_{\rm kin} \ll m_\pi$, so pions are not needed in this description, hence the name.

$\not\pi$ EFT for interacting nucleons has been fleshed out in a series of papers [36–38], but for this lecture I want to focus on an even simpler version of $\not\pi$ EFT: pure neutron $\not\pi$ EFT. While inappropriate for describing nuclei such as the deuteron, this theory would be relevant for a very neutron rich environment. Can you think of one?

To build the EFT, we note that neutrons are fermions, and since we consider non-relativistic neutrons, we describe them as two-component spinors $\psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}$. Only neutrons, no anti-neutrons are included, because the energy scales for pair-production are much above the relevant scale of the theory. Free non-relativistic neutrons obey the Schrödinger

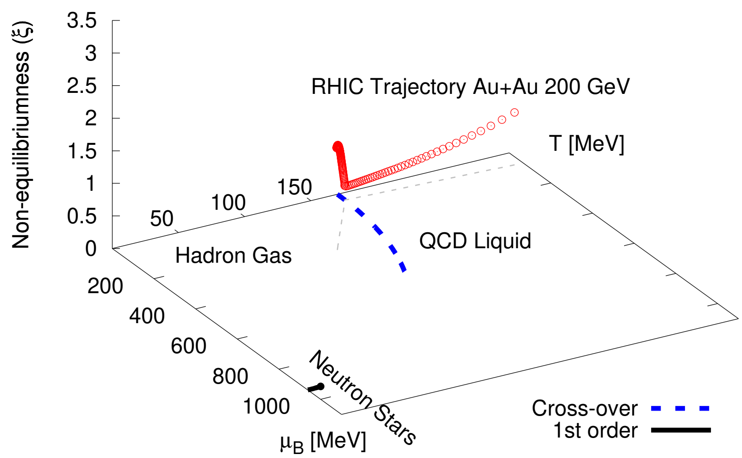

FIG. 2. Sketch of what we know about the QCD phase diagram, adapted from Ref. [35]. Axis are the equilibrium temperature T, baryon chemical potential μ_B and the parameter ξ corresponding to deviations from equilibrium. Deconfinement cross-over and liquid-gas first order phase transitions are marked. Areas relevant to neutron stars and relativistic heavy ion collisions – such as gold ion collisions at center-of-mass energies of $\sqrt{s} = 200$ GeV per nucleon pair at the Relativistic Heavy Ion Collider (RHIC) as well as their projection on the equilibrium T, μ_B plane (grey dashed lines) – are indicated. See original reference for details.

equation, which can be turned into a Lagrangian density:

$$\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\vec{\nabla}^2}{2M} \right) \psi \,. \tag{35}$$

Field theorists accustomed to relativistic fields will find that this form also arises from taking the non-relativistic limit of the free Dirac fermion Lagrange density $\bar{\Psi} i \partial\!\!\!/ \Psi$.

The above Lagrangian describes free (non-interacting) non-relavistic neutrons. This is boring. In order to have something of interest, we need to include interactions. In an EFT, one writes down all possible interactions allowed by symmetry, such as two-neutron, three-neutron, four-neutron, etc. interactions. All of these come with unknown coefficients that need to be fixed by other means, e.g. experiment. However, the lowest-order interaction is that of a two-neutron singlet "contact term" (no derivatives), such that [34]

$$\mathcal{L}_I = -\frac{C_0}{4} \left(\psi \sigma_y \psi \right)^\dagger \left(\psi \sigma_y \psi \right) \,, \tag{36}$$

where $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ is the second Pauli matrix. As promised, C_0 is a coefficient that needs to be fixed by other means. In the present case, this can be done by calculating the scattering amplitude and comparing to the corresponding scattering amplitude resulting from solving the Schrödinger equation (see appendix B for the explicit matching in the case of bosons). One finds

$$C_0 = \frac{4\pi a_0}{M}\,, \tag{37}$$

where a_0 is the s-wave scattering length for neutrons. Fortunately, the s-wave scattering lengths for neutrons is well known experimentally [39] as

$$a_0 \simeq -18.5 \text{ fm}\,, \tag{38}$$

which together with the known nucleon mass M fixes the parameters of the theory. We are now ready to calculate!

Let's jump right in and write down the grand-canonical partition function for pure-neutron $\not\pi$ EFT with spin-singlet interaction:

$$Z = \int \mathcal{D}\psi e^{-S_E + (\mu_B - M)N}\,, \tag{39}$$

where

$$S_E = \int_0^\beta d\tau \int d^3x \left[\psi^\dagger \left(\partial_\tau - \frac{\vec{\nabla}^2}{2m}\right)\psi + \frac{C_0}{4}\left(\psi\sigma_y\psi\right)^\dagger\left(\psi\sigma_y\psi\right)\right]\,, \tag{40}$$

is the Euclidean action corresponding to analytically continuing the Lagrangian density $\mathcal{L}$ above to Euclidean time $\tau \in [0,\beta]$, μ_B is the baryon chemical potential, and

$$N = \int_0^\beta d\tau \int d^3x \psi^\dagger\psi\,, \tag{41}$$

is the neutron number. Since the baryon chemical potential only appears in the combination $\mu_B - M$, it is useful to denote this 'excess' chemical potential as

$$\mu \equiv \mu_B - M\,. \tag{42}$$

With the theory defined by the grand-canonical partition function (39), obtaining observables such as the pressure $p \equiv \frac{\ln Z}{\beta V}$, the baryon density $n \equiv \frac{\partial}{\partial\mu}p$ and the excess energy density (equal to energy density minus nucleon rest mass) $\epsilon = \mu n - p$ is "'just" a matter of solving the many-body partition function.

However, even for this admittedly simple EFT, exact solutions for Z are hard because of the 4-fermi interaction term in (39):

$$\left(\psi\sigma_y\psi\right)^\dagger\left(\psi\sigma_y\psi\right) = 4\left(\psi_\downarrow\psi_\uparrow\right)^\dagger\left(\psi_\downarrow\psi_\uparrow\right)\,. \tag{43}$$

But we learned in lecture 1 how to deal with such quartic interactions in a large N framework! Let's make use of this knowledge!

Instead of a single neutron species, consider N neutron species $\psi \to \psi_f = (\psi_1, \psi_2, \ldots, \psi_N)$. You may think of these either as fictitious extra particles, or for $N = 2$, as a very crude way of including the proton into the description. In either case, we will use $\frac{1}{N} \ll 1$ as a small expansion parameter unrelated to any other parameter in the theory, which allows us to perform non-perturbative calculations of the theory.

