New Wilson line based action for gluodynamics

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Introduction

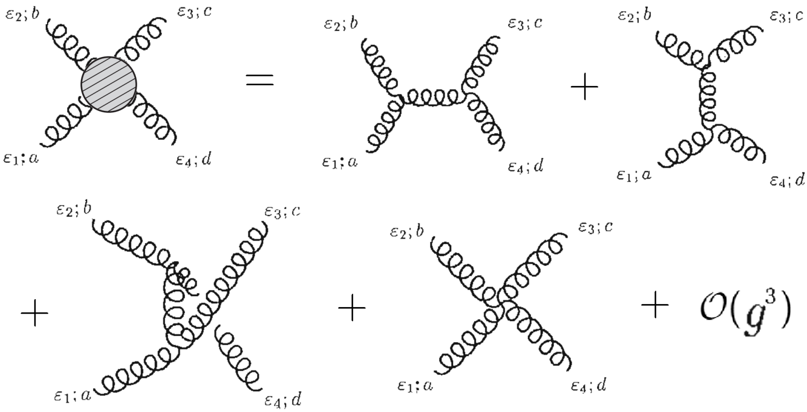
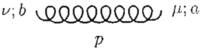
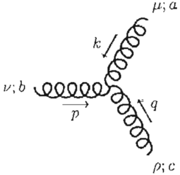


Figure: $gg \to gg$ scattering

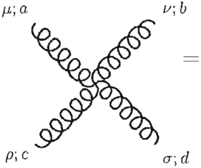
Feynman rules for QCD



$$= i\frac{-g^{\mu\nu} + (1-\xi)\frac{p^{\mu}p^{\nu}}{p^2}}{p^2 + i\varepsilon}\delta^{ab}$$



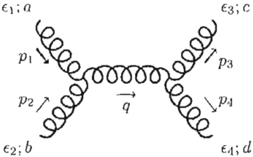
$$= g f^{abc}\left[g^{\mu\nu}(k-p)^{\rho} + g^{\nu\rho}(p-q)^{\mu} + g^{\rho\mu}(q-k)^{\nu}\right]$$



$$= -ig^2 \times \left[f^{abe}f^{cde}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + f^{ace}f^{bde}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + f^{ade}f^{bce}\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma}\right)\right]$$

4-point Green's function calculation

$$i\mathcal{M}_s\left(p_1p_2 \to p_3p_4\right) =$$



$$= -i\frac{g_s^2}{s} f^{abe} f^{cde}\left[(\epsilon_1\cdot\epsilon_2)(p_1-p_2)^\mu + \epsilon_2^\mu(p_2+q)\cdot\epsilon_1 + \epsilon_1^\mu(-q-p_1)\cdot\epsilon_2\right]$$
$$\times\left[(\epsilon_4^\star\cdot\epsilon_3^\star)(p_4-p_3)^\mu + \epsilon_3^{\star\mu}(p_3+q)\cdot\epsilon_4^\star + \epsilon_4^{\star\mu}(-q-p_4)\cdot\epsilon_3^\star\right],$$

Figure: Exemplary contribution to the $gg \to gg$ process.

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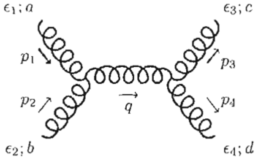

$$= -i\frac{g_s^2}{s} f^{abe} f^{cde} \left[(\epsilon_1 \cdot \epsilon_2)(p_1 - p_2)^\mu + \epsilon_2^\mu (p_2 + q) \cdot \epsilon_1 + \epsilon_1^\mu (-q - p_1) \cdot \epsilon_2\right]$$
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Figure: Exemplary contribution to the $gg \to gg$ process.

Obtaining cross section

To obtain cross section, one needs to add all other contributions, take their module squared and then sum over polarizations and color factors, which yields around 1000 terms at just tree level!

4-point Green's function calculation cont.

The squared amplitude for $gg \to gg$

$$\frac{1}{256}\sum_{\substack{\text{pols.}\\\text{colors}}} |\mathcal{M}|^2 = g_s^4 \frac{9}{2}\left(3 - \frac{tu}{s^2} - \frac{su}{t^2} - \frac{st}{u^2}\right)$$

Remarkably simple!

4-point Green's function calculation cont.

