

# Semi-Classical Yang-Mills with $n_f = 12$ at $T = 0$

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But wait...

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Methods used were derived from Euler/Heisenberg-1936 and Swinger-1951 & 1954

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AdS/CFT has stolen the spotlight for the last 30 years,

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$$A^a = B^a + a^a$$

Dropping terms linear in  $a^a$ , the gauge fixed Lagrangian is

$$L = \frac{1}{4}(F^a)^2 + \frac{1}{2}a^a (D^2)^{ac} g^{bc} f^{abc} F^b a^c + \frac{1}{2} B^2 + c^a (D^2)^{ac} c^c: \quad (3)$$

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$$Z = \int dB^a e^{-\int_x \left[ \frac{1}{4} (F^a)^2 \right] - \frac{1}{2} \ln \det(\text{Glue}) + \frac{n_f}{2} \ln \det(\text{Dirac}) + \ln \det(\text{Ghosts})} \quad (4)$$



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Leads to a partition functions for gauge theories which can be solved analytically (quadratic in all fields):

$$Z = \int dB^a e^{-\int d^4x \left[ \frac{1}{4} (F^a)^2 \right] - \frac{1}{2} \ln \det(\text{Glue}) + \frac{n_f}{2} \ln \det(\text{Dirac}) + \ln \det(\text{Ghosts})} \quad (4)$$

These  $B^a$ 's are just the operators found in (3)

I use

$$\ln \det( ) = \frac{d}{ds} \Big|_{s=0} \quad (5)$$

where  $\zeta$  is the zeta-function of an operator

# Solving Methods

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$$\ln \det(\Delta) = \left. \frac{d}{ds} \zeta(s) \right|_{s=0} \quad (5)$$

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$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} = \sum_{n=1}^{\infty} \frac{1}{s_n^s} = \int_0^{\infty} dt \, t^{s-1} \text{Tr} K(t); \quad (6)$$

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and  $K(t)$  is the heat kernels of the operator (Bertlmann Anomalies in QFT ch. 5)

$$K(t) = \text{Tr}[e^{-t\Delta}]; \quad (7)$$

The heat kernel is obtained by tracing out  $x$  which naturally leads to a partition function for each operator

$$\text{Tr}[e^{-\beta H}] = \int \mathcal{D}x \int \mathcal{D}j [e^{-S[x, j]}] \quad (8)$$

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$$\text{Tr}[e^{-\beta H}] = \int dx \langle x | e^{-\beta H} | x \rangle \quad (8)$$

$$\text{Tr}[e^{-\beta H}] = \int Dx Dp e^{\int_0^{\beta} ds (i p \dot{x} - H(x, p))} \quad (9)$$

The heat kernel is obtained by tracing out  $\psi$  which naturally leads to a partition function for each operator

$$\text{Tr}[e^{-\beta H}] = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \int \mathcal{D}x [e^{-\beta H} \psi \bar{\psi}] \quad (8)$$

$$\text{Tr}[e^{-\beta H}] = \int \mathcal{D}x \mathcal{D}p e^{-\int_0^{\beta} ds (\frac{1}{2} p^2 + i p \dot{x})} \quad (9)$$

HW: find the heat kernels need for (3) for SU(3).

# Results

Using all of this machinery the  $\mathfrak{su}(3)$  abelian sub-sector of  $SU(3)$  gives

$$\begin{aligned}
 & \frac{1}{(s)_0} \int_{\mathbb{R}^3} d^3s \int_{\mathbb{R}^3} d^3Z \left[ \frac{1}{2} K_{\text{Gluon}} + \frac{n_f}{2} K_{\text{Dirac}} + K_{\text{Ghosts}} \right] \\
 &= \frac{3 \text{Vol}}{16^2} \frac{B^2}{(s)_0} \int_{\mathbb{R}^3} d^3s \int_{\mathbb{R}^3} d^3Z \left[ \frac{1}{4} \frac{d}{2} + B^2 \frac{3dn_f}{4} \right] 9d \quad (10) \\
 &+ \frac{dn_f}{2} \frac{1}{\sinh^2(B)} + \frac{dn_f}{4} \frac{1}{\sinh^2\left(\frac{B}{2}\right)} + 9 \frac{1}{2} \frac{1}{\sinh^2\left(\frac{3B}{2}\right)} \quad \#
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Using all of this machinery the  $8$  abelian sub-sector of  $SU(3)$  gives

$$\begin{aligned}
 & \frac{1}{(s)_0} \int_{\mathbb{S}^d} d^s s \int_{\mathbb{S}^1} Z_1 \left[ \frac{1}{2} K_{\text{Gluon}} + \frac{n_f}{2} K_{\text{Dirac}} + K_{\text{Ghosts}} \right] \\
 &= \frac{3 \text{Vol}}{16^2} \frac{B^2}{(s)_0} \int_{\mathbb{S}^d} d^s s \int_{\mathbb{S}^1} Z_1 \left[ \frac{d}{2} + B^2 \frac{3dn_f}{4} \right] \quad (10) \\
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 \end{aligned}$$

For  $d = 4$  and  $n_f = 12$  this gives

$$\frac{d}{ds} \text{total}_{s=0} = \frac{B^2 \text{Vol}}{16^2} \left[ 36 \log(A) + 3 \log \frac{2}{B} + 3 + \log \frac{27}{4} \right] \quad (11)$$

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# Polyakov loop

the Polyakov loop

$$h_i = \lim_{\beta \rightarrow \infty} \text{Tr} \frac{1}{N} \int_{\mathbb{R}^3} d\mathbf{B} e^{i \int_{\mathbb{R}^3} S[\mathbf{B}] + \int_{\mathbb{R}^3} \mathbf{B}_0^a t^a} \quad (12)$$

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Notice we are just exponentiating a constant with some spatial dependence times the generators. Only gives a vanishing Polyakov loop in the  $\beta \rightarrow \infty$  limit.

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$$h_i = \lim_{\beta \rightarrow \infty} \text{Tr} \frac{1}{N} \int_{\mathbb{S}^1} d\mathbf{B}_0^a \exp(i \int_0^{2\pi} dt \mathbf{B}_0^a) \quad (12)$$

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recall  $SU(3)$  is three copies of  $SU(2)$  embedded which overlap (in the adjoint representation) via  $\lambda_3$  and  $\lambda_8$ . These two diagonal generators can give all eigenvalues of  $SU(3)$ . Without them we have  $SU(2) \times SU(2) \times SU(2)$  which would not give a vanishing Polyakov loop for constant  $\mathbf{B}$ .

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