In complete analogy to the case of quantum mechanics studied in lecture 1, we generalize the interaction term to the N-component case as

$$C_0\left(\psi_\downarrow\psi_\uparrow\right)^\dagger\left(\psi_\downarrow\psi_\uparrow\right) \to \frac{C_0}{N}\left(\psi_{\downarrow,f}\psi_{\uparrow,f}\right)^\dagger\left(\psi_{\downarrow,g}\psi_{\uparrow,g}\right)\,, \tag{44}$$

where the "flavor" indices f, g run from 1 to N and Einstein sum convention is used to suppress the summation symbols.

Next, introduce the complex auxiliary field ζ through inserting the identity

$$1 = \int \mathcal{D}\zeta e^{N\int_x \frac{\zeta^*\zeta}{C_0}} \tag{45}$$

(note that this makes sense because $C_0 \propto a_0$ is negative for neutrons, cf. (38)). Now shifting

$$\zeta \to \zeta - \frac{iC_0}{N}\psi_{\downarrow,f}\psi_{\uparrow,f} \tag{46}$$

then leads to the auxiliary-field formulation for N-component pure-neutron $\not\pi$ EFT:

$$Z = \int \mathcal{D}\psi\mathcal{D}\zeta e^{-\int_x\left[\psi_f^\dagger\left(\partial_\tau - \frac{\vec{\nabla}^2}{2M} - \mu\right)\psi_f + i\zeta^*\psi_{\downarrow,f}\psi_{\uparrow,f} - i\zeta\psi^\dagger_{\uparrow,f}\psi^\dagger_{\downarrow,f} - \frac{N\zeta\zeta^*}{C_0}\right]}\,. \tag{47}$$

In this form, all the fermions enter as bilinears into the path integral action. They can be compactly brought into the form

$$\Psi_f^\dagger G^{-1}\Psi_f\,, \tag{48}$$

with the two-component composite (Nambu-Gorkov) spinor

$$\Psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow^\dagger \end{pmatrix}\,, \tag{49}$$

and the inverse propagator in matrix form

$$G^{-1} = \begin{pmatrix} \partial_\tau - \frac{\vec{\nabla}^2}{2M} - \mu & -i\zeta \\ i\zeta^* & \partial_\tau + \frac{\vec{\nabla}^2}{2M} + \mu \end{pmatrix} . \tag{50}$$

Since the fermions enter the action quadratically, they can be integrated out:

$$Z = \int \mathcal{D}\zeta e^{N \ln \det G^{-1} + \frac{N}{C_0} \int_x \zeta^* \zeta} . \tag{51}$$

So far, everything has been exact. However, in the large N limit, the remaining path integral simplifies considerably because of the same reason outlined in quantum mechanics after Eq. (14): the leading large N saddle corresponds to constant ζ, or equivalently the zero mode ζ_0. In the literature, it is customary to denote $i\zeta_0^* \equiv \Delta$, and (with hindsight) assume Δ to be real. Then the large N partition function becomes

$$\lim_{N \gg 1} Z = \int d\Delta e^{N\beta V p(T,\Delta)} , \tag{52}$$

with

$$p(T,\Delta) = \frac{\Delta^2}{C_0} + T \sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln\left[\tilde{\omega}_n^2 + (\epsilon_k - \mu)^2 + \Delta^2\right] , \tag{53}$$

where $\tilde{\omega}_n = \pi T (2n+1)$ the fermionic Matsubara frequencies and $\epsilon_k = \frac{\mathbf{k}^2}{2M}$ the non-relativistic kinetic energy.

In the zero temperature limit, the thermal sum in (53) becomes an integral which is straightforward to solve:

$$p(0,\Delta) = \frac{\Delta^2}{C_0} + \int \frac{d^3\mathbf{k}}{(2\pi)^3} \sqrt{(\epsilon_k - \mu)^2 + \Delta^2} . \tag{54}$$

The remaining integral over momenta $\mathbf{k}$ can likewise be calculated in closed form when using dimensional regularization. Expanding the square root and using the identities from Ref. [40], one finds [19]

$$p(0,\Delta) = \frac{\Delta^2}{C_0} + \frac{2\mu}{5} \frac{(2M\mu)^{\frac{3}{2}}}{3\pi^2} g\left(\frac{\mu}{\sqrt{\mu^2 + \Delta^2}}\right) , \tag{55}$$

where the function $g(y) = y^{-\frac{5}{2}}\left[(4y^2 - 3)E\left(\frac{1+y}{2}\right) + \frac{3+y-4y^2}{2} K\left(\frac{1+y}{2}\right)\right]$ is expressed using E, K, the complete elliptic integrals of the first and second kind, respectively.

To leading order in the large N and low temperature limit, the grand-canonical path integral is then given as

$$\lim_{\beta \gg 1} \lim_{N \gg 1} Z = e^{N\beta V p(0,\Delta)} , \tag{56}$$

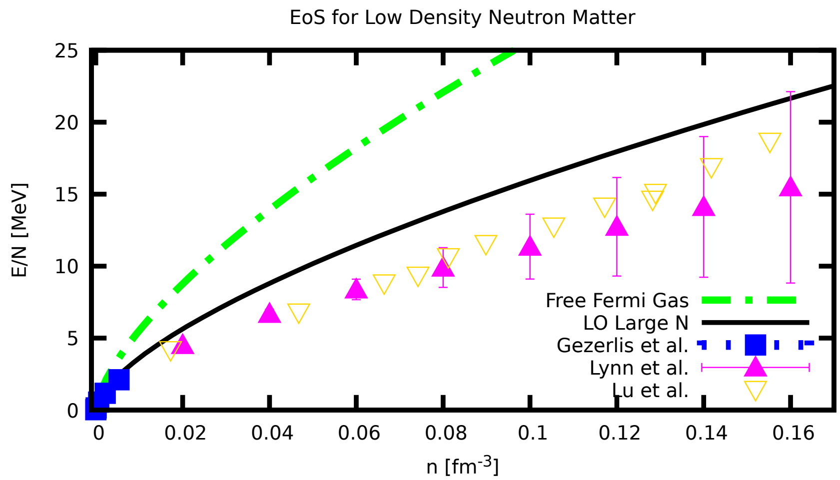

FIG. 3. Energy per particle for pure neutron matter as a function of density. Shown are results for the free Fermi gas $\frac{E}{N} = \frac{3\mu}{5}$, the LO large N result (59) and Monte Carlo results from three different groups: Gezerlis et al. [41, 42], Lynn et al. [43, 44] and Lu et al. [45].

with Δ being the solution of the saddle point condition

$$0 = \frac{dp(0,\Delta)}{d\Delta}\,. \tag{57}$$

We have a solution!