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The same result can be obtained in a much simpler way through the use of *spinor helicity formalism* in which only two amplitudes contribute

$$\widetilde{\mathcal{M}}(1^-2^-3^+4^+) = \frac{\langle 12 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}, \quad \widetilde{\mathcal{M}}(1^-2^+3^-4^+) = \frac{\langle 13 \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 41 \rangle}$$

4-point Green's function calculation cont.

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There surely must be some redundancy in our description.

Spinor helicity formalism

Helicity spinors

They are a useful tool to compactly decribe two degrees of freedom (momentum and helicity) in one object with properties that are useful during computation of amplitudes

$$\mathcal{M}(p_1, ..., p_n; \epsilon_1(p_1), ..., \epsilon_n(p_n)) \to \mathcal{M}(\lambda_1, ..., \lambda_n)$$

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Any massless(!) four-vector can be written as a helicity spinor outer product

$$p^{\alpha\dot{\alpha}} \equiv p^{\mu}\sigma_{\mu}^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}$$

with $\alpha = 1, 2$, σ - Pauli matrices, λ^{α} - helicity spinors.

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with $\alpha = 1, 2$, σ - Pauli matrices, λ^{α} - helicity spinors. Introducing notation

$$\lambda^{\alpha} = p\rangle, \quad \lambda_{\alpha} = \langle p, \quad \tilde{\lambda}_{\dot{\alpha}} = p], \quad \tilde{\lambda}^{\dot{\alpha}} = [p$$

$$p^{\alpha\dot{\alpha}} = p\rangle[p, \quad p_{\dot{\alpha}\alpha} = p]\langle p, \quad \text{and} \quad p \cdot q = \frac{1}{2}\langle pq\rangle[qp].$$

Color Decomposition

The idea is to disentangle the color degrees of freedom from the rest of an amplitude. This is done by utilizing the $SU(N)$ generators properties.

Color decomposed amplitudes

$$\mathcal{M}_n(\{\lambda_i, a_i\}) = \sum_{\sigma \in S_n/\mathbb{Z}_n} Tr(t^{a_{\sigma_1}} \ldots t^{a_{\sigma_n}}) \widetilde{\mathcal{M}}_n(\lambda_{\sigma_1} \ldots \lambda_{\sigma_n})$$

$\widetilde{\mathcal{M}}$'s are called *color-ordered* amplitudes

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Color-ordered amplitudes contain less diagrams, as the only contributions are from *planar diagrams*, meaning that no external legs cross.

external legs	4	5	6	7	8
diagrams	4	25	220	2485	34300
planar diagrams	3	10	38	154	654

Rapid scaling of diagram amount

In addition to many terms in a given diagram, the amount of the latter grows very quickly with increase of external legs.

Tree level diagrams

$n = 4$	4 diagrams
$n = 5$	25 diagrams
$n = 6$	220 diagrams
$n = 7$	2845 diagrams
$n = 8$	34300 diagrams

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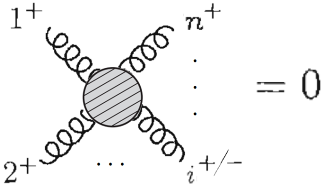
But fixing the helicities, we recognize that some of them do not give any contributions! The first nontrivial ones are the so-called MHV amplitudes.



MHV amplitudes

MHV amplitudes

Maximally Helicity Violating (MHV) amplitudes are ones where two of the external particles have helicity - or + and the rest has the opposite.

Due to Parke, Taylor (1986) we know that in the spinor helicity formalism the MHV amplitudes are surprisingly simple

$$\widetilde{\mathcal{M}}(1^+, ..., i^-, ..., j^-, ..., n^+) \sim g^{n-2} \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle ... \langle n1 \rangle}.$$

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Problem

But how can non local object like scattering amplitudes serve as vertices that are local?

CSW Method

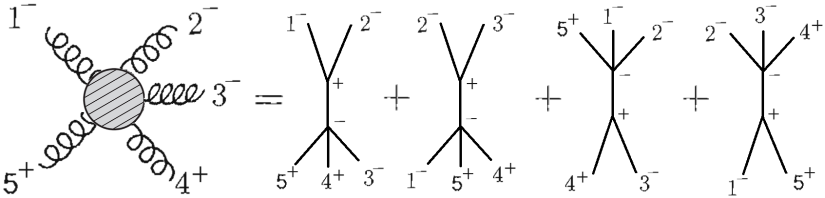

Figure: Example of CSW method application to a NMHV ($\overline{MHV}$) 5-point amplitude.