Now let's see if the solution is any good. We need the neutron density, which we can calculate as

$$n = \frac{dp(0,\Delta)}{d\mu} = \frac{(2M\mu)^{\frac{3}{2}}}{3\pi^2} g(y)\,, \tag{58}$$

where $y = \frac{\mu}{\sqrt{\mu^2+\Delta^2}}$ and we have used the saddle point condition (57) to simplify the expression. Using n and the zero-temperature pressure $p(0,\Delta)$ we can construct the energy density ϵ, and in particular the energy per particle

$$\frac{E}{N} = \frac{\epsilon}{n} = \mu\left(\frac{3}{5} + \frac{3\pi\Delta^2}{8\mu^2 g(y)\sqrt{2M\mu a_s^2}}\right)\,. \tag{59}$$

For a given value of μ, we can numerically calculate the value of Δ from solving the saddle point condition (57). With μ, Δ, we can then calculate n and $\frac{E}{N}$. How do our leading order large N results compare to other methods?

The relevant comparison is shown in Fig. 3, where the lO large N result for the energy per particle for pure neutron matter is compared to the results from three other groups.

One finds that the LO large N result for $\frac{E}{N}$ is about 30 percent higher than the considerably more complex calculations from Refs. [41–45].

The 30 percent difference is surprisingly similar to what we found when comparing the LO large N result to the N=1 ground state energy for the quartic oscillator in quantum mechanics in the first lecture. In that lecture, we found that going to NLO in the large N expansion was straightforward, and just involved a Gaussian integral, yet reduced the difference with the N=1 value by a factor of two.

Not surprisingly, calculating the NLO large N correction to the grand-canonical partition function can be done with similar ease here [46]. What is surprising, though, is that the equivalent NLO large N result for Fig. 3 is not available in the literature!

Maybe you can help?

Guide to further reading

- Transport coefficients can be calculated for the pure neutron matter theory in the large N limit for any coupling/density. Currently, only the LO large N result for so-called thermodynamic transport coefficients are known [19, 47], but calculating shear viscosity along the lines of Ref. [14, 15] is doable.
- Calculating the zero temperature limit of the grand canonical partition function to NLO in the large N limit exhibits a concrete example of non-commutative limits that was uncovered in Ref. [48].

Homework Problems Lecture 2

1. In the literature, the strong coupling limit $a_0 \to -\infty$ is called the 'Unitary Fermi Gas' limit, whereas the weak coupling limit $a_0 \to 0$ is called the 'Free Fermi Gas' limit. Calculate the large N 'superfluid gap' Δ from solving (57) in both of these limits and show that

$$\lim_{a_0\to-\infty} \Delta \simeq 1.1622 \times \mu \,, \quad \lim_{a_0\to 0} \Delta \simeq e^{-\frac{\pi}{\sqrt{8M\mu a_0^2}} -2+3\ln 2} \times \mu \,. \tag{60}$$

2. In the Unitary Fermi Gas limit, the energy density can be expressed as

$$\lim_{a_0\to-\infty} \epsilon = \frac{3}{5} n^{\frac{5}{3}} \frac{(3\pi^2)^{\frac{2}{3}}}{2M} \times \xi \,, \tag{61}$$

with ξ a pure number (the 'Bertsch parameter'). Calculate ξ in the large N approximation and show that

$$\lim_{N\gg 1} \xi \simeq 0.59\,. \tag{62}$$

LECTURE 3: NEGATIVE COUPLING AND TRIVIALITY

In lecture 1, we considered large N techniques for N-dimensional quantum mechanics, and found that the large N calculations gave improvable and reasonably accurate results for finite N, including down to N=1.

In lecture 2, we considered large N techniques for a four-dimensional (non-relativistic) quantum field theory of interacting neutrons, and we found that also here large N gave reasonable results even for N=1.

There are plenty of other examples I could cite about successes of large N calculations applied to observables at finite (and sometimes quite small) N.

It seems the method is sound and the math is trustworthy.

So how about we trust the math, even if its implications are non-intuitive?

Let's see where this 'trust the math' axiom leads in the case of four-dimensional scalar field theory.

To be concrete, let's consider N-component scalars $\vec{\phi} = (\phi_1, \phi_2, \ldots, \phi_N)$ interacting via a quartic coupling with Euclidean action

$$S_E = \int d^4x \left[\frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial_\mu\vec{\phi} + \frac{\lambda}{N}\left(\vec{\phi}^2\right)^2\right]\,. \tag{63}$$

This theory referred to as the O(N) model in the literature.

If you want to have a concrete physical system in mind, consider the Standard Model Higgs field is a two-component complex scalar $\Phi = \begin{pmatrix}\phi_1 + i\phi_2\\ \phi_3 + i\phi_4\end{pmatrix}$, which is equivalent to considering the O(N) model for $N = 4$. Since $N = 4$ is not that small, we might even expect our large N techniques to be quantitatively better in describing the Higgs sector than for instance the pure neutron case in lecture 2.

The Euclidean action then defines the partition function for the theory in terms of a path integral $Z = \int \mathcal{D}\vec{\phi} e^{-S_E}$. Using exactly the same steps as in lecture 1, we can introduce an auxiliary field ζ to make the action quadratic in the field $\vec{\phi}$, so the path integral over $\vec{\phi}$ can be done in closed form:

$$Z = \int \mathcal{D}\vec{\phi}\mathcal{D}\zeta e^{-\int_x \frac{1}{2}\vec{\phi}[-\partial_\mu\partial_\mu + 2i\zeta]\vec{\phi} - \frac{N}{4\lambda}\int d^4x\zeta^2} = \int \mathcal{D}\zeta e^{-\frac{N}{2}\mathrm{Tr}\ln[-\partial_\mu\partial_\mu + 2i\zeta] - \frac{N}{4\lambda}\int d^4x\zeta^2}\,. \tag{64}$$

Also, again just as in the case of quantum mechanics, when splitting the auxiliary field into a global zero mode ζ_0 and fluctuations ζ', the path integral over fluctuations does not contribute to the LO large N partition function, hence

$$\lim_{N\gg 1} Z = \int d\zeta_0 e^{-\frac{N}{2}\mathrm{Tr}\ln[-\partial_\mu\partial_\mu+2i\zeta_0]-\frac{N}{4\lambda}\int d^4x\zeta_0^2}, \tag{65}$$

where the quantum field theory partition function is now given in terms of a single integral (and not a path integral!).