CSW Method

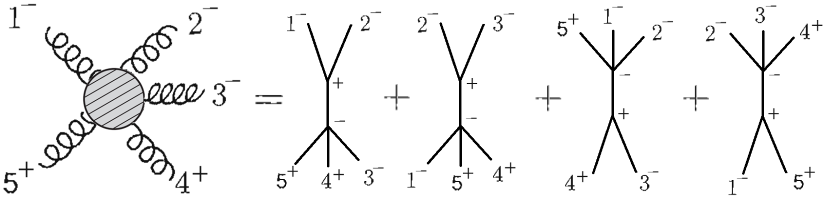

Figure: Example of CSW method application to a NMHV ($\overline{MHV}$) 5-point amplitude.

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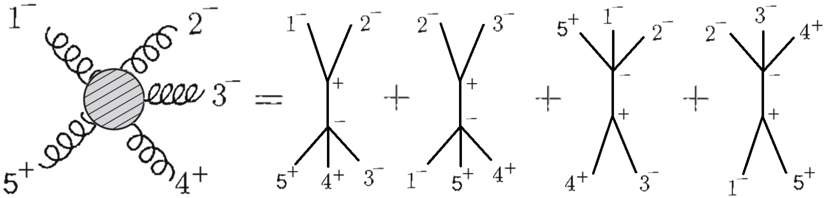

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Comment

Note that we have reduced the problem from a four-vector field to two fields of helicity either + or -. This is equivalent to a specific complex scalar field theory with an internal $SU(3)$ symmetry.

Lagrangian formulation

Light Cone Yang-Mills Lagrangian

$$\mathcal{L}^{LC}[A_+, A_-] = \mathcal{L}_{+-} + \mathcal{L}_{++-} + \mathcal{L}_{+--} + \mathcal{L}_{++--}$$

A_+, A_- are the positive and negative helicity fields, respectively.

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MHV Lagrangian

$$\mathcal{L}^{MHV}[B_+, B_-] = \mathcal{L}_{+-} + \mathcal{L}_{+--} + \mathcal{L}_{++--} + \mathcal{L}_{+++--} + \dots$$

There is an infinite series of $\mathcal{L}^{MHV}$ vertex terms with ever increasing amount of $+$ helicity legs - each vertex is MHV!

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The B fields can be further canonically transformed to remove the other three-vertex. This, however, generates new terms.

Z field transformation - H. Kakkad, P. Kotko, A. Stasto

Z theory Lagrangian

$$\begin{aligned}\mathcal{L}^Z[Z^+,Z^-] = \mathcal{L}_{+-} &+ \mathcal{L}_{++--} + \mathcal{L}_{++---} + \mathcal{L}_{++----} + \dots \\ &+ \mathcal{L}_{+++--} + \mathcal{L}_{+++---} + \dots \\ &+ \mathcal{L}_{++++--} + \dots \\ &\vdots\end{aligned}$$

No triple vertices! This reduces the amount of diagrams greatly, while the vertices are still written in a simple fashion.

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&+ \mathcal{L}_{++++--} + \dots \\
&\vdots
\end{aligned}
$$

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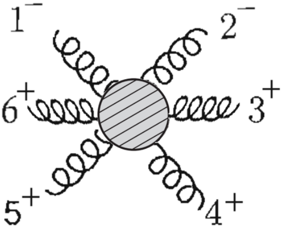
Consistency check

All plus/minus amplitudes (and ones with one helicity flipped) simply do not exist, since there are no such vertices in the theory!

$$
\mathcal{A}(1^+ \dots + i^\pm + \dots + n^+) = 0
$$

Z theory amplitude example



$$= g^4 \left(\frac{p_1^+}{p_2^+}\right)^2 \frac{\tilde{v}_{21}^{4*}}{\tilde{v}_{16}^{*}\,\tilde{v}_{65}^{*}\,\tilde{v}_{54}^{*}\,\tilde{v}_{43}^{*}\,\tilde{v}_{32}^{*}\,\tilde{v}_{21}^{*}}$$

Figure: The expression for a 6 point MHV amplitude in the Z theory.