Because ζ_0 does not depend on position, it is a constant as far as the operator $[-\partial_\mu\partial_\mu + 2i\zeta_0]$ is concerned. Hence we can treat $2i\zeta_0 = m^2$ as a constant mass term and directly evaluate the trace of the operator, e.g. via dimensional regularization [21, Eq. 2.72]

$$\frac{1}{2\mathrm{vol}}\mathrm{Tr}\ln\left[-\partial_\mu\partial_\mu + m^2\right] = \frac{1}{2}\int \frac{d^{4-2\varepsilon}k}{(2\pi)^{4-2\varepsilon}} \ln\left[k^2+m^2\right] = -\frac{m^4}{64\pi^2}\left(\frac{1}{\varepsilon} + \ln\frac{\bar{\mu}^2 e^{\frac{3}{2}}}{m^2}\right), \tag{66}$$

where vol $= \int d^4x$ denotes the spacetime volume and $\bar{\mu}$ is the $\overline{\mathrm{MS}}$ renormalization scale. (Of course, using any other consistent regularization scheme for the ultraviolet divergencies of the integral gives equivalent results, see e.g. Ref. [49] for cut-off regularization.)

The large N partition function then is given by

$$\lim_{N\gg 1} Z = \int d\zeta_0 e^{-\mathrm{vol}\times\frac{N\zeta_0^2}{4}\left[\frac{1}{\lambda}+\frac{1}{4\pi^2\varepsilon}+\frac{1}{4\pi^2}\ln\frac{\bar{\mu}^2 e^{\frac{3}{2}}}{2i\zeta_0}\right]}. \tag{67}$$

After regularization, the expression for the partition function still has an uncanceled UV divergence for $\varepsilon \to 0$. This divergence can be canceled by introducing a suitable coupling-constant counterterm to the bare coupling λ in a renormalization procedure. For the case at hand, we can non-perturbatively renormalize the theory by introducing the renormalized (running) coupling λ_R as

$$\frac{1}{\lambda} + \frac{1}{4\pi^2\varepsilon} \equiv \frac{1}{\lambda_R(\bar{\mu})}. \tag{68}$$

Note that this renormalization procedure is non-perturbative because λ contains an infinite number of terms with powers of λ_R. Also note that this renormalization procedure does not recover the LO perturbative renormalization when expanded in powers of the coupling, because the LO large N theory does not contain the full LO perturbative contribution (actually only one third of it, whereas the remaining 2/3 originate at NLO in the large N limit, cf. R1/R2 level resummation in cf. Refs. [50]).

Given the renormalization (68), one obtains the running coupling as

$$\lambda_R(\bar{\mu}) = \frac{4\pi^2}{\ln\frac{\Lambda_{LP}^2}{\bar{\mu}^2}}, \tag{69}$$

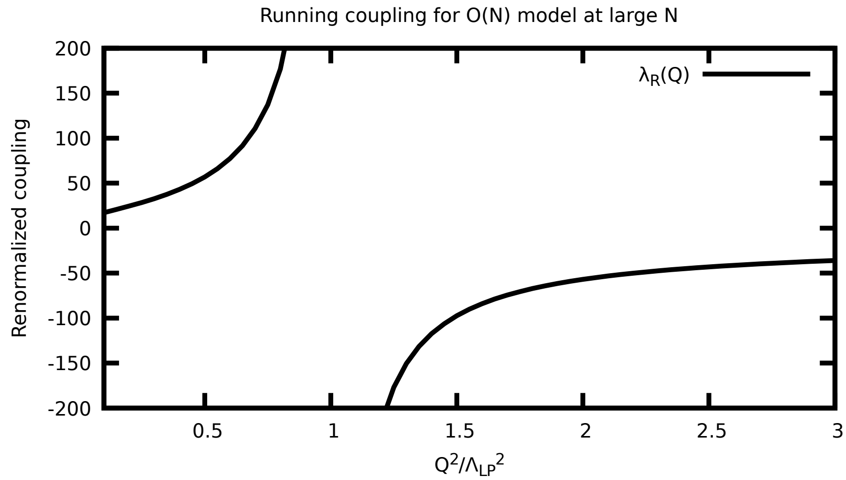

FIG. 4. Exact large N running coupling $\lambda_R(Q)$ from Eq. (69). Figure from Ref. [49]. See text for details.

where Λ_{LP} is an emergent scale of the theory. It is defined by the value of the scale $\bar{\mu}$ at which λ_R diverges, e.g.

$$\lambda_R(\Lambda_{LP}) = \infty \,. \tag{70}$$

The scale Λ_{LP} is commonly referred to as the 'Landau pole' of the theory, even though it is clear from (69) that λ_R does not have a pole, but rather a logarithmic singularity at $\bar{\mu} = \Lambda_{LP}$.

A plot of the running coupling is shown in Fig. 4. In particular, note that in the UV limit, the running coupling approaches zero from below:

$$\lim_{\bar{\mu}\to\infty} \lambda_R(\bar{\mu}) = 0^- \,. \tag{71}$$

It is straightforward to calculate the β function for this theory as

$$\beta \equiv \frac{d\lambda_R(\bar{\mu})}{d\ln\bar{\mu}^2} = \frac{4\pi^2}{\ln^2\frac{\Lambda_{LP}^2}{\bar{\mu}^2}} = \frac{\lambda_R^2(\bar{\mu})}{4\pi^2} \geq 0 \quad \forall\ \lambda_R \in \mathbb{R}\,. \tag{72}$$

Obviously, the β function is positive, consistent with an ever-increasing running coupling, cf. Fig. 4.

Before trying to make sense of these results, let me stress that the running coupling (69), its negative value in the UV (71, the Landau pole (70) and the β function (72) are exact results in the large N limit. In particular, their validity is not limited to a weak coupling domain, because we did not use a weak coupling expansion in obtaining them.

Let's review the prevailing interpretation for these findings first, before heeding my advice of 'trust the math, even if it's non-intuitive'.

By far the majority opinion of theoretical physicists is that a negative coupling, a positive β function and/or a Landau pole are all fatal flaws of a continuum interacting quantum field theory. Reviewing these one-by-one, it is possible to understand how the verdict 'fatal' arises in each case. However, in the interest of keeping the lecture to its allotted time frame, I relegate this to the guide to further reading at the end.

For now, let's ignore 'fatal flaw' majority opinion, trust the math, and see where it leads us.

So instead of giving up, we can ask the question: is there actually something wrong with the theory?

In order to answer this question, we better calculate observables, so let's do that.