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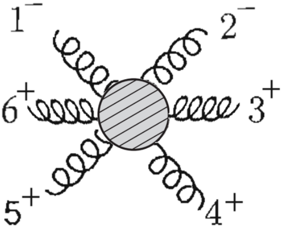

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Preserved simplicity

We use the off-shell continuation of spinors, but since

$$\tilde{v}_{ij} \sim \langle ij \rangle, \qquad \tilde{v}_{ij}^* \sim [ij],$$

the simple algebraic structure of amplitudes still holds.

Expansion in diagrams

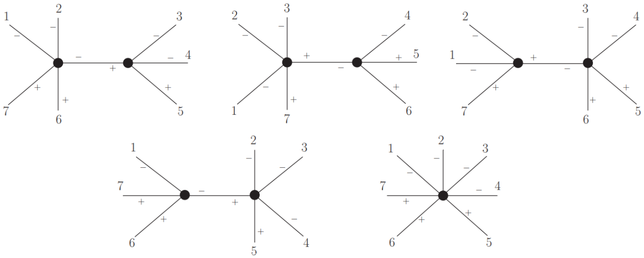

Figure: Diagrams contributing to an exemplary 7-point NNMHV amplitude [2].

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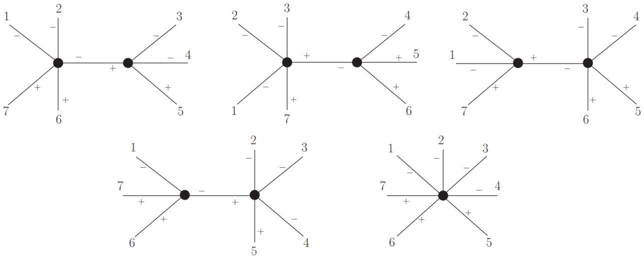

Figure: Diagrams contributing to an exemplary 7-point NNMHV amplitude [2].

external legs	4	5	6	7	8
pure QFT	4	25	220	2485	34300
Z theory	1	2	5	12	29

Table: The Z theory generates significantly less diagrams for a given amplitude, and the scaling is greatly quenched!

Conclusions

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Outlook

Further work will focus on extending the methods to loop-level and finding a suitable renormalization scheme.

Bonus Slides

Duality of Minkowski and Twistor space

Revisiting the problem of locality

Due to Witten, we know that tree level amplitudes localize on algebraic curves in twistor space. The degree of the curve is $d = n_- - 1$, where n_- is a number of negative helicity legs.

MHV aplitudes are lines in $\mathbb{PT}$

$$\Downarrow$$

MHV amplitudes are points (local!) in $\mathbb{M}^4$!

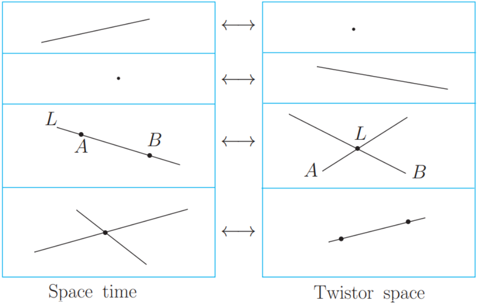

Figure: Correspondence of objects in Minkowski $\mathbb{M}^4$ and Twistor $\mathbb{PT}$ spaces [2].

Delannoy numbers

Surprising observation

The amount of diagrams for a given amplitude seems to follow the Delannoy numbers $D(n, m) = \sum_{i=0}^{n} \binom{m}{i}\binom{n+m-i}{m}$.

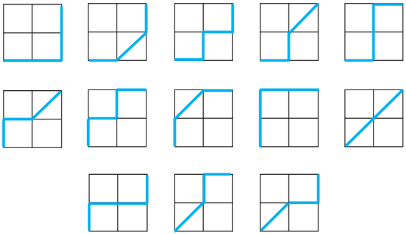

Figure: Exemplary Delannoy number determination for $\mathcal{L}_{++++----}$ [2].

# legs	helicity	# diagrams
4 point	MHV	1
	$\overline{\text{MHV}}$	1
5 point	MHV	1
	$\overline{\text{MHV}}$	1
6 point	MHV	1
	NMHV	3
	$\overline{\text{MHV}}$	1
7 point	MHV	1
	NMHV	5
	NNMHV	5
	$\overline{\text{MHV}}$	1
8 point	MHV	1
	NMHV	7
	NNMHV	13
	NNNMHV	7
	$\overline{\text{MHV}}$	1

Figure: The amount of diagrams in the Z theory [2].