The first observable we can look at is the mass of the field $\vec{\phi}$, which for $N=4$ would be nothing else but the Higgs boson mass. The large N Euclidean Green's function for $\vec{\phi}$ is given by $[-\partial_\mu\partial_\mu+2i\zeta_0]$, so at large N, the vector mass is determined through

$$m^2=2i\zeta_0\,, \tag{73}$$

where ζ_0 is the location of the saddle point. After renormalization, the large N partition function (67) is given by

$$\lim_{N\gg 1} Z=\int d\zeta_0 e^{-\mathrm{vol}\times\frac{N\zeta_0^2}{4}\left[\frac{1}{4\pi^2}\ln\frac{\Lambda_{LP}^2 e^{\frac{3}{2}}}{2i\zeta_0}\right]}\,, \tag{74}$$

from which the saddle point condition becomes[1]

$$\frac{\zeta_0}{8\pi^2}\ln\frac{\Lambda_{LP}^2 e^1}{2i\zeta_0}=0\,. \tag{75}$$

This saddle point condition implies two solutions for the vector mass squared:

$$m^2=0\,,\quad m^2=e\Lambda_{LP}^2\,. \tag{76}$$

The first of these corresponds to a vanishing vector mass expectation value, which corresponds to the prevailing assumption for the perturbative vacuum for the theory defined

[1] As an aside, note that any physical observable $\mathcal{O}$ must be renormalization-scale independent, $\frac{d\mathcal{O}}{d\bar{\mu}}=0$. It is gratifying to find that both the large N partition function (74) and the saddle point condition for the vector mass are explicitly renormalization scale independent.

by (63). In the perturbative setup of the Electroweak sector of the Standard Model, one introduces a 'negative mass squared' term $-m^2\vec{\phi}^2$ into the action in order to get spontaneous symmetry breaking, and one obtains a non-vanishing vector mass only after this construction.

By contrast, the second solution (76) corresponds to a non-perturbative vacuum where the vector mass is non-vanishing even though O(N) symmetry remains unbroken. This is clearly different from the Standard Model, already because the mass does not get put in 'by hand' through the addition of a tachyon to the theory. In this situation, the Higgs mass becomes a prediction of the theory, not a parameter.

But which of the two solutions (76) is the right one?

There is an easy way to decide this question, and hinges on calculating a second observable, the free energy F of the theory. Namely, each of the two solutions will lead to a different value of the large N partition function, and hence the large N free energy. The correct solution to (76) then is the one that has the lower free energy.

Let's calculate: in the two cases, we get for the large N free energy

$$F_{m^2=0} = 0\,, \quad F_{m^2=e\Lambda_{LP}^2} = -\mathrm{vol}\times\frac{Ne^2\Lambda_{LP}^4}{128\pi^2}\,. \tag{77}$$

Clearly, the non-perturbative solution has the lower free energy, and hence the perturbative vacuum must be unstable.

We thus find for the two observables (vector mass and free energy density) in the O(N) model:

$$m = \sqrt{e}\Lambda_{LP}\,, \quad \frac{F}{\mathrm{vol}} = \frac{Ne^2\Lambda_{LP}^4}{128\pi^2}\,. \tag{78}$$

Both of these are finite, non-vanishing and renormalization scale independent, despite the decidedly weird properties of the theory (71), (70), (72). Even better, they are parameter-free predictions for the Higgs mass and Higgs free energy in the case of N=4!

How's that for a theory that doesn't exist/is trivial/is fatally flawed?

Maybe trusting the math is not such a crazy suggestion after all.

Could it be that the 'fatal flaw' reveals itself only when we look at scattering?

So let's calculate scattering cross-sections at large N. To this end, consider the connected, amputated four-point function

$$\mathcal{M} = -\langle\phi_a(x_1)\phi_b(x_2)\phi_c(x_3)\phi_d(x_4)\rangle_{\mathrm{conn.,amp.}} \tag{79}$$

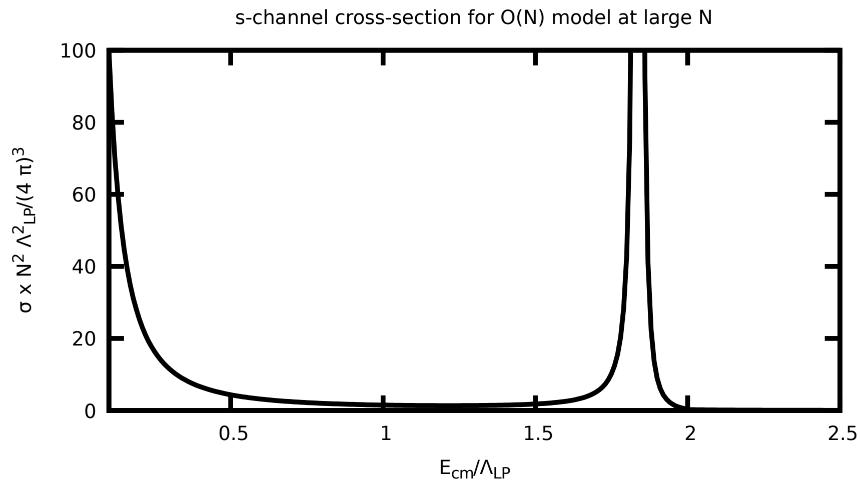

FIG. 5. s-channel cross section for scattering in the 4d O(N) model to LO in large N, reproduced from Ref. [51].

at large N. From (64), this becomes for the s-channel amplitude in momentum space

$$\mathcal{M}(k) = D(k)\,, \tag{80}$$

where $D(x-y) = \langle \zeta(x)\zeta(y)\rangle$ is the auxiliary field propagator. The auxiliary field propagator can be calculated by again integrating out the vector field $\vec{\phi}$, and then expanding the action to second order in the fluctuation field ζ'. In complete analogy to Eq. (29), one finds

$$D(k) = \frac{1}{\frac{N}{8\lambda} + N\Pi(k)}\,, \quad \Pi(k) = \frac{1}{2}\int \frac{d^{4-2\varepsilon}p}{(2\pi)^{4-2\varepsilon}} \frac{1}{p^2+m^2}\frac{1}{(p+k)^2+m^2}\,, \tag{81}$$

where m is the large N vector mass for the dominant saddle (78). The momentum integral can be done in closed form in dimensional regularization, the UV divergence for $\varepsilon \to 0$ cancels when using the renormalization condition (68). One finds [51]

$$D^{-1}(k) = \frac{N}{32\pi^2}\left[\ln\frac{\Lambda_{LP}^2 e^2}{m^2} - 2\sqrt{1+\frac{4m^2}{k^2}}\,\mathrm{atanh}\sqrt{\frac{k^2}{k^2+4m^2}}\right]\,. \tag{82}$$

The s-channel scattering amplitude is then simply found by analytically continuing $D(k)$ to Minkowski space as $k^2 \to -E^2 + \mathbf{k}^2 - \mathrm{sgn}(E)i0^+$. A plot of the s-channel cross section is shown in Fig. 5. Note again the explicit independence of $\mathcal{M}$ from the renormalization scale $\bar{\mu}$, as expected for a physical observable.

No pathologies are observed for scattering in the LO large N limit. The only curious finding is the presence of a stable bound state with a mass of $m_2 \simeq 1.84m$.

Where are all the pathologies hiding that everyone is so scared about?

I do not know....