Thank you for your attention!

Any questions are welcome.

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Kakkad Hiren, Kotko Piotr, Stasto Anna
A new Wilson line-based action for gluodynamics
https://arxiv.org/abs/2102.11371

Spinor representation of momentum

Momentum in the amplitudes can also be represented as a 2x2 matrix

$$p^{\alpha\dot{\alpha}} \equiv p^{\mu}\sigma_{\mu}^{\alpha\dot{\alpha}} = \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \\ -p^1 - ip^2 & p^0 + p^3 \end{pmatrix}, \quad det(p^{\alpha\dot{\alpha}}) = (p^0)^2 - \mathbf{p}^2 = \mu^2$$

In the massless case the determinant is equal to 0, so the matrix can be written as an outer product of spinors with metric $\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

$$p^{\alpha\dot{\alpha}} = \lambda^{\alpha}\tilde{\lambda}^{\dot{\alpha}}$$

with explicit decomposition

$$\lambda^{\alpha} = \frac{z}{\sqrt{p^0 - p^3}} \begin{pmatrix} p^0 - p^3 \\ -p^1 - ip^2 \end{pmatrix}, \quad \tilde{\lambda}^{\dot{\alpha}} = \frac{z^{-1}}{\sqrt{p^0 - p^3}} \begin{pmatrix} p^0 - p^3 & -p^1 + ip^2 \end{pmatrix}$$

Color Decomposition

Due to the identity

$$i\sqrt{2}f^{abc} = Tr(t^a t^b t^c) - Tr(t^c t^b t^a),$$

where t^a - $SU(3)$ generators. The contraction $f^{abe}f^{ecd}$ can be written as

$$Tr(t^a t^b t^e)\,Tr(t^e t^c t^d) \sim Tr(t^a t^b t^c t^d)$$

owing to the Fierz identity $\sum_a (t^a)^{j_1}_{i_1}(t^a)^{j_2}_{i_2} = \delta^{j_2}_{i_1}\delta^{j_1}_{i_2} - \cancel{\frac{1}{N}\delta^{j_1}_{i_1}\delta^{j_2}_{i_2}}$ tree level

Vanishing of the trivial MHV amplitudes

Polarization vectors can depend (due to gauge invariance) on some reference vector. Product of the former with the same helicity is given in spinor formalism by

$$\epsilon_i^+(p) \cdot \epsilon_j^+(q) = \frac{\langle pq \rangle [ji]}{\langle ri \rangle [rj]}.$$

The spinor products are asymmetric so, choosing $p = q$ yields $\epsilon_i^+(p) \cdot \epsilon_j^+(p) = 0$, which can always be done, and there is such factor in every MHV amplitude term (because there is always greater amount of external legs than vertices at tree level).

For one negative helicity, say $\epsilon_j^-(p)$, we choose the reference momentum $q = p_1$ for all other polarization vectors so that

$$\epsilon_i^+(j) \cdot \epsilon_j^-(p) = \frac{[ip]\langle jj \rangle}{\langle ji \rangle [jr]} = 0.$$

While the both-positive helicity combinations still vanish since they have the same reference.(Notice different form of product due to different helicity combination!)

Light Cone Yang-Mills Lagrangian

The Lagrangian is obtained by considering the Yang-Mills Lagrangian with double null-cone coordinates

$$v^{+} = v \cdot \eta \qquad v^{-} = v \cdot \tilde{\eta}$$
$$v^{\bullet} = v \cdot \varepsilon_{\perp}^{+} \qquad v^{\star} = v \cdot \varepsilon_{\perp}^{-},$$

with

$$\eta = (1,0,0,1), \qquad \tilde{\eta} = (1,0,0,-1), \qquad \varepsilon_{\perp}^{\pm} = (0,1,\pm i,0).$$

Then one can use the light cone gauge $A^{+} = 0$ and integrate out the A^{-} field. The vectors become

$$v = v^{+}\tilde{\eta} + v^{-}\eta - v^{\star}\varepsilon_{\perp}^{+} - v^{\bullet}\varepsilon_{\perp}^{-}.$$

Caution!

In the main presentation the notation in such that we have $A^{\pm}$ fields. They are actually $A^{\bullet/\star}$ fields renamed to intuitively connect them to $\pm$ helicity fields (which is entirely valid!).