Guide to further reading

- Obtaining a non-vanishing Higgs mass without introducing a negative mass squared term into the theory was considered a long time ago by Coleman and Weinberg in a famous paper on radiative corrections [52]. The prediction for the Higgs mass in the so-called Coleman-Weinberg mechanism came out wrong, but that may be partly a consequence of doing the calculation perturbatively and throwing away terms 'not under perturbative control'.

- For many people, the Landau pole is a showstopper because perturbation theory breaks down, which on the other hand is not an issue if using techniques not limited to weak coupling (such as large N). Other people co-mingle the Landau pole with Landau's ghost, a tachyonic excitation that appears in perturbative QED. However, as discussed in Ref. [53], the large N O(N) model in four dimensions does not have a Landau ghost (even though it has a Landau pole), in contradistinction to perturbative QED.

- The original studies of the O(N) model in four dimensions date back to the 1970s [2, 3, 54], with Ref. [2, 3] pointing out that the tachyon (Landau's ghost) found in Ref. [54] simply was a consequence of expanding around the wrong vacuum, namely the $m = 0$ solution in (76).

- There are mathematical proofs of triviality of scalar field theories in four dimensions, in particular by Aizenman and Duminil-Copin in Ref. [55]. Note that these proofs are limited to $N \leq 2$ and positive bare coupling, so they do not apply to the O(N) model in the large N limit. Using analytic continuation of the path integral contour, it is possible (but numerically challenging) to study negative coupling field theory on the lattice [49].

- The proof by Coleman and Gross [56] that only non-abelian gauge theories in four dimensions can have asymptotic freedom rests on the same assumption as quantum triviality, namely that the bare coupling is positive.

- Scalar field theory with negative coupling was considered a long time ago by Symanzik [57]. For quantum mechanics, there is a whole literature surrounding negative coupling Hamiltonians which was opened up by Bender and Böttcher in Ref. [58]. In quantum mechanics, strong numerical evidence for the equivalence of so-called PT-symmetric spectra and contour-deformed partition functions can be obtained [59].

Homework Problems Lecture 3

1. Consider the 4d O(N) model with Euclidean action (63) at finite temperature. Calculate the finite-temperature corrections to the saddle point condition (75) and show that real-valued solutions for $2i\zeta_0 = m^2$ of this equation cease to exist for

$$T > T_c \simeq 0.616\Lambda_{LP}\,. \tag{83}$$

ACKNOWLEDGMENTS

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Appendix A: Numerically calculating the spectrum of quartic oscillator in multi-dimensional quantum mechanics

In this appendix, I review a simple numerical scheme to solve for the eigenvalue spectrum (really mostly the ground-state energy E_0) of the Hamiltonian operator for quantum mechanics in N dimensions,

$$\mathcal{H} = \frac{\vec{p}^2}{2} + \frac{\lambda}{N}\left(\vec{x}^2\right)^2\,. \tag{A1}$$

I assume a discrete eigenspectrum for the Hamiltonian $\mathcal{H}|n\rangle = E_n|n\rangle$. Using spherical coordinates, the angular part of the Laplace operator may be separated off whereas the radial part of the Schrödinger equation becomes

$$-\psi''(r) - \frac{N-1}{r}\psi'(r) + \frac{l(l+N-2)}{r^2}\psi(r) + \frac{2\lambda}{N}r^4\psi(r) - 2E\psi(r) = 0\,, \tag{A2}$$

with l the angular quantum number using the eigenvalues of the Laplacian on a $N-1$-dimensional sphere [31, Eq. (3.3)]. The boundary condition at $r=0$ for the wave function

N	1	2	3	4	5	6	7	8	9	10
$n=3$	0.209987	0.276277	0.500798	0.847447	1.30688	1.87549	2.55159	3.33429	4.22307	5.21761
$n=7$	0.781176	0.591071	0.472416	0.429663	0.438973	0.486257	0.564257	0.669113	0.79867	0.951676
$n=11$	0.657241	0.602466	0.583693	0.544103	0.498688	0.469752	0.461883	0.47327	0.501472	0.544575
$n=15$	0.668003	0.581889	0.552725	0.543857	0.536674	0.520139	0.498165	0.480726	0.473021	0.476145
$n=19$	0.668383	0.586494	0.551802	0.533517	0.524978	0.521291	0.516602	0.507002	0.494293	0.483397
$n=23$	0.667887	0.586368	0.553458	0.534786	0.522767	0.51547	0.511692	0.50933	0.505614	0.498988
$n=27$	0.667991	0.586166	0.553281	0.535322	0.523688	0.5154	0.509576	0.505916	0.503775	0.501855
$n=31$	0.66799	0.586204	0.553199	0.535197	0.523837	0.515885	0.509889	0.505314	0.502048	0.499932

TABLE I. Estimates for the spectral gap $\frac{E_0}{\lambda^{\frac{1}{3}}N}$ for various values of N resulting from solving $c_n = 0$ for different approximation levels n. One should note that results stabilize as $n \to \infty$ as well as for $N \to \infty$.

is

$$\lim_{r\to 0} r\psi(r) = 0\,, \tag{A3}$$

because otherwise $\vec{\nabla}^2\left(\frac{1}{r}\right) = -4\pi\delta(\vec{x})$ is not a solution to the Schrödinger equation. Rescaling of coordinates and energy values as $r = (2\lambda)^{-\frac{1}{6}}\,\hat{r}$, $E = (2\lambda)^{\frac{1}{3}}\,\hat{E}$, and rescaling ψ as $\psi(\hat{r}) = \frac{u(\hat{r})}{\sqrt{\hat{r}^{N-1}}}$, the Schrödinger equation becomes

$$-u''(\hat{r}) + \frac{4l(l+N-2)+(N-1)(N-3)}{4\hat{r}^2}u(\hat{r}) + \frac{\hat{r}^4}{N}u(\hat{r}) - 2\hat{E}u(r) = 0\,. \tag{A4}$$

For large $\hat{r}$, the $\hat{r}^4$ term in the potential dominates, so we choose a bounded wave-function by setting $u(\hat{r}) = e^{-\frac{\hat{r}^3}{3\sqrt{N}}}v(\hat{r})$, with $v(\hat{r})$ fulfilling

$$-v''(\hat{r}) + \frac{2\hat{r}^2}{\sqrt{N}}v'(\hat{r}) + \left[\frac{4l(l+N-2)+(N-1)(N-3)}{4\hat{r}^2} + \frac{2\hat{r}}{\sqrt{N}} - 2\hat{E}\right]v(\hat{r}) = 0\,. \tag{A5}$$

The spectral gap is given by setting the angular quantum number to zero, $l = 0$. It is then convenient to compactify the interval $\hat{r} \in [0,\infty)$ by introducing

$$\hat{r} = \frac{y}{1-y}\,, \quad y \in [0,1)\,, \quad v(\hat{r}) = w(y) \tag{A6}$$

and subsequently solving the Schrödinger equation by expanding $w(y)$ in a power series in y. However, because of the boundary condition at $r = 0$, the series expansion must be taken as

$$w(y) = y^{\frac{N-1}{2}}\sum_{n=0}^{\infty} c_n y^n\,. \tag{A7}$$

The resulting recursion relation for the coefficients c_n is somewhat unenlightening. For the first few coefficients we find

$$c_1 = c_0 \frac{(N-1)}{2}\,,$$
$$c_2 = c_0 \frac{N^3 - N - 8\hat{E}}{8N}\,,$$
$$c_3 = c_0 \frac{N^4 + 3N^3 - N^2 - 3N + 16\sqrt{N} - 24\hat{E}(3+N)}{48N}\,. \tag{A8}$$

A simple yet effective way to obtain the spectrum $\hat{E}$ is by demanding that $c_n = 0$ for sufficiently large n. For instance, setting $c_2 = 0$ leads to the crude estimate $E_0^{(n=2)} = (2\lambda)^{\frac{1}{3}} \frac{N^3-N}{8}$ for the spectral gap. In practice, we find that the larger N, the higher n needs to be in order for the spectral gap from $c_n = 0$ to stabilize. Our results for the spectral gap for different N, n are summarized in table I. One should note that the result for the spectral gap for the one component theory $N = 1$ is consistent with the result from [23, Eq.(IV.16)]

Appendix B: Fixing Parameters in EFTs

For the case of $\not\pi$ EFT, the two-neutron parameter C_0 was identified with the s-wave scattering length in Eq. (37). In this appendix, I derive the corresponding relation for bosons. To this end, consider a simple example theory with an effective Lagrangian density

$$\mathcal{L} = \phi\left(i\partial_t + \frac{\nabla^2}{2M}\right)\phi - \frac{2C_0}{4!}\phi^4\,, \tag{B1}$$

where for illustrative purposes we take ϕ to be a boson. The Lagrangian obeys Galilean invariance, and corresponds to an interacting non-relativistic field theory if C_0 is non-vanishing.

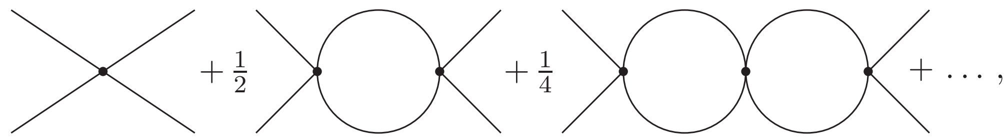
A standard calculation in quantum-field theory is the S-matrix

$$S = 1 + iT\,, \tag{B2}$$

where the interaction part T (also referred to as "T-matrix") may be expressed in terms of Feynman diagrams, see for example section 4.6 in Ref. [60]. Let us consider two-particle scattering: dividing $\mathcal{L}$ into a free field theory part $\mathcal{L}_0 = \phi\left(i\partial_t + \frac{\nabla^2}{2M}\right)\phi$ and an interaction part $\mathcal{L}_I = \mathcal{L} - \mathcal{L}_0$, the T-matrix can be written as

$$T = \langle\phi_1\phi_2|e^{i\int d^4x\mathcal{L}_I}|\phi_A\phi_B\rangle_{\text{amputated, fully connected}}\,, \tag{B3}$$

where time-ordering is implicit and the attributes "amputated" and "fully connected" refer to the class of Feynman diagrams contributing to T. Here ϕ_1, ϕ_2, ϕ_A, ϕ_B are shorthand for the properties of the scattered particles, e.g. incoming particles 1 and 2, while A and B are outgoing particles. Examples for diagrams contributing to T are

$T =$  (B4)

where the symmetry factors for the diagrams have been made explicit and the Feynman rules in momentum space are:

- There is a factor of $-2iC_0$ for every vertex
- Energy and momentum are conserved at each vertex: $(2\pi)^4\delta(E_{\rm in} - E_{\rm out})\delta^3(\mathbf{p}_{\rm in} - \mathbf{p}_{\rm out})$
- Integrate over every loop momentum: $\int \frac{d^4p}{(2\pi)^4}$
- Each propagator is given by $\Delta(E, \mathbf{p})$ with $E, \mathbf{p}$ positive in the direction of momentum flow
- All external lines are set to unity

The propagator $\Delta(E, \mathbf{p})$ in momentum space may be calculated by performing a Fourier transform $\phi(x) = \int \frac{dEd\mathbf{p}}{(2\pi)^4} e^{-iEt+i\mathbf{p}\cdot\mathbf{x}}\phi(E, \mathbf{p})$ in

$$- i \int d^4x\mathcal{L}_0 = -i \int \frac{dEd\mathbf{p}}{(2\pi)^4}|\phi(E, \mathbf{p})|^2 \left(E - \frac{\mathbf{p}^2}{2M}\right) = \frac{dEd\mathbf{p}}{(2\pi)^4}|\phi(E, \mathbf{p})|^2\Delta^{-1}(E, \mathbf{p})\,,$$
$$\Delta(E, \mathbf{p}) = \frac{i}{E - \frac{\mathbf{p}^2}{2M} + i0^+} \equiv \Delta(P)\,, \tag{B5}$$

where we collectively denote four-momenta as $P \equiv (E, \mathbf{p})$.

With these set of Feynman rules, the T-matrix for the above diagrams can be evaluated. One finds that there is an overall factor of momentum conservation, which for the two-particle scattering case at hand implies

$$iT = (2\pi)^4\delta(E_1 + E_2 - E_A - E_B)\delta^3\left(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_A - \mathbf{p}_B\right) i\mathcal{M}\,. \tag{B6}$$

(Note that our normalization convention differs from standard relativistic quantum field theory, cf. Ref. [60], but this difference does not play a role for the results found below).

Here $\mathcal{M}$ is the scattering amplitude as defined in quantum field theory, and for the set of diagrams given in , it is given by

$$i\mathcal{M} = 2(-iC_0) + 2(-iC_0)^2 \int \frac{d^4P}{(2\pi)^4} \Delta(P)\Delta(P_1 + P_2 - P) + \dots . \tag{B7}$$

It is convenient to evaluate $\mathcal{M}$ in the center of mass frame, e.g. $E_1 = E_2 = E_A = E_B = \frac{E}{2}$, $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{k}$, $\mathbf{p}_A = -\mathbf{p}_B = \mathbf{k}'$. Because these particles are on-shell, $E = \frac{\mathbf{k}^2}{M} = \frac{\mathbf{k}'^2}{M}$. With these choices, the relevant loop integral in the scattering amplitude becomes

$$\int \frac{dp_0 d^3\mathbf{p}}{(2\pi)^4} \Delta(p_0, \mathbf{p})\Delta(E - p_0, -\mathbf{p}) = i \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{E - \frac{\mathbf{p}^2}{M} + i0^+} . \tag{B8}$$

The integral is linearly divergent, so a regularization scheme has to be chosen. We will follow Ref. [61] by employing dimensional regularization where $D = 3 \to D = 3 - 2\epsilon$ such that

$$\begin{aligned} \int \frac{d^D\mathbf{p}}{(2\pi)^D} \frac{1}{\mathbf{p}^2 - k^2 - i0^+} \quad &= \quad \frac{1}{(4\pi)^{D/2}} \Gamma(1 - \frac{D}{2})(-k^2)^{D/2-1} , \\ &=_{D\to 3} -\frac{ik}{4\pi} . \end{aligned} \tag{B9}$$

Therefore, the scattering amplitude in the center of mass frame is given by

$$\mathcal{M} = -2C_0 - 2C_0^2 M \frac{ik}{4\pi} + \dots . \tag{B10}$$

Now let us redo the calculation in the context of the Schrödinger equation for two-particle scattering. For two particles with mass M interacting with a two-body potential V, the Hamiltonian is given by

$$\mathcal{H} = \frac{2p^2}{2M} + \hat{V} = \frac{p^2}{M} + V , \tag{B11}$$

where $\mathcal{H}_0 = \frac{p^2}{M}$ is the free Hamiltonian. The free retarded Greens function operator is given by

$$G_0 = \frac{1}{E - \mathcal{H}_0 + i0^+} , \tag{B12}$$

which can be used to write a solution to the full time-independent Schrödinger equation $\mathcal{H}|\phi\rangle = E|\phi\rangle$ as

$$|\phi\rangle = |\mathbf{k}\rangle + G_0 V|\phi\rangle , \tag{B13}$$

where $|\mathbf{k}\rangle$ is the solution to the free Schrödinger equation which we take to be normalized as $\langle x|\mathbf{k}\rangle = e^{i\mathbf{k}\cdot\mathbf{x}}$. The free retarded Greens function may be calculated with standard methods, finding

$$\langle x|G_0|x'\rangle = -\frac{M}{4\pi} \frac{e^{ik|\mathbf{x}-\mathbf{x}'|}}{|\mathbf{x} - \mathbf{x}'|} , \quad \langle \mathbf{p}'|G_0|\mathbf{p}\rangle = \frac{(2\pi)^3\delta^3(\mathbf{p} - \mathbf{p}')}{E - p^2/M + i0^+} , \tag{B14}$$

such that the solution (B13) to the full Schrödinger equation for short range potentials V becomes

$$\phi(x) = e^{i\mathbf{k}\cdot\mathbf{x}} - \frac{M}{4\pi}\frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|}\int d^3x' e^{-i\mathbf{k}'\cdot\mathbf{x}'} V(x')\phi(x')\,, \tag{B15}$$

where $\mathbf{k}' \equiv k\frac{\mathbf{x}}{|\mathbf{x}|}$. This form may be compared to that of a scattered wave with scattering amplitude $f(\mathbf{k},\mathbf{k}')$:

$$\phi(x) = e^{i\mathbf{k}\cdot\mathbf{x}} + \frac{e^{ik|\mathbf{x}|}}{|\mathbf{x}|} f(\mathbf{k},\mathbf{k}')\,, \tag{B16}$$

from which it follows that

$$f(\mathbf{k},\mathbf{k}') = -\frac{M}{4\pi}\langle \mathbf{k}'|V|\phi\rangle\,. \tag{B17}$$

We will find that the scattering amplitude f as used in the Schrödinger equation is related to the scattering amplitude $\mathcal{M}$ calculated in quantum field theory (B7) up to a normalization. For a spherically symmetric scattering potential, the scattering amplitude may be decomposed entirely in partial waves as

$$f(\mathbf{k},\mathbf{k}') = \sum_{l=0}^{\infty} \frac{(2l+1)P_l(\cos\theta)}{k\cot\delta_l(k) - ik}\,, \tag{B18}$$

where $\mathbf{k}\cdot\mathbf{k}' = k^2\cos\theta$ and $\delta_l(k)$ are the energy-dependent scattering phase shifts. For low energy scattering $k \to 0$, the higher partial waves are suppressed and s-wave scattering $l = 0$ dominates the scattering amplitude. One finds that in this case, the form of the s-wave phase shift is universally given by

$$k\cot\delta_0(k) = -\frac{1}{a_0} + \frac{r_0}{2}k^2 + \mathcal{O}(k^3)\,, \tag{B19}$$

where a_0, r_0 are the s-wave scattering length and effective range, respectively. The scattering length and effective range are reliably measured experimentally for a variety of systems.

Using again the result for the scattering amplitude in the Schrödinger calculation given in Eq. (B17), where $|\phi\rangle$ is given by Eq. (B13), we have

$$f = -\frac{M}{4\pi}\left(\langle \mathbf{k}'|V|\mathbf{k}\rangle + \langle \mathbf{k}'|VG_0V|\mathbf{k}\rangle + \ldots\right)\,. \tag{B20}$$

Using $\langle \mathbf{k}'|V|\mathbf{k}\rangle = V(\mathbf{k},\mathbf{k}')$ and the known form of the Green's function (B14) leads to

$$f = -\frac{M}{4\pi}\left(V(\mathbf{k}',\mathbf{k}) + \int \frac{d^3p}{(2\pi)^3} V(\mathbf{k}',\mathbf{p})\frac{1}{E - p^2/M + i0^+}V(\mathbf{p},\mathbf{k}) + \ldots\right)\,. \tag{B21}$$

Comparing (B21) to Eq. (B7) when using (B8), one finds that the structure of the integrals is very similar. In fact one finds that

$$f = \frac{M}{4\pi}\frac{\mathcal{M}}{2} \tag{B22}$$

if $V(\mathbf{p},\mathbf{q}) = C_0$ such that if we focus on low-energy (s-wave) scattering, we have

$$\frac{4\pi}{M}\frac{1}{-\frac{1}{a_0} - ik + \frac{r_0}{2}k^2 + \dots} = -C_0 + C_0^2 M \frac{ik}{4\pi} + \dots\,, \tag{B23}$$

which implies

$$C_0 = \frac{4\pi a_0}{M}\,. \tag{B24}$$

Note that the matching includes the term linear in k in (B23) which is a non-trivial consistency check. Equation (B24) implies that we have matched the leading low-energy constant C_0 to an experimentally measurable quantity, the scattering length a_0.